

SUPPLEMENT TO “JUMP REGRESSIONS”  
 (Econometrica, Vol. 85, No. 1, January 2017, 173–195)

BY JIA LI, VIKTOR TODOROV, AND GEORGE TAUCHEN

This document contains all proofs of the paper.

THROUGHOUT THIS APPENDIX, we use  $K$  to denote a generic constant that may change from line to line; we sometimes emphasize the dependence of this constant on some parameter  $q$  by writing  $K_q$ . We use  $0_{k \times q}$  to denote a  $k \times q$  matrix of zeros, and when  $q = 1$ , we write  $0_k$  for notational simplicity;  $0_k$  is understood to be empty when  $k = 0$ . For any sequence of variables  $(\xi_{n,p})_{p \geq 1}$ , the convergence  $(\xi_{n,p})_{p \geq 1} \rightarrow (\xi_p)_{p \geq 1}$  is understood as  $n \rightarrow \infty$  under the product topology. We write w.p.a.1 for “with probability approaching 1.”

By a standard localization procedure (see Section 4.4.1 of Jacod and Protter (2012)), we can strengthen Assumption 1 to the following stronger version without loss of generality.

ASSUMPTION S1: We have Assumption 1. Moreover, the processes  $X_t$ ,  $b_t$ , and  $\sigma_t$  are bounded.

SUPPLEMENTAL APPENDIX A: PROOF OF PROPOSITION 1

(a) Since the jumps of  $Z$  have finite activity, we can assume without loss of generality that each interval  $((i - 1)\Delta_n, i\Delta_n]$  contains at most one jump; otherwise we can restrict our calculation to the w.p.a.1 set of sample paths on which this condition holds. We denote the continuous part of  $Z$  by  $Z^c$ , that is,

$$(SA.1) \quad Z_t^c = Z_t - \sum_{s \leq t} \Delta Z_s, \quad t \geq 0.$$

Note that  $\mathcal{I}_n(\mathcal{D})$  is the union of two disjoint sets  $\mathcal{I}_{1n}(\mathcal{D})$  and  $\mathcal{I}_{2n}(\mathcal{D})$  that are defined as

$$(SA.2) \quad \begin{aligned} \mathcal{I}_{1n}(\mathcal{D}) &= \mathcal{I}_n(\mathcal{D}) \cap \{i(p) : p \in \mathcal{P}\}, \\ \mathcal{I}_{2n}(\mathcal{D}) &= \mathcal{I}_n(\mathcal{D}) \setminus \mathcal{I}_{1n}(\mathcal{D}). \end{aligned}$$

It suffices to show that, w.p.a.1,

$$(SA.3) \quad \mathcal{I}_{1n}(\mathcal{D}) = \mathcal{I}(\mathcal{D}), \quad \mathcal{I}_{2n}(\mathcal{D}) = \emptyset.$$

First consider  $\mathcal{I}_{1n}(\mathcal{D})$ . Since  $v_n \rightarrow 0$ , we have  $|\Delta_{i(p)}^n Z| > v_n$  for all  $p \in \mathcal{P}$ , when  $n$  is large enough. Therefore,

$$(SA.4) \quad \mathcal{I}_{1n}(\mathcal{D}) = \{i(p) : p \in \mathcal{P}, ((i(p) - 1)\Delta_n, \Delta_{i(p)}^n Z) \in \mathcal{D}\} \quad \text{w.p.a.1.}$$

Now, observe that

$$(SA.5) \quad \sup_{p \in \mathcal{P}} \|((i(p) - 1)\Delta_n, \Delta_{i(p)}^n Z) - (\tau_p, \Delta Z_{\tau_p})\| \rightarrow 0 \quad \text{a.s.}$$

Indeed, almost surely,

$$\begin{aligned}
 \text{(SA.6)} \quad & \sup_{p \in \mathcal{P}} \left\| \left( (i(p) - 1) \Delta_n, \Delta_{i(p)}^n Z \right) - (\tau_p, \Delta Z_{\tau_p}) \right\| \\
 &= \sup_{p \in \mathcal{P}} \left\| \left( (i(p) - 1) \Delta_n - \tau_p, \Delta_{i(p)}^n Z^c \right) \right\| \\
 &\leq \Delta_n + \sup_{s, t \leq T, |s-t| \leq \Delta_n} |Z_t^c - Z_s^c| \rightarrow 0.
 \end{aligned}$$

