# SUPPLEMENT TO "RISK SHARING IN PRIVATE INFORMATION MODELS WITH ASSET ACCUMULATION: EXPLAINING THE EXCESS SMOOTHNESS OF CONSUMPTION"
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This supplement contains two main sections. In Appendix B, we formally derive the closed forms presented in Sections 3.2 and 3.3 in the main text. First, we present the derivation for the closed forms in levels for CARA and quadratic utility. Then we derive the closed form in logs, assuming isoelastic preferences and multiplicative (or Cobb–Douglas) $f$. Finally, we extend the model to derive the closed form for the model with two types of income shocks: a permanent and a temporary shock.

In Appendix C, we derive the expression for the bias in the variance induced by the use of a pseudopanel (such as the one we consider in our estimations) and explain how we corrected our estimates and tests to take this bias into account.

## APPENDIX B: CLOSED FORMS

THIS APPENDIX contains the formal derivation of the closed forms presented in Sections 3.2 and 3.3 in the main text. First, we present the derivation for the closed forms in levels for CARA and quadratic utility. Then we derive the closed form in logs, assuming isoelastic preferences and multiplicative (or Cobb–Douglas) $f$. Finally, we extend the model to derive the closed form for the model with two types of income shocks: a permanent and a temporary shock.

The outcome of this appendix is a set of closed-form solutions to our model which give a structural interpretation, in terms of the marginal cost/return of effort, of the coefficient $\phi$ that comes from a generalized permanent income equation of the form

$$\Delta c_t = \Gamma_t + \phi \Delta y_t^p,$$

where the variable is expressed in levels or in logs, depending on the specification, and

$$\phi = \frac{1}{a},$$

with $a \geq 1$ and where $\frac{1}{a}$ is the marginal return to shirking. Since in our model wealth effects are absent (at least in the chosen space), the equilibrium contract implements a constant effort level in all periods, which is normalized

to a given number: the first best level of effort. So the whole margin in welfare comes from risk sharing. The incentive compatibility constraint hence dictates the degree of such insurance as a function of the marginal cost of effort. A lower effort cost/return allows the firm to insure the agent a lot without inducing him to shirk, and the firm uses the entire available margin to impose transfers and obtain consumption smoothing.

## B.1. *Closed Form in Levels: CARA Utility*

### B.1.1. *Model*

Recall that we can perform a change in variable and assume $y_t = \theta_t + e_t$ and $u(c, e) = u(c - v(e))$, where

$$(S1) \qquad v(e_t) := \frac{1}{a} \min\{e_t, 0\} + \frac{1}{b} \max\{e_t, 0\}, \quad \text{with} \quad a \geq 1 \geq b.$$

Interestingly, as we saw in Section 3.1 for $a = b = 1$, we are in the standard ACK case, hence there is no room for risk sharing at all (on top of self-insurance) and the allocation replicates that of the Bewley model.

Finally, notice that as long as $a > 1$ (and $b < 1$), the first-best effort level is zero. However, the first-best allocation also implies a constant consumption. This allocation can only be obtained by imposing a constant tax rate such that $\tau'_t = -1$. Obviously, this allocation is not incentive feasible in a world where effort and productivity are private information of the agent.

The main steps toward the derivation of our closed form are as follows. First, we consider a relaxed optimization problem. More precisely, we consider an auxiliary problem for the firm that imposes strictly less stringent incentive constraints, the same objective function, and the same technological constraints. Then we show that the solution for the relaxed problem corresponds to our closed form. Finally, we show that our closed form satisfies the original incentive compatibility constraint. This implies that the closed-form solution solves the original maximization problem of the firm.

### B.1.2. *The Relaxed Problem*

We eliminate the time subscript whenever possible. Consider the problem

$$(R) \qquad \max_{\tau, \mathbf{y}} \mathbf{E}_0 \left[ \sum_{t=1}^{T} \delta^{t-1} u(y(\theta^t) + \tau(\theta^t) - v(y(\theta^t) - \theta_t)) \right]$$

subject to, for all $\theta^t$, $t \geq 1$,

$$\max_{b, \hat{\theta}_t \leq \theta_t} u\big(y(\theta^{t-1}, \hat{\theta}_t) + \tau(\theta^{t-1}, \hat{\theta}_t) - q_t b - v(y(\theta^{t-1}, \hat{\theta}_t) - \theta_t)\big)$$

$$+ \delta V_t((\theta^{t-1}, \theta_t), \hat{\theta}_t, b)$$

$$\leq u(y(\theta^t) + \tau(\theta^t) - v(y(\theta^t) - \theta_t)) + \delta U_t(\theta^t),$$

$$0 \geq \mathbf{E}_0\left[\sum_t \left(\prod_{n=0}^t q_s\right)\tau(\theta^t)\right].$$

