

SUPPLEMENT TO “SOLVING ASSET PRICING MODELS WHEN
THE PRICE-DIVIDEND FUNCTION IS ANALYTIC”
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THIS APPENDIX PROVIDES PROOFS of some results stated in our paper. First, we prove the existence and uniqueness of the price-dividend function (Proposition 1) to the integral equation (6) in the vector space \mathcal{S} (Definition 1). Second, we calculate the system of linear equations (22) for the coefficients of the power series for the price-dividend function. Finally, we use Cauchy’s integral formula to bound all the derivatives of the price-expected dividend function (23) so that we can calculate the error terms (24), (25), and (26).

We use the following notation in the proofs: $C(n) = \sum_{i=1}^n |\phi|^i = \frac{|\phi|(1-|\phi|^n)}{1-|\phi|}$ for $n = 0, 1, 2, \dots$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-s^2/2} ds$, for all $x \in \mathbb{R}$, $\theta = x_0 + \sigma^2(1 + \phi - \alpha)(1 - \gamma)$, $K_3 = K_0 \exp(x_0 K_1 + \frac{\sigma^2(1-\gamma)^2(\phi-\alpha)(2+\phi-\alpha)}{2})$, $K_4 = K_3[1 + 2\Phi(\frac{|\phi K_1| \sigma}{1-|\phi|})]$, $K_5 = K_4 \exp(\frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|^2)} + \frac{|\phi K_1 \theta|}{1-|\phi|})$, and $K_6 = \frac{1}{\sqrt{\pi}} K_0 \exp(-\frac{\sigma^2}{2}(1 - \gamma)^2 + \frac{\sigma^2}{2}[1 - \gamma + K_1]^2)$. The term Q is defined in (18), which satisfies the integral equation (19).

A series of lemmas are now proved which are used in the proof of Proposition 1.

LEMMA 2: *Let $a > 0$ and $\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-x-a}^{-x+a} e^{-s^2/2\sigma^2} ds$ for all $x \in \mathbb{R}$. Then $\varphi(x) \leq 2\Phi(\frac{a}{\sigma})$.*

PROOF: These proof follows from the fact that φ has a unique global maximum at $x = 0$. Q.E.D.

LEMMA 3: *For any real numbers A and k with $k \geq 0$, we have*

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-1/(2\sigma^2)(s-A)^2+k|s|} ds \leq e^{k^2\sigma^2/2+k|A|} [1 + 2\Phi(k\sigma)].$$

PROOF: Note that

$$-\frac{1}{2\sigma^2}(s-A)^2 + ks = -\frac{1}{2\sigma^2}(s-A-k\sigma^2)^2 + \frac{k^2\sigma^2}{2} + kA,$$

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-1/(2\sigma^2)(s-A)^2+k|s|} ds \\ &= \frac{1}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^0 e^{-1/(2\sigma^2)(s-A)^2-ks} ds + \int_0^{\infty} e^{-1/(2\sigma^2)(s-A)^2+ks} ds \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{k^2\sigma^2/2}}{\sqrt{2\pi\sigma}} \\
&\quad \times \left(e^{-kA} \int_{-\infty}^0 e^{-1/(2\sigma^2)(s-A+k\sigma^2)^2} ds + e^{kA} \int_0^{\infty} e^{-1/(2\sigma^2)(s-A-k\sigma^2)^2} ds \right) \\
&\leq \frac{e^{k^2\sigma^2/2+k|A|}}{\sqrt{2\pi\sigma}} \left(\int_{-\infty}^0 e^{-1/(2\sigma^2)(s-A+k\sigma^2)^2} ds + \int_0^{\infty} e^{-1/(2\sigma^2)(s-A-k\sigma^2)^2} ds \right) \\
&= \frac{e^{k^2\sigma^2/2+k|A|}}{\sqrt{2\pi\sigma}} \left(\int_{-\infty}^{-A+k\sigma^2} e^{-s^2/(2\sigma^2)} ds + \int_{-A-k\sigma^2}^{\infty} e^{-s^2/(2\sigma^2)} ds \right) \\
&= e^{k^2\sigma^2/2+k|A|} \left(1 + \frac{1}{\sqrt{2\pi\sigma}} \int_{-A-k\sigma^2}^{-A+k\sigma^2} e^{-s^2/(2\sigma^2)} ds \right).
\end{aligned}$$

By Lemma 2, we achieve $\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2\sigma^2}(s-A)^2 + k|s|) ds \leq \exp(\frac{k^2\sigma^2}{2} + k|A|)[1 + 2\Phi(k\sigma)]$. *Q.E.D.*

LEMMA 4: For any $f \in \mathcal{S}$, the function Tf defined by

$$(1) \quad (Tf)(x) = \frac{K_3 e^{\phi K_1 x}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} f(s) e^{-1/(2\sigma^2)[s-(\phi x+\theta)]^2} ds$$

lies in the vector space \mathcal{S} . Thus, T defines a linear transformation from the vector space \mathcal{S} to itself.