By Assumption 2, the marks  $(\tau_p, \Delta Z_{\tau_p})_{p \in \mathcal{P}_{\mathcal{D}}}$  are contained in the interior of  $\mathcal{D}$  a.s. Then, by (SA.5),  $((i(p) - 1) \Delta_n, \Delta_{i(p)}^n Z)_{p \in \mathcal{P}_{\mathcal{D}}} \subseteq \mathcal{D}$  w.p.a.1. With the same argument but with  $\mathcal{D}^c$  (i.e., the complement of  $\mathcal{D}$ ) replacing  $\mathcal{D}$ , we deduce  $((i(p) - 1) \Delta_n, \Delta_{i(p)}^n Z)_{p \in \mathcal{P} \setminus \mathcal{P}_{\mathcal{D}}} \subseteq \mathcal{D}^c$  w.p.a.1. Therefore, the set on the right-hand side of (SA.4) coincides with  $\mathcal{I}(\mathcal{D})$  w.p.a.1. From here, the first claim of (SA.3) readily follows.

It remains to show that  $\mathcal{I}_{2n}(\mathcal{D})$  is empty w.p.a.1. Note that for  $i \in \mathcal{I}_{2n}(\mathcal{D})$ ,  $\Delta_i^n Z = \Delta_i^n Z^c$ . Hence, for any  $q > 2/(1 - 2\varpi)$ ,

$$\text{(SA.7)} \quad \mathbb{P}(\mathcal{I}_{2n}(\mathcal{D}) \neq \emptyset) \leq \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{P}(|\Delta_i^n Z^c| > v_n) \leq K_q \Delta_n^{-1} \frac{\Delta_n^{q/2}}{v_n^q} \rightarrow 0,$$

where the second inequality is by Markov's inequality and  $\mathbb{E}|\Delta_i^n Z^c|^q \leq K_q \Delta_n^{q/2}$ ; the convergence is due to (2.12) and our choice of  $q$ . The proof of part (a) is now complete.

(b) By part (a), it suffices to show that

$$\text{(SA.8)} \quad \left( (i - 1) \Delta_n, \Delta_i^n X \right)_{i \in \mathcal{I}(\mathcal{D})} - (\tau_p, \Delta X_{\tau_p})_{p \in \mathcal{P}_{\mathcal{D}}} = o_p(1).$$

Observe that  $((i - 1) \Delta_n, \Delta_i^n X)_{i \in \mathcal{I}(\mathcal{D})}$  is simply  $((i(p) - 1) \Delta_n, \Delta_{i(p)}^n X)_{p \in \mathcal{P}_{\mathcal{D}}}$ . We deduce the desired convergence via the same argument as that for (SA.5). *Q.E.D.*

#### SUPPLEMENTAL APPENDIX B: PROOF OF THEOREM 1

(a) Let

$$\text{(SB.9)} \quad \bar{\beta}(\mathcal{D}) \equiv \frac{Q_{ZY}(\mathcal{D})}{Q_{ZZ}(\mathcal{D})}.$$

For each  $p \geq 1$ , we set

$$\text{(SB.10)} \quad R_{n,p} = \Delta_n^{-1/2} (\Delta_{i(p)}^n X - \Delta X_{\tau_p}) \quad \text{and} \quad \mathfrak{s}_{n,p} = (-\bar{\beta}(\mathcal{D}), 1) R_{n,p}.$$

With these notations, we have in restriction to  $\Omega_0(\mathcal{D})$ ,

$$\text{(SB.11)} \quad \Delta_{i(p)}^n Y = \beta_0 \Delta_{i(p)}^n Z + \Delta_n^{1/2} \mathfrak{s}_{n,p}.$$

By Proposition 4.4.10 in Jacod and Protter (2012),  $(R_{n,p})_{p \geq 1} \xrightarrow{\mathcal{L}-s} (R_p)_{p \geq 1}$ , where  $R_p$  is defined in (3.2). Consequently (recall the notation (3.12)),

$$\text{(SB.12)} \quad (\mathfrak{s}_{n,p})_{p \geq 1} \xrightarrow{\mathcal{L}-s} (\mathfrak{s}_p)_{p \geq 1}.$$