In the above formulation, $U_t(\theta^t)$ is the equilibrium utility and it solves

$$U_t(\theta^t) := \mathbf{E}_t\left[\sum_{s=1}^{T-t} \delta^{s-1} u(y(\theta^{t+s}) + \tau(\theta^{t+s}) - v(y(\theta^{t+s}) - \theta_{t+s}))\right]$$

$$= \int_\Theta \left[u(y(\theta^t, \theta_{t+1}) + \tau(\theta^t, \theta_{t+1}) - v(y(\theta^{t+1}) - \theta_{t+1}))\right.$$

$$\left. + \delta U_{t+1}(\theta^t, \theta_{t+1})\right] d\Phi(\theta_{t+1}|\theta^t),$$

while $V_t((\theta^{t-1}, \theta_t), \hat{\theta}_t, b)$ represents the highest utility the agent can get by freely choosing the plan of bonds but telling the truth,

$$V_t((\theta^{t-1}, \theta_t), \hat{\theta}_t, b)$$

$$= \max_{c,b} \mathbf{E}\left[\sum_{s=1}^{T-t} \delta^{s-1} u\big(c(\theta^{t+s}) - v(y(\hat{\theta}^{t+s}) - \theta_{t+s})\big)\bigg|\theta^t\right]$$

subject to

$$c_{t+s}(\theta^{t+s}) = y(\hat{\theta}^{t+s}) + \tau(\hat{\theta}^{t+s}) + q_{t+s}b_{t+s+1}(\theta^{t+s}) - b_{t+s}(\theta^{t+s}),$$

where in the previous expression, for all $s \geq 1$, we denote $\hat{\theta}^{t+s} := (\theta^{t-1}, \hat{\theta}_t, \theta_{t+1}, \ldots, \theta_{t+s})$. Clearly, $U_{T+1} \equiv V_{T+1} \equiv 0$.

The maximization problem is relaxed with respect to the original problem solved by the firm in equilibrium in a number of dimensions. First, the incentive constraints are only for downward deviations. Second, the deviation is "local" because it assumes that the agent never lies for more than one period although he is allowed to deviate for more than one period in the bond decisions after a first deviation. Finally, bond deviations are also local since the agent starts with zero wealth at each node in equilibrium.

LEMMA 1: *The contract solving problem* (R) *implements* $e(\theta^t) = 0$ *for all* $\theta^t$.

PROOF: Take any contract and suppose that for some history $\theta^t$, we have $e(\theta^t) > 0$. Then consider the contract that keeps all transfers and recommendations as the previous contract, but at history $\theta^t$, where it recommends income $\tilde{y}(\theta^t) = y(\theta^t) - e(\theta^t)$ and zero effort: $\tilde{e}(\theta_{T-1}, \theta_T) = 0$ and transfers $\tilde{\tau}(\theta^t) = \tau(\theta^t) + (1 - \frac{1}{b})e(\theta^t)$. It is easy to see that, at all histories, the new contract delivers exactly the same argument of the utility function $u$ in equilibrium.

We have to show that the incentive constraints are all satisfied under the new contract. The fact that, in the equilibrium for all histories, the arguments of $u$ are all the same implies that $U_s(\theta^s)$ are unchanged for all $s$ and $\theta^s$. Moreover, it should also be clear that future values $V_{t+s}((\theta^{t+s-1}, \theta_{t+s}), \hat{\theta}_{t+s}, b)$ for all $s \geq 0$, $\theta^{t+s}$ and $\hat{\theta}_{t+s}, b$ do not change either. Finally, since the modification of the contract leaves the equilibrium utilities unchanged at all nodes (including node $\theta^t$), the values $V_{t-k}((\theta^{t-k-1}, \theta_{t-k}), \hat{\theta}_{t-k}, b)$ for all $k \geq 1$ are also unaffected by the change: since the argument of the utility flow $u$ in equilibrium is unchanged—and $V_{t-k}((\theta^{t-k-1}, \theta_{t-k}), \hat{\theta}_{t-k}, b)$ does not contemplate deviations over declarations after period $t - k$—the set of consumption plans available by deviating only in the bond are unchanged by the modification to the contract.

