Econometrica Supplementary Material

SUPPLEMENT TO "STABILITY AND PREFERENCE ALIGNMENT IN MATCHING AND COALITION FORMATION"

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THIS SUPPLEMENT (i) shows that a preference profile can be embedded in a rich domain of pairwise-aligned profiles if and only if it does not admit n-cycles and (ii) discusses the trade-offs involved in relaxation of the assumptions in Theorems 1 and 2.

A CHARACTERIZATION OF PROFILES IN RICH PAIRWISE-ALIGNED DOMAINS

PROPOSITION 6: *If the family of coalitions satisfies* C1 *and* C2, *then a preference profile* $\precsim_A$ *belongs to a rich domain of pairwise-aligned profiles if and only if* $\precsim_A$ *admits no* n*-cycles*, $n = 2, 3, \ldots$.

PROOF: One implication follows from Lemmas 3 and 4. To prove the remaining implication, we show that the domain of all preference profiles that do not admit n-cycles, $n = 2, 3, \ldots$, satisfies R1 and R2. Fix a profile $\precsim_A$ that does not admit n-cycles, $n = 2, 3, \ldots$.

To prove that R1 is satisfied, we construct an extension of the partial ordering $\trianglelefteq$ constructed in the proof of Lemma 5 (the construction did not rely on the strictness of preferences assumed in the lemma). We recursively construct the extension so that all proper coalitions are comparable while maintaining transitivity and acyclicity. Let relation $\trianglelefteq^k \subset \mathcal{C} \times \mathcal{C}$ be a transitive and acyclic extension of $\trianglelefteq$. Take any proper coalitions C and C' that are not comparable under $\trianglelefteq^k$ (if there are no such coalitions, then the extension is complete). Let $\trianglelefteq^{k+1} \subset \mathcal{C} \times \mathcal{C}$ be the smallest transitive extension of $\trianglelefteq^k \cup \{(C, C')\}$. Relation $\trianglelefteq^{k+1}$ is transitive by definition and is acyclic, as otherwise either $\trianglelefteq^k$ would violate acyclicity or C and C' would be comparable under $\trianglelefteq^k$. Since there is a finite number of proper coalitions, this process terminates, producing the postulated extension of $\trianglelefteq$. The extension satisfies relation (3) from the proof of Lemma 5 because all pairs of proper coalitions C and C' with nonempty intersection are comparable under $\trianglelefteq$. In the remainder of the proof, let us refer to the extension as $\trianglelefteq$.[1]

To prove R1, we take three different coalitions C_0, C, and C_1, and agent a such that $C_0 \precsim_a C_1$, and construct a preference profile $\precsim'_A$ whose existence is postulated in R1. At least one of the coalitions C_0 or C_1 is proper, and because of symmetry, we may assume that $C_1 \neq A$. Define the preference profile $\precsim'_A$

[1]For preference profiles $\precsim_A$ generated by equal sharing, examples of $\trianglelefteq$ and $\trianglelefteq^k$ can be obtained by restricting the common ranking of Farrell and Scotchmer (1988).

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so that it coincides with $\precsim_A$ for pairs of coalitions $C', C'' \in \mathcal{C} - \{C\}$, and for coalitions $C' \in \mathcal{C} - \{C\}$ and $a \in C' \cap C$, let $C' \prec_a C$ if $C' = A$ and otherwise let

$$C \prec'_a C' \quad \text{if} \quad C_1 \lhd C',$$
$$C \succ'_a C' \quad \text{if} \quad C_1 \rhd C',$$
$$C \sim'_a C' \quad \text{otherwise.}$$

The profile $\precsim'_A$ proves R1 because $C_0 \precsim'_a C \sim'_a C_1$ and $\precsim'_A$ does not admit n-cycles. The latter claim is true because if there was an n-cycle $C_{m,1} \prec'_{a_1} \cdots \precsim'_{a_{m-1}} C_{m-1,m} \precsim'_{a_m} C_{m,1}$, then $C'_{m,1} \lhd \cdots \unlhd C'_{m-1,m} \unlhd C'_{m,1}$ for coalitions $C'_{i,i+1} = C_{i,i+1}$ when $C_{i,i+1} \neq C$ and $C'_{i,i+1} = C_1$ when $C_{i,i+1} = C$. This is, however, impossible as $\unlhd$ is acyclic.