PROOF: By the definition of \mathcal{S} , we can find positive constants M and k such that $|f(x)| \leq M e^{k|x|}$. Using Lemma 3, we deduce

$$\begin{aligned}
|(Tf)(x)| &\leq \frac{K_3 e^{\phi K_1 x}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} |f(s)| e^{-1/(2\sigma^2)[s-(\phi x+\theta)]^2} ds \\
&\leq \frac{MK_3 e^{\phi K_1 x}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-1/(2\sigma^2)[s-(\phi x+\theta)]^2+k|s|} ds \\
&\leq MK_3 [1 + 2\Phi(k\sigma)] e^{\phi K_1 x + k^2\sigma^2/2 + k(|\phi||x| + |\theta|)} \\
&\leq MK_3 e^{k^2\sigma^2/2+k|\theta|} [1 + 2\Phi(k\sigma)] e^{|\phi|(k+|K_1|)|x|}.
\end{aligned}$$

The continuity of Tf can be easily checked by applying the standard argument in real analysis; for example, see Theorem 56 in Kaplan (1956). *Q.E.D.*

Now construct a sequence of functions $\{Q_n | n = 0, 1, 2, \dots\}$ by setting $Q_0 = 0$ and

$$(2) \quad Q_{n+1}(x) = K_0 + \frac{K_3 e^{\phi K_1 x}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} Q_n(s) \exp\left(-\frac{1}{2\sigma^2}[s - (\phi x + \theta)]^2\right) ds,$$

and set $Q = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n]$.

Recall that $C(n) = \sum_{i=1}^n |\phi|^i = \frac{|\phi|(1-|\phi|^n)}{1-|\phi|}$ satisfies (a) $C(n+1) = |\phi| \times (C(n) + 1)$, (b) $C(n) < \frac{|\phi|}{1-|\phi|}$, and (c) $\lim_{n \rightarrow \infty} C(n) = \frac{|\phi|}{1-|\phi|}$.

LEMMA 5: For $n = 0, 1, 2, \dots$,

$$\begin{aligned} 0 &\leq Q_{n+1}(x) - Q_n(x) \\ &\leq K_0 K_4^n \exp\left(\frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i)\right) e^{C(n) |K_1| |x|}. \end{aligned}$$

PROOF: These inequalities can be easily proven by induction on n . *Q.E.D.*

LEMMA 6: The series $Q = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n]$ is uniformly convergent on any bounded closed interval.

PROOF: Fix a bounded closed interval $[a, b]$. For any $x \in [a, b]$, by Lemma 5 we see

$$\begin{aligned} 0 &\leq \sum_{n=0}^{\infty} [Q_{n+1}(x) - Q_n(x)] \\ &\leq K_0 \sum_{n=0}^{\infty} K_4^n \exp\left(\frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i)\right) e^{C(n) |K_1| (|a|+|b|)}. \end{aligned}$$

The ratio test together with the Weierstrass M -test (see Theorem 30 of Kaplan (1956)) implies the uniform convergence of the series $Q = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n]$ on $[a, b]$, provided

$$\begin{aligned} &\lim_{n \rightarrow \infty} K_4 \exp\left(\frac{\sigma^2 K_1^2}{2} C(n)^2 + |K_1 \theta| C(n)\right) e^{[C(n+1) - C(n)] |K_1| (|a|+|b|)} \\ &= K_4 \exp\left(\frac{\sigma^2 K_1^2 \phi^2}{2(1-|\phi|)^2} + \frac{|\phi K_1 \theta|}{1-|\phi|}\right) = K_5 < 1. \end{aligned} \quad \text{Q.E.D.}$$

LEMMA 7: The function Q lies in the vector space \mathcal{S} .