Consider now how the change in the contract might affect the incentive constraint in period $t$. Clearly it can affect the incentives for productivity levels above $\theta_t$: call these values $\bar{\theta}_t \geq \theta_t$ (we include $\bar{\theta}_t = \theta_t$ since the agent with productivity $\theta_t$ might find the bond deviation profitable under the new contract). Since the equilibrium utilities (both flows $u$ and values $U_t$) do not change, to verify that the new transfer scheme solves the period $t$ incentive constraint, it suffices to show that for all $b$ and $\bar{\theta}_t > \theta_t$, we have

$$u\big(y(\theta^{t-1}, \theta_t) + \tau(\theta^{t-1}, \theta_t) - q_t b - v(y(\theta^{t-1}, \theta_t) - \bar{\theta}_t)\big)$$
$$+ \delta U_t((\theta^{t-1}, \theta_t), \bar{\theta}_t, b)$$
$$\geq u\big(\tilde{y}(\theta^{t-1}, \theta_t) + \tilde{\tau}(\theta^{t-1}, \theta_t) - q_t b - v(\tilde{y}(\theta^{t-1}, \theta_t) - \bar{\theta}_t)\big)$$
$$+ \delta U_t((\theta^{t-1}, \theta_t), \bar{\theta}_t, b).$$

But again, since $U_t((\theta^{t-1}, \theta_t), \bar{\theta}_t, b)$ is unaffected by the change, it suffices to show that

$$u\big(x(\theta^{t-1}, \theta_t) + \tau(\theta^{t-1}, \theta_t) - q_t b - v(x(\theta^{t-1}, \theta_t) - \bar{\theta}_t)\big)$$
$$\geq u\big(\tilde{y}(\theta^{t-1}, \theta_t) + \tilde{\tau}(\theta^{t-1}, \theta_t) - q_t b - v(\tilde{y}(\theta^{t-1}, \theta_t) - \bar{\theta}_t)\big).$$

The last inequality is true because of the following conditions: If $y(\theta^{t-1}, \theta_t) - \bar{\theta}_t > e(\theta^{t-1}, \theta_t) > 0$, then the change in transfer scheme generates exactly the same utility to the deviating agent. If $y(\theta^{t-1}, \theta_t) - \bar{\theta}_t < e(\theta^{t-1}, \theta_t)$, then the utility from deviation decreases. We show this by assuming that $y(\theta^{t-1}, \theta_t) - \bar{\theta}_t < 0$. The case where $e(\theta^t) > y(\theta^t) - \bar{\theta}_t > 0$ is a combination of this cases and the case we just analyzed. We have

$$u\left( y(\theta^{t-1}, \theta_t) + \tau(\theta^{t-1}, \theta_t) - \frac{1}{a}(y(\theta^{t-1}, \theta_t) - \bar{\theta}_t) \right)$$
$$= u\left( y(\theta^{t-1}, \theta_t) - \frac{1}{a}y(\theta^{t-1}, \theta_t) + \tau(\theta^{t-1}, \theta_t) + \frac{1}{a}\bar{\theta}_t \right)$$

$$> u\bigg( y(\theta^{t-1}, \theta_t) - \frac{1}{a}y(\theta^{t-1}, \theta_t) + \tau(\theta^{t-1}, \theta_t)$$

$$+ \bigg(\frac{1}{a} - \frac{1}{b}\bigg)e(\theta^{t-1}, \theta_t) + \frac{1}{a}\bar{\theta}_t \bigg),$$

since $(\frac{1}{a} - \frac{1}{b})e(\theta^{t-1}, \theta_t) < 0$. The case against $e(\theta^t) < 0$ follows from a similar line of proof. This implies that the equilibrium utility of the agent is unchanged. At this point, we can try to actually increase the agent's utility, but we only need to show that our closed-form solution belongs to the set of optimal contracts.                *Q.E.D.*

Lemma 1 implies that we can rewrite problem (R) as

(R′)    $\displaystyle\max_{\tau(\theta^t)} \mathbf{E}_0\bigg[ \sum_{t=1}^{T} \delta^{t-1}u(\theta_t + \tau(\theta^t)) \bigg]$    subject to, for all $\theta^t$, $t \geq 1$,