To prove R2(i), take two different coalitions C and C_1, and define the preference profile $\precsim'_A$ so that it coincides with $\precsim_A$ for pairs of coalitions $C', C'' \in \mathcal{C} - \{C\}$, and $C \prec'_a C'$ whenever $C' \in \mathcal{C} - \{C\}$ and $a \in C' \cap C$. If $\precsim'_A$ admitted an n-cycle $C_{m,1} \prec'_{a_1} \cdots \precsim'_{a_{m-1}} C_{m-1,m} \precsim'_{a_m} C_{m,1}$, then one of the coalitions $C_{k,k+1}$, $k = 1, \ldots, m$, would need to be C (since $\precsim_A$ does not admit n-cycles), but C is not weakly preferred to any coalition.

To prove R2(ii), take three different coalitions C_0, C, and C_1, take agents a and b such that $C_0 \prec_a C \sim_b C_1$, and define the preference profile $\precsim'_A$ so that it coincides with $\precsim_A$ for pairs of coalitions $C', C'' \in \mathcal{C} - \{C\}$; in addition, for $C' \in \mathcal{C} - \{C\}$ and $a \in C' \cap C$, let $C \prec'_a C'$ if $C \precsim_a C'$ and $C \succ'_a C'$ otherwise. If $\precsim'_A$ admitted an n-cycle $C_{m,1} \prec'_{a_1} \cdots \precsim'_{a_{m-1}} C_{m-1,m} \precsim'_{a_m} C_{m,1}$, then we would need to have $C = C_{m,1}$ (as otherwise the coalitions would form an n-cycle under $\precsim_A$) and, hence, $C_{m,1} \prec'_{a_1} \cdots \precsim'_{a_{m-1}} C_{m-1,m} \prec'_{a_m} C_{m,1}$ (as no agent is $\precsim'_A$-indifferent between C and another coalition). Then, however, $C_{m,1} \precsim_{a_1} \cdots \precsim_{a_{m-1}} C_{m-1,m} \prec_{a_m} C_{m,1}$ would be an n-cycle, a contradiction. *Q.E.D.*

The above argument also shows that a preference profile $\precsim_A$ belongs to an R1-rich domain of pairwise-aligned profiles iff $\precsim_A$ admits no n-cycles, $n = 2, 3, \ldots$. The argument can be adapted to show that the domain of all strict preference profiles that do not admit n-cycles is also rich.

RELAXING ASSUMPTIONS ON THE FAMILY OF COALITIONS $\mathcal{C}$, AND THE ROOMMATE PROBLEM

We can relax C2 and C3 at the cost of replacing pairwise alignment by less transparent conditions. For instance, for families of coalitions satisfying C1 and preference-profile domains satisfying R1, if no profile admits a 3-cycle, then all profiles admit a stable coalition structure (by Lemmas 2 and 4). For the roommate problem, assumption R2 and Lemma 6 provide the reverse implication, and we obtain a corollary.

COROLLARY 3: *Suppose that* $\mathcal{C} = \{C \subseteq A, |C| \leq 2\}$ *and that domain of preferences* $\mathbf{R}$ *satisfies* R1 *and* R2. *Then the following three statements are equivalent*:

- *All profiles in* $\mathbf{R}$ *admit a stable matching*.
- *No profile in* $\mathbf{R}$ *admits a* 3*-cycle*.
- *No profile in* $\mathbf{R}$ *admits an n-cycle*, $n = 3, 4, \ldots$.

RELAXING RICHNESS IN MATCHING

In many-to-one matching, one might expect that the division of output in coalitions of a firm and a worker is different than the division of output in larger coalitions if workers compete against each other in the internal bargaining. We can take this into account by imposing an R1-like condition only on coalitions with two or more workers.

PROPOSITION 7: *Assume that* $\mathbf{R}$ *is a domain of preferences in a many-to-one matching problem satisfying* C2, *and that for any* $\precsim_A \in \mathbf{R}$, *any agent* $a \in A$, *and coalitions* C, C' *such that* $a \in C, C'$ *and* $|C|, |C'| \geq 3$, *there exists a profile* $\precsim'_A \in \mathbf{R}$ *such that* $C \sim'_a C'$ *and all agents'* $\precsim'_A$*-preferences between pairs of coalitions not including* C *are the same as their* $\precsim_A$*-preferences*. *If all preference profiles in* $\mathbf{R}$ *are pairwise-aligned*, *then they admit stable coalition structures*.

The working paper draft (Pycia (2007)), gives the proof.

STABILITY CONCEPTS

The existence results of this paper are not specific to core stability. In many-to-one matching, the results hold true for pairwise stability and group stability. The working paper draft (Pycia (2007)), gives details.

REFERENCES

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