By Proposition 1(a), w.p.a.1,

$$(SB.13) \quad \det[Q_n(\mathcal{D})] = \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \Delta_{i(p)}^n Z^2 \right) \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \Delta_{i(p)}^n Y^2 \right) - \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \Delta_{i(p)}^n Z \Delta_{i(p)}^n Y \right)^2.$$

Plug (SB.11) into (SB.13). After some algebra, we deduce

$$(SB.14) \quad \Delta_n^{-1} \det[Q_n(\mathcal{D})] = \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \Delta_{i(p)}^n Z^2 \right) \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} s_{n,p}^2 \right) - \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \Delta_{i(p)}^n Z s_{n,p} \right)^2.$$

Note that for each  $p \geq 1$ ,  $\Delta_{i(p)}^n Z \rightarrow \Delta Z_{\tau_p}$ . Combining this convergence with (SB.12), we use the property of stable convergence to derive the joint convergence

$$(SB.15) \quad (s_{n,p}, \Delta_{i(p)}^n Z)_{p \geq 1} \xrightarrow{\mathcal{L}\text{-}s} (s_p, \Delta Z_{\tau_p})_{p \geq 1}.$$

Since the set  $\mathcal{P}_{\mathcal{D}}$  is a.s. finite, the assertion of part (a) follows from (SB.14), (SB.15), and the continuous mapping theorem.

(b) By a standard localization argument (see Section 4.4.1 of Jacod and Protter (2012)), we assume that Assumption S1 holds without loss of generality. Since  $\mathcal{P}_{\mathcal{D}}$  is a.s. finite, we can also assume that  $|\mathcal{P}_{\mathcal{D}}| \leq M$  for some constant  $M > 0$  for the purpose of proving convergence in probability; otherwise, we can fix some large  $M$  to make  $\mathbb{P}(|\mathcal{P}_{\mathcal{D}}| > M)$  arbitrarily small and restrict the calculation below on the set  $\{|\mathcal{P}_{\mathcal{D}}| \leq M\}$ .

By Theorem 9.3.2 in Jacod and Protter (2012), we have

$$(SB.16) \quad \hat{c}_{n,i(p)-} \xrightarrow{\mathbb{P}} c_{\tau_{p-}}, \quad \hat{c}_{n,i(p)+} \xrightarrow{\mathbb{P}} c_{\tau_p} \quad \text{all } 1 \leq p \leq M.$$

By Proposition 1(b),

$$(SB.17) \quad Q_n(\mathcal{D}) \xrightarrow{\mathbb{P}} Q(\mathcal{D}),$$

which further implies (with  $\tilde{\beta}_n \equiv Q_{ZY,n}(\mathcal{D})/Q_{ZZ,n}(\mathcal{D})$ )

$$(SB.18) \quad \tilde{\beta}_n \xrightarrow{\mathbb{P}} \bar{\beta}(\mathcal{D}).$$

Furthermore, by essentially the same argument as in the proof of Proposition 1(a), we deduce

$$(SB.19) \quad \mathcal{I}'_n(\mathcal{D}) = \mathcal{I}(\mathcal{D}) \quad \text{w.p.a.1.}$$

Therefore,

$$(SB.20) \quad \tilde{\zeta}_n(\mathcal{D}) = \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \Delta_{i(p)}^n Z^2 \right) \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \tilde{s}_{n,i(p)}^2 \right) - \left( \sum_{p \in \mathcal{P}_{\mathcal{D}}} \Delta_{i(p)}^n Z s_{n,i(p)} \right)^2 \quad \text{w.p.a.1.}$$

Fix any subsequence  $\mathbb{N}_1 \subseteq \mathbb{N}$ . By (SB.16) and (SB.18), we can extract a further subsequence  $\mathbb{N}_2 \subseteq \mathbb{N}_1$ , such that along  $\mathbb{N}_2$ ,

$$(SB.21) \quad ((\hat{c}_{n,i(p)-}, \hat{c}_{n,i(p)+})_{1 \leq p \leq M}, \tilde{\beta}_n) \rightarrow ((c_{\tau_{p-}}, c_{\tau_p})_{1 \leq p \leq M}, \bar{\beta}(\mathcal{D}))$$