PROOF: The continuity of Q follows from Theorem 31 in Kaplan (1956) and Lemma 6. By Lemma 5, we also have

$$|Q(x)| \leq K_0 \sum_{n=0}^{\infty} K_4^n \exp\left(\frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i)\right) e^{C(n) |K_1| |x|}$$

$$\begin{aligned} &\leq \left(K_0 \sum_{n=0}^{\infty} K_4^n \exp\left(\frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i) \right) \right) \\ &\quad \times \exp\left(\frac{|\phi K_1|}{1 - |\phi|} |x| \right). \end{aligned} \quad Q.E.D.$$

LEMMA 8: For any $x \in \mathbb{R}$, we have $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} [Q(s) - Q_n(s)] \times e^{-1/(2\sigma^2)[s - (\phi x + \theta)]^2} ds = 0$.

PROOF: This is an immediate consequence of Theorem 32 in Kaplan (1956) and Lemma 6. Q.E.D.

PROOF OF PROPOSITION 1: Equivalently, we are going to show that Equation (19) has a unique solution Q in the vector space \mathcal{S} .

Existence. Applying the limit as n approaches ∞ to Equation (2) shows that the function $Q = \sum_{n=0}^{\infty} [Q_{n+1} - Q_n] = \lim_{n \rightarrow \infty} Q_n$ is a solution to (19) in the vector space \mathcal{S} .

Uniqueness. Suppose that $\tilde{Q} \in \mathcal{S}$ is another solution to (19). Then Q and \tilde{Q} satisfy the functional equation $Q - \tilde{Q} = T(Q - \tilde{Q})$. By the definition of \mathcal{S} , we can also find positive constants M and k such that $|Q(x) - \tilde{Q}(x)| \leq M e^{k|x|}$. Using the results in the proof of Lemma 4, we see

$$\begin{aligned} |Q(x) - \tilde{Q}(x)| &\leq (T|Q - \tilde{Q}|)(x) \\ &\leq MK_3 e^{k^2 \sigma^2 / 2 + k|\theta|} [1 + 2\Phi(k\sigma)] e^{(|\phi|k + C(1)|K_1|)|x|} \end{aligned}$$

for all $x \in \mathbb{R}$. Applying this computation n times gives rise to

$$\begin{aligned} |Q(x) - \tilde{Q}(x)| &\leq MK_3^n \left\{ \prod_{i=0}^{n-1} \exp\left(\frac{(|\phi|^i k + C(i)|K_1|)^2 \sigma^2}{2} + (|\phi|^i k + C(i)|K_1|)|\theta| \right) \right. \\ &\quad \left. \times [1 + 2\Phi(\sigma|\phi|^i k + \sigma C(i)|K_1|)] \right\} e^{(|\phi|^n k + C(n)|K_1|)|x|}. \end{aligned}$$

Note that

$$\begin{aligned} &\lim_{n \rightarrow \infty} K_3^n \exp\left(\frac{(|\phi|^n k + C(n)|K_1|)^2 \sigma^2}{2} + (|\phi|^n k + C(n)|K_1|)|\theta| \right) \\ &\quad \times [1 + 2\Phi(\sigma|\phi|^n k + \sigma C(n)|K_1|)] \end{aligned}$$

$$\begin{aligned}
&= K_3 \exp\left(\frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|)^2} + \frac{|\phi K_1 \theta|}{1-|\phi|}\right) \left[1 + 2\Phi\left(\frac{|\phi K_1 \sigma|}{1-|\phi|}\right)\right] \\
&= K_5 < 1.
\end{aligned}$$

We can find a positive integer N and a positive real number $\delta < 1$ so that for any $n \geq N$,

$$\begin{aligned}
&K_3 \exp\left(\frac{(|\phi|^n k + C(n)|K_1|)^2 \sigma^2}{2} + (|\phi|^n k + C(n)|K_1|)|\theta|\right) \\
&\quad \times [1 + 2\Phi(\sigma|\phi|^n k + \sigma C(n)|K_1|)|\theta|] \leq \delta.
\end{aligned}$$

Then for all $n \geq N$, we have

$$\begin{aligned}
&|Q(x) - \tilde{Q}(x)| \\
&\leq MK_3^N \left\{ \prod_{i=0}^{N-1} \exp\left(\frac{(|\phi|^i k + C(i)|K_1|)^2 \sigma^2}{2} + (|\phi|^i k + C(i)|K_1|)|\theta|\right) \right. \\
&\quad \left. \times [1 + 2\Phi(\sigma|\phi|^i k + \sigma C(i)|K_1|)|\theta|] \right\} \delta^{n-N} e^{(|\phi|^n k + C(n)|K_1|)|x|}.
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} \delta^{n-N} = 0$ and $\lim_{n \rightarrow \infty} e^{(|\phi|^n k + C(n)|K_1|)|x|} = \exp\left(\frac{|\phi||K_1|}{1-|\phi|}|x|\right)$, we obtain $Q(x) = \tilde{Q}(x)$ for all $x \in \mathbb{R}$. This completes the proof of Proposition 1. *Q.E.D.*