on some set  $\tilde{\Omega}$  with  $\mathbb{P}(\tilde{\Omega}) = 1$ . Then, for each  $\omega \in \tilde{\Omega}$  fixed, the transition kernel of  $\tilde{\zeta}_n(\mathcal{D})$  given  $\mathcal{F}$  converges weakly to the  $\mathcal{F}$ -conditional law of  $\zeta(\mathcal{D})$ . Moreover, observe that the  $\mathcal{F}$ -conditional law of the variables  $(s_p)_{1 \leq p \leq M}$  does not have atoms and has full support on  $\mathbb{R}^M$ . Therefore, the  $\mathcal{F}$ -conditional distribution function of  $\zeta(\mathcal{D})$  is continuous and strictly increasing. By Lemma 21.2 in van der Vaart (1998), we deduce that on each path  $\omega \in \tilde{\Omega}$ , along the subsequence  $\mathbb{N}_2$ ,  $cv_n^\alpha \rightarrow cv^\alpha$ , where  $cv^\alpha$  is the  $\mathcal{F}$ -conditional  $(1 - \alpha)$ -quantile of  $\zeta(\mathcal{D})$ . Since the subsequence  $\mathbb{N}_1$  is arbitrarily chosen, we further deduce that  $cv_n^\alpha \xrightarrow{\mathbb{P}} cv^\alpha$  by the subsequence characterization of convergence in probability. The proof for part (b) is now complete.

(c) By part (a) and part (b), as well as the property of stable convergence, we have

$$(SB.22) \quad (\Delta_n^{-1} \det[Q_n(\mathcal{D})], cv_n^\alpha, 1_{\Omega_0(\mathcal{D})}) \xrightarrow{\mathcal{L}\text{-}s} (\zeta(\mathcal{D}), cv^\alpha, 1_{\Omega_0(\mathcal{D})}).$$

In particular,

$$(SB.23) \quad \mathbb{P}(\{\Delta_n^{-1} \det[Q_n(\mathcal{D})] > cv_n^\alpha\} \cap \Omega_0(\mathcal{D})) \rightarrow \mathbb{P}(\{\zeta(\mathcal{D}) > cv^\alpha\} \cap \Omega_0(\mathcal{D})).$$

Since  $\mathbb{P}(\zeta(\mathcal{D}) > cv^\alpha | \mathcal{F}) = \alpha$  and  $\Omega_0(\mathcal{D}) \in \mathcal{F}$ , the right-hand side of (SB.23) equals to  $\alpha \mathbb{P}(\Omega_0(\mathcal{D}))$ . The first assertion of part (c) then follows from (SB.23). To show the second assertion of part (c), we first observe that (SB.17) implies  $\det[Q_n(\mathcal{D})] \xrightarrow{\mathbb{P}} \det[Q(\mathcal{D})]$ . In restriction to  $\Omega_a(\mathcal{D})$ ,  $\det[Q(\mathcal{D})] > 0$  and, hence,  $\Delta_n^{-1} \det[Q_n(\mathcal{D})]$  diverges to  $+\infty$  in probability. Part (b) implies that  $cv_n^\alpha$  is tight in restriction to  $\Omega_a(\mathcal{D})$ . Consequently,  $\mathbb{P}(\Delta_n^{-1} \det[Q_n(\mathcal{D})] > cv_n^\alpha | \Omega_a(\mathcal{D})) \rightarrow 1$  as asserted. Q.E.D.

#### SUPPLEMENTAL APPENDIX C: PROOF OF THEOREM 2

(a) Observe that

$$(SC.24) \quad Q_{ZY,n}(\mathcal{D}, w) - \beta_0 Q_{ZZ,n}(\mathcal{D}, w) \\ = \sum_{i \in \mathcal{I}_n(\mathcal{D})} w(\hat{c}_{i-}^n, \hat{c}_{i+}^n, \tilde{\beta}_n) \Delta_i^n Z (\Delta_i^n Y - \beta_0 \Delta_i^n Z).$$

Recall the notation  $s_{n,p}$  from (SB.10). By (SB.19), we further deduce that, w.p.a.1,

$$(SC.25) \quad \Delta_n^{-1/2} (Q_{ZY,n}(\mathcal{D}, w) - \beta_0 Q_{ZZ,n}(\mathcal{D}, w)) \\ = \sum_{p \in \mathcal{P}_{\mathcal{D}}} w(\hat{c}_{n,i(p)-}, \hat{c}_{n,i(p)+}, \tilde{\beta}_n) \Delta_{i(p)}^n Z s_{n,p}.$$

By (SB.16), (SB.18), and Assumption 3,

$$(SC.26) \quad w(\hat{c}_{i(p)-}^n, \hat{c}_{i(p)+}^n, \tilde{\beta}_n) \xrightarrow{\mathbb{P}} w(c_{\tau_{p-}}, c_{\tau_p}, \beta_0), \quad p \geq 1.$$

Since  $\mathcal{P}_{\mathcal{D}}$  is a.s. finite, we use properties of stable convergence to deduce from (SB.12) and (SC.26) that

$$(SC.27) \quad \Delta_n^{-1/2}(Q_{ZY,n}(\mathcal{D}, w) - \beta_0 Q_{ZZ,n}(\mathcal{D}, w)) \xrightarrow{\mathcal{L}\text{-}s} \sum_{p \in \mathcal{P}_{\mathcal{D}}} w(c_{\tau_{p-}}, c_{\tau_p}, \beta_0) \Delta Z_{\tau_p} s_p.$$

Note that

$$(SC.28) \quad \Delta_n^{-1/2}(\hat{\beta}_n(\mathcal{D}, w) - \beta_0) = \frac{\Delta_n^{-1/2}(Q_{ZY,n}(\mathcal{D}, w) - \beta_0 Q_{ZZ,n}(\mathcal{D}, w))}{Q_{ZZ,n}(\mathcal{D}, w)}.$$

By (SB.19),

$$(SC.29) \quad Q_n(\mathcal{D}, w) = \sum_{p \in \mathcal{P}_{\mathcal{D}}} w(\hat{c}_{i(p)-}^n, \hat{c}_{i(p)+}^n, \tilde{\beta}_n) \Delta_{i(p)}^n X \Delta_{i(p)}^n X^\top.$$

By  $\Delta_{i(p)}^n X \rightarrow \Delta X_{\tau_p}$  and (SC.26), we deduce

$$(SC.30) \quad Q_n(\mathcal{D}, w) \xrightarrow{\mathbb{P}} \sum_{p \in \mathcal{P}_{\mathcal{D}}} w(c_{\tau_{p-}}, c_{\tau_p}, \beta_0) \Delta X_{\tau_p} \Delta X_{\tau_p}^\top.$$

The first assertion of part (a), that is,  $\Delta_n^{-1/2}(\hat{\beta}_n(\mathcal{D}, w) - \beta_0) \xrightarrow{\mathcal{L}\text{-}s} \zeta_\beta(\mathcal{D}, w)$ , readily follows from (SC.27), (SC.28), and (SC.30).

Turning to the second assertion of part (a), we first observe that when  $c_t$  does not jump at the same time as  $Z_t$ , each  $s_p$  is  $\mathcal{F}$ -conditionally centered Gaussian; moreover, the variables  $(s_p)_{p \geq 1}$  are  $\mathcal{F}$ -conditionally independent. Therefore, the limiting variable  $\zeta_\beta(\mathcal{D})$  is centered Gaussian conditional on  $\mathcal{F}$ , with conditional variance given by  $\Sigma(\mathcal{D}, w)$ . This finishes the proof of the second assertion.

(b) For notational simplicity, we denote

$$A_p = \frac{(-\beta_0, 1)(c_{\tau_{p-}} + c_{\tau_p})(-\beta_0, 1)^\top}{2\Delta Z_{\tau_p}^2}, \quad B_p = w(c_{\tau_{p-}}, c_{\tau_p}, \beta_0) \Delta Z_{\tau_p}^2.$$

Then we can rewrite  $\Sigma(\mathcal{D}, w)$  and  $\Sigma(\mathcal{D}, w^*)$  as

$$\Sigma(\mathcal{D}, w) = \frac{\sum_{p \in \mathcal{P}_{\mathcal{D}}} B_p^2 A_p}{\left(\sum_{p \in \mathcal{P}_{\mathcal{D}}} B_p\right)^2}, \quad \Sigma(\mathcal{D}, w^*) = \left(\sum_{p \in \mathcal{P}_{\mathcal{D}}} A_p^{-1}\right)^{-1}.$$

The assertion of part (b) is then proved by observing

$$\sqrt{\frac{\Sigma(\mathcal{D}, w)}{\Sigma(\mathcal{D}, w^*)}} = \frac{\sqrt{\sum_{p \in \mathcal{P}_{\mathcal{D}}} B_p^2 A_p} \sqrt{\sum_{p \in \mathcal{P}_{\mathcal{D}}} A_p^{-1}}}{\sum_{p \in \mathcal{P}_{\mathcal{D}}} B_p} \geq 1,$$

where the inequality is by the Cauchy–Schwarz inequality.