Solving for the Coefficients in the Analytic Price–Dividend Function

The integral equation (6) may be written as

$$P(x) = K_0 e^{K_1 x} + K_0 e^{K_1 x - \sigma^2 / 2(1-\gamma)^2} I(x),$$

where $I(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{(1-\gamma)s - s^2 / (2\sigma^2)} P(x_0 + \phi x + s) ds$. To solve the integral equation we posit that $P(x)$ is analytic, in which case it has the functional form (21). Consequently we may write $P(x_0 + \phi x + s)$ as

$$P(x_0 + \phi x + s) = e^{K_1(x_0 + \phi x + s)} \sum_{k=0}^n b_k (x_0 + \phi x + s - x_*)^k.$$

Using the binomial theorem²⁶ we may rewrite this equation as

$$P(x_0 + \phi x + s) = e^{K_1(x_0 + \phi x + s)} \left(\sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} s^i \right).$$

Substituting this result into $I(x)$ we obtain

$$I(x) = e^{K_1(x_0 + \phi x)} \sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} \gamma_i,$$

where $\gamma_i = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp((1 - \gamma + K_1)s - \frac{s^2}{2\sigma^2}) s^i ds$. The evaluation of the integral γ_i entails the use of a change of variable followed by the evaluation of all the moments of the normal distribution: $\gamma_i = K_3 a_i$, where $K_7 = \frac{1}{\sqrt{\pi}} e^{\sigma^2/2[1-\gamma+K_1]^2}$ and $a_i \equiv \sum_{j=0}^i \binom{i}{j} (\sigma^2[1-\gamma+K_1])^{i-j} (\sqrt{2}\sigma)^j (1 + (-1)^j) (1/2) \Gamma[\frac{j+1}{2}]$.

We can now write $I(x)$ as $K_7 e^{K_1(x_0 + \phi x)} \sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} a_i$ so that the integral equation becomes

$$\begin{aligned} e^{-K_1 x} P(x) &= K_0 + K_6 e^{K_1(x_0 + \phi x)} \sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} a_i \\ &= \sum_{k=0}^n b_k (x - x_*)^k. \end{aligned}$$

Next we find the undetermined coefficients b_i , $i = 0, \dots, n$. To do this, we collect all the functions of x on the left-hand side of this equation and use Taylor's theorem to take an n th order Taylor expansion around x_* . This produces a system of $(n + 1)$ linear equations in the variables b_i , $i = 0, \dots, n$.

Define the functions $q(x) = K_1(x_0 + \phi x)$, $r_{i,k}(x) = (x_0 - x_* + \phi x)^{k-i}$, and $w_{i,k}(x) = \exp(q(x)) r_{i,k}(x) \approx \sum_{l=0}^n \frac{1}{l!} w_{i,k}^{(l)}(x_*) (x - x_*)^l$. Substituting for $w_{i,k}(x)$ into the integral equation gives us

$$\begin{aligned} K_0 + K_6 \sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} \left(\sum_{l=0}^n \frac{1}{l!} w_{i,k}^{(l)}(x_*) (x - x_*)^l \right) a_i \\ = \sum_{k=0}^n b_k (x - x_*)^k. \end{aligned}$$

²⁶The binomial theorem states that we may write $(x_0 + \phi x + s - x_*)^k = \sum_{i=0}^k \binom{k}{i} (x_0 + \phi x - x_*)^{k-i} s^i$.

Finally, equate the coefficients on the left- and right-hand sides of this equation to yield

$$b_0 = K_0 + K_6 \sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} w_{i,k}^{(0)}(x_*) a_i \quad \text{and}$$

$$b_l = K_6 \sum_{k=0}^n b_k \sum_{i=0}^k \binom{k}{i} \frac{1}{l!} w_{i,k}^{(l)}(x_*) a_i,$$

where $l = 1, \dots, n$. These equations are an $n + 1$ system of linear equations in the b_i 's.

Analytic Error of the Taylor Polynomial Approximation

The complex function $Q(z) = e^{-K_1 z} P(z)$ is analytic and it is expressible as a Taylor series

$$Q(z) = \sum_{k=0}^{\infty} \frac{Q^{(k)}(z_*)}{k!} (z - z_*)^k \quad \text{for any complex number } z,$$

where $z_* = x_*$. We may use Cauchy's integral formula to estimate the error of the Taylor polynomial approximation $Q(z) \approx \sum_{k=0}^n \frac{Q^{(k)}(z_*)}{k!} (z - z_*)^k$ near $z = z_*$.

Write C_r for the circle of radius $r > 0$ centered at $z = z_*$ in the complex plane. Cauchy's integral formula (see Corollary 5.9 in Conway (1973)) gives

$$Q^{(k)}(z_*) = \frac{k!}{2\pi i} \oint_{C_r} \frac{Q(z)}{(z - z_*)^{k+1}} dz \quad \text{for } k = 0, 1, 2, \dots$$

Each point $z = x + iy$ on C_r satisfies $x_* - r \leq x \leq x_* + r$ and $-r \leq y \leq r$. Set

$$A = K_0 \sum_{n=0}^{\infty} K_4^n \exp\left(\frac{\sigma^2 K_1^2}{2} \sum_{i=1}^{n-1} C(i)^2 + |K_1 \theta| \sum_{i=1}^{n-1} C(i)\right).$$

Using the result in the proof of Lemma 7, we obtain $0 \leq Q(s) \leq A \exp\left(\frac{|\phi K_1|}{1-|\phi|} |s|\right)$ for all $s \in \mathbb{R}$, and

$$\begin{aligned} |Q(z)| &\leq K_0 + \frac{K_3 |e^{\phi K_1(x+iy)}|}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} Q(s) |e^{-1/(2\sigma^2)[s-\phi x-\theta-i\phi y]^2}| ds \\ &= K_0 + \frac{K_3 e^{\phi K_1 x + \phi^2 y^2 / (2\sigma^2)}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} Q(s) e^{-1/(2\sigma^2)[s-\phi x-\theta]^2} ds \\ &\leq K_0 + \frac{AK_3 e^{|\phi K_1 x| + \phi^2 y^2 / (2\sigma^2)}}{\sqrt{2\pi\sigma}} \end{aligned}$$

$$\begin{aligned}
& \times \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}[s - \phi x - \theta]^2 + \frac{|\phi K_1|}{1-|\phi|}|s|\right) ds \\
& \leq K_0 + AK_3 \exp\left(|\phi K_1 x| + \frac{\phi^2 |K_1 x|}{1-|\phi|}\right. \\
& \quad \left. + \frac{\phi^2 y^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|)^2} + \frac{|\phi K_1 \theta|}{1-|\phi|}\right) \\
& \quad \times \left[1 + 2\Phi\left(\frac{|\phi K_1 \sigma|}{1-|\phi|}\right)\right] \\
& = K_0 + AK_3 \exp\left(\frac{|\phi K_1|(|x| + |\theta|)}{1-|\phi|} + \frac{\phi^2 y^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|)^2}\right) \\
& \quad \times \left[1 + 2\Phi\left(\frac{|\phi K_1 \sigma|}{1-|\phi|}\right)\right] \\
& \leq K_0 + AK_3 \exp\left(\frac{|\phi K_1|(r + |x_*| + |\theta|)}{1-|\phi|} + \frac{\phi^2 r^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|)^2}\right) \\
& \quad \times \left[1 + 2\Phi\left(\frac{|\phi K_1 \sigma|}{1-|\phi|}\right)\right].
\end{aligned}$$

Define

$$\begin{aligned}
B_r &= K_0 + AK_3 \exp\left(\frac{|\phi K_1|(r + |x_*| + |\theta|)}{1-|\phi|} + \frac{\phi^2 r^2}{2\sigma^2} + \frac{\phi^2 K_1^2 \sigma^2}{2(1-|\phi|)^2}\right) \\
& \quad \times \left[1 + 2\Phi\left(\frac{|\phi K_1 \sigma|}{1-|\phi|}\right)\right].
\end{aligned}$$

Cauchy's integral formula for $Q^{(k)}(z_*)$ together with the above estimate to $|Q(z)|$ gives rise to

$$\begin{aligned}
|Q^{(k)}(z_*)| &\leq \left| \frac{k!}{2\pi i} \oint_{C_r} \frac{Q(z)}{(z - z_*)^{k+1}} dz \right| = \frac{k!}{2\pi} \left| \int_0^{2\pi} \frac{Q(z_* + re^{i\theta})}{(re^{i\theta})^{k+1}} ire^{i\theta} d\theta \right| \\
&\leq \frac{k!}{2\pi} \int_0^{2\pi} \frac{|Q(z_* + re^{i\theta})|}{r^k} d\theta \leq \frac{B_r k!}{r^k}.
\end{aligned}$$

This bound allows us to estimate the analytic error (24).

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