(c) By (SB.19) and (SC.26), as well as  $\Delta_{i(p)}^n Z \rightarrow \Delta Z_{\tau_p}$ , we deduce that the  $\mathcal{F}$ -conditional law of  $\tilde{\xi}_{n,\beta}(\mathcal{D}, w)$  converges in probability to that of  $\zeta_\beta(\mathcal{D}, w)$  under any metric for weak

convergence. From here, by using an argument similar to that in the proof of Theorem 1(b), we further deduce that

$$(SC.31) \quad cv_{n,\beta}^{\alpha/2} \xrightarrow{\mathbb{P}} cv_{\beta}^{\alpha/2},$$

where  $cv_{\beta}^{\alpha/2}$  denotes the  $(1 - \alpha/2)$ -quantile of the  $\mathcal{F}$ -conditional law of  $\zeta_{\beta}(\mathcal{D}, w)$ . It is easy to see that the  $\mathcal{F}$ -conditional law of  $\zeta_{\beta}(\mathcal{D}, w)$  is symmetric. The assertion of part (c) then follows from part (a) and (SC.31). *Q.E.D.*

#### SUPPLEMENTAL APPENDIX D: PROOF OF THEOREM 3

(a) Fix  $S \in \mathbf{S}$  and let  $m = \dim(S) - 1$ . We consider a sequence of subsets  $\Omega_n$  defined by

$$\Omega_n = \left\{ \begin{array}{l} \text{For every } 1 \leq i \leq \lfloor T/\Delta_n \rfloor, \text{ if } ((i-1)\Delta_n, i\Delta_n] \text{ contains} \\ \text{some jump of } Z, \text{ then this interval is contained in } (S_{j-1}, S_j] \\ \text{for some } 1 \leq j \leq m \text{ and it contains exactly one jump of } Z \end{array} \right\}.$$

Under Assumption 1, the process  $Z$  has finitely active jumps without any fixed time of discontinuity. Hence,  $\mathbb{P}(\Omega_n) \rightarrow 1$ , so we can restrict our calculation below on  $\Omega_n$  without loss of generality.

Below, we write  $h = (h_0, \dots, h_m)^\top$  and denote the log likelihood ratio by

$$L_n(h) = \log \frac{dP_{\theta_0 + \Delta_n^{1/2}h}^n}{dP_{\theta_0}^n}.$$

For each  $i \geq 1$ , we set  $h(n, i) = h_j$ , where  $j$  is the unique integer in  $\{1, \dots, m\}$  such that  $i\Delta_n \in (S_{j-1}, S_j]$ . On the set  $\Omega_n$ , with  $\theta = \theta_0 + \Delta_n^{1/2}h$ , we have

$$\begin{aligned} \Delta_i^n X &= \int_{(i-1)\Delta_n}^{i\Delta_n} b_s ds + \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dW_s \\ &+ \left( \begin{array}{c} (1 + \Delta_n^{1/2}h(n, i))\Delta_i^n J_Z \\ (\beta_0 + \Delta_n^{1/2}h_0)(1 + \Delta_n^{1/2}h(n, i))\Delta_i^n J_Z + \Delta_i^n \varepsilon \end{array} \right). \end{aligned}$$

To simplify notations, we denote, for each  $i \geq 1$ ,

$$\begin{aligned} x_{n,i} &\equiv \Delta_n^{-1/2} \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dW_s, \\ \bar{b}_{n,i} &\equiv \int_{(i-1)\Delta_n}^{i\Delta_n} b_s ds, \\ \bar{c}_{n,i} &\equiv \Delta_n^{-1} \int_{(i-1)\Delta_n}^{i\Delta_n} c_s ds, \\ J_{n,i} &\equiv \left( \begin{array}{c} \Delta_i^n J_Z \\ \beta_0 \Delta_i^n J_Z + \Delta_i^n \varepsilon \end{array} \right), \\ d_{n,i} &\equiv \left( \begin{array}{c} h(n, i) \\ h_0 + \beta_0 h(n, i) + \Delta_n^{1/2} h_0 h(n, i) \end{array} \right). \end{aligned}$$

Note that under Assumption 4,  $(x_{n,i})_{i \geq 1}$  are independent conditional on  $(b_i, \sigma_i, J_{Z,t}, \varepsilon_i)_{t \geq 0}$  and each  $x_{n,i}$  is distributed as  $\mathcal{N}(0, \bar{c}_{n,i})$ . With these notations, we can write the log likelihood ratio explicitly as

$$(SD.32) \quad L_n(h) = \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \Delta_i^n J_Z d_{n,i}^\top \bar{c}_{n,i}^{-1} x_{n,i} - \frac{1}{2} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \Delta_i^n J_Z^2 d_{n,i}^\top \bar{c}_{n,i}^{-1} d_{n,i}.$$

Note that on  $\Omega_n$ ,  $\Delta_i^n J_Z \neq 0$  only if  $((i-1)\Delta_n, i\Delta_n]$  contains one (and only one) jump of  $Z$ . Therefore,

$$(SD.33) \quad L_n(h) = \sum_{p \in \mathcal{P}} \Delta Z_{\tau_p} d_{n,i(p)}^\top \bar{c}_{n,i(p)}^{-1} x_{n,i(p)} - \frac{1}{2} \sum_{p \in \mathcal{P}} \Delta Z_{\tau_p}^2 d_{n,i(p)}^\top \bar{c}_{n,i(p)}^{-1} d_{n,i(p)}.$$

By Proposition 4.4.10 in Jacod and Protter (2012),  $(x_{n,i(p)})_{p \geq 1} \xrightarrow{\mathcal{L}-s} (R_p)_{p \geq 1}$ . Under Assumption 5, the variables  $(R_p)_{p \geq 1}$  are  $\mathcal{F}$ -conditionally independent, where the  $\mathcal{F}$ -conditional law of  $R_p$  is  $\mathcal{N}(0, c_{\tau_p})$ ; moreover,  $\bar{c}_{n,i(p)} \rightarrow c_{\tau_p}$  a.s. for each  $p \geq 1$ . Further note that for each  $p \geq 1$ ,

$$(SD.34) \quad d_{n,i(p)} \rightarrow D_p h,$$

where the matrix  $D_p$  is defined as

$$(SD.35) \quad D_p \equiv \begin{pmatrix} 0 & 0_{j-1}^\top & 1 & 0_{m-j}^\top \\ 1 & 0_{j-1}^\top & \beta_0 & 0_{m-j}^\top \end{pmatrix} \quad \text{for } j \text{ such that } \tau_p \in (S_{j-1}, S_j].$$

Since  $\mathcal{P}$  is a.s. finite, we deduce (4.9) from (SD.33) and (SD.34), that is,

$$(SD.36) \quad L_n(h) = h^\top \Gamma_n^{1/2} \psi_n - \frac{1}{2} h^\top \Gamma_n h + o_p(1),$$

where

$$(SD.37) \quad \Gamma_n \equiv \sum_{p \in \mathcal{P}} \Delta Z_{\tau_p}^2 D_p^\top \bar{c}_{n,i(p)}^{-1} D_p, \quad \psi_n = \Gamma_n^{-1/2} \sum_{p \in \mathcal{P}} \Delta Z_{\tau_p} D_p^\top \bar{c}_{n,i(p)}^{-1} x_{n,i(p)}.$$

In addition, (4.10) follows with

$$(SD.38) \quad \Gamma \equiv \sum_{p \in \mathcal{P}} \Delta Z_{\tau_p}^2 D_p^\top c_{\tau_p}^{-1} D_p, \quad \psi \equiv \Gamma^{-1/2} \sum_{p \in \mathcal{P}} \Delta Z_{\tau_p} D_p^\top c_{\tau_p}^{-1} R_p.$$

It is easy to verify that  $\Gamma$  defined in (SD.38) equals to  $\Gamma(S)$  defined by (4.17). To see, we make the following explicit calculation using (SD.35):

$$(SD.39) \quad D_p^\top c_{\tau_p}^{-1} D_p = \begin{pmatrix} \frac{1}{v_{\tau_p}^c} & 0_{j-1}^\top & \frac{\beta_0 - \beta_{\tau_p}^c}{v_{\tau_p}^c} & 0_{m-j}^\top \\ 0_{j-1} & 0_{(j-1) \times (j-1)} & 0_{j-1} & 0_{(j-1) \times (m-j)} \\ \beta_0 - \beta_{\tau_p}^c & 0_{j-1}^\top & \frac{(\beta_0 - \beta_{\tau_p}^c)^2}{v_{\tau_p}^c} + \frac{1}{c_{ZZ, \tau_p}} & 0_{m-j}^\top \\ 0_{m-j} & 0_{(m-j) \times (j-1)} & 0_{m-j} & 0_{(m-j) \times (m-j)} \end{pmatrix}.$$

Finally, we note that conditional on  $\mathcal{F}$ ,  $\psi$  has a standard normal distribution and, hence, is independent of  $\mathcal{F}$ . The proof for the LAMN property is now complete.

From the proof of Theorem 3 of Jeganathan (1982), we see that the convolution theorem can be applied in restriction to the set  $\Omega(S) \equiv \{\Gamma(S) \text{ is nonsingular}\}$ . The information bound for estimating  $\beta$ , that is, the first diagonal element of  $\Gamma(S)^{-1}$ , can then be easily computed by using the inversion formula for partitioned matrices.

(b) Since the jumps of  $Z$  have finite activity, on each sample path  $\omega \in \Omega$  there exists some  $S^*(\omega) \in \mathbf{S}$  that shatters its jumps. That is, each interval  $(S_{j-1}^*(\omega), S_j^*(\omega)]$  contains exactly one jump time of  $Z$ . We can then evaluate  $\bar{\Sigma}_\beta(\cdot)$  at  $S^*$  on each sample path and obtain

$$(SD.40) \quad \bar{\Sigma}_\beta(S^*) = \left( \sum_{s \leq T} \begin{pmatrix} \Delta Z_s^2 & \gamma_{1s}^2 \\ v_s^c & \gamma_{2s} \end{pmatrix} \right)^{-1}.$$

Plugging the definitions of  $\gamma_{1s}$  and  $\gamma_{2s}$  (see (4.16)) into (SD.40), we can verify that

$$(SD.41) \quad \bar{\Sigma}_\beta(S^*) = \left( \sum_{s \leq T} \frac{\Delta Z_s^2}{c_{YY,s} - 2\beta_0 c_{ZY,s} + \beta_0^2 c_{ZZ,s}} \right)^{-1}.$$

Recall that we fix  $\mathcal{D} = [0, T] \times \mathbb{R}_*$  and  $\Sigma^* \equiv \Sigma(\mathcal{D}, w^*)$ , with the latter given by (4.8). Under Assumption 5, we see  $\bar{\Sigma}_\beta(S^*) = \Sigma^*$ .

It remains to verify that  $\bar{\Sigma}_\beta(S^*) \geq \bar{\Sigma}_\beta(S)$  for all  $S \in \mathbf{S}$ . By the Cauchy–Schwarz inequality,

$$(SD.42) \quad \frac{\left( \sum_{S_{j-1} < s \leq S_j} \gamma_{1s} \right)^2}{\sum_{S_{j-1} < s \leq S_j} \gamma_{2s}} \leq \sum_{S_{j-1} < s \leq S_j} \frac{\gamma_{1s}^2}{\gamma_{2s}}.$$

From (4.19), (SD.40), and (SD.42),  $\bar{\Sigma}_\beta(S^*) \geq \bar{\Sigma}_\beta(S)$  readily follows. Q.E.D.

#### REFERENCES

- JACOD, J., AND P. PROTTER (2012): *Discretization of Processes*. Berlin: Springer. [1-3,7]  
 JEGANATHAN, P. (1982): “On the Asymptotic Theory of Estimation When the Limit of the Log-Likelihood Is Mixed Normal,” *Sankhya, Ser. A*, 44, 173–212. [8]  
 VAN DER VAART, A. W. (1998): *Asymptotic Statistics*. Cambridge: Cambridge University Press. [4]

*Dept. of Economics, Duke University, Durham, NC 27708, U.S.A.; j1410@duke.edu,*  
*Dept. of Finance, Kellogg School of Management, Northwestern University, Evanston, IL*  
*60208, U.S.A.; v-todorov@northwestern.edu,*

and

*Dept. of Economics, Duke University, Durham, NC 27708, U.S.A.; george.tauchen@duke.edu.*

*Co-editor Elie Tamer handled this manuscript.*

*Manuscript received 3 November, 2014; final version accepted 30 August, 2016; available online 13 September, 2016.*