(S2)    $\displaystyle\max_{b, \hat{\theta}_t \leq \theta_t} u\bigg( \hat{\theta}_t + \tau(\theta^{t-1}, \hat{\theta}_t) - q_t b - \frac{1}{a}(\hat{\theta}_t - \theta_t) \bigg) + \delta V_t((\theta^{t-1}, \theta_t), \hat{\theta}_t, b)$

$$\leq u(\theta_t + \tau(\theta^t)) + \delta U_t(\theta^t),$$

$$0 \geq \mathbf{E}_0\bigg[ \sum_t q^t \tau(\theta^t) \bigg],$$

where

$$U_t(\theta^t) := \mathbf{E}_t\bigg[ \sum_{s=1}^{T-t} \delta^{s-1}u(\theta_{t+s} + \tau(\theta^{t+s})) \bigg]$$

$$= \int_{\Theta} \big[ u(\theta_{t+1} + \tau(\theta^t, \theta_{t+1})) + \delta U_{t+1}(\theta^t, \theta_{t+1}) \big] d\Phi(\theta_{t+1}|\theta^t)$$

and

$$V_t((\theta^{t-1}, \theta_t), \hat{\theta}_t, b) := \max_{c, b} \mathbf{E}\bigg[ \sum_{s=1}^{T-t} \delta^{s-1}u(c(\hat{\theta}^{t+s})) \bigg| \theta^t \bigg]$$

subject to

$$c_{t+s}(\theta^{t+s}) = \theta_{t+s} + \tau(\hat{\theta}^{t+s}) + q_{t+s}b_{t+s+1}(\theta^{t+s}) - b_{t+s}(\theta^{t+s}).$$

ASSUMPTION 1: *The utility function takes the exponential* (CARA) *form*

$$u(c - v(e)) = -\frac{1}{\rho} \exp\{-\rho(c - v(e))\}$$

*with $\rho > 0$ and the function $v$ is as in* (S1).

PROPOSITION 2: *If preferences are CARA, for each given $\theta^{t-1}$, the present value of transfers (PVT) solving problem* (R$'$), *which are defined as* $\text{PVT}_t = \sum_{n=0}^{T-t}(\prod_{s=0}^{n} q_{t+s-1})\tau_{t+n}(\theta^{t+n})$, *obeys the following criteria. There are $\theta^{t-1}$, measurable functions $\{\eta_t\}_{t=1}^{T}$ such that for all $\theta^t, t \geq 1$,*

$$(S3) \qquad \sum_{n=0}^{T-t}\left(\prod_{s=0}^{n} q_{t+s-1}\right)\tau(\theta^{t+n}) = \eta_t(\theta^{t-1}) + \sum_{n=0}^{T-t}\left(\prod_{s=0}^{n} q_{t+s-1}\right)\left[\left(\frac{1}{a}-1\right)\theta_{t+n}\right]$$

*or, equivalently, for all $\theta^t, t \geq 1$,*

$$\tau_t(\theta^t) + \sum_{n=1}^{T-t}\left(\prod_{s=1}^{n} q_{t+s-1}\right)\eta_{t+n}(\theta^{t+n}) = \eta_t(\theta^{t-1}) + \left(\frac{1}{a}-1\right)\theta_t.$$

*In particular, $\prod_{n=0}^{T-t}(\prod_{s=0}^{n} q_{t+s-1})\tau(\theta^{t+n})$ admits a partial derivative with respect to $\theta_t$, and for each fixed past history $\theta^{t-1}$ and fixed future $\theta_{t+1}, \dots, \theta_T$, we have*

$$\frac{\partial}{\partial \theta_t} \sum_{n=0}^{T-t}\left(\prod_{s=0}^{n} q_{t+s-1}\right)\tau(\theta^{t+n}) = \left(\frac{1}{a}-1\right).$$

PROOF:  Keep in mind that we must show the set of equations

$$\tau_T(\theta^T) = \eta_T(\theta^{T-1}) + \left(\frac{1}{a}-1\right)\theta_T,$$

$$\tau_{T-1}(\theta^{T-1}) + q_{T-1}\eta_T(\theta^{T-1}) = \eta_{T-1}(\theta^{T-2}) + \left(\frac{1}{a}-1\right)\theta_{T-1},$$

$$\vdots$$

$$\tau_1(\theta_1) + \sum_{n=1}^{T-1}\left(\prod_{s=1}^{n} q_{t+s-1}\right)\eta_{n+1}(\theta^n) = \eta_1 + \left(\frac{1}{a}-1\right)\theta_1.$$

We prove our proposition backward. Let us consider our problem in the last two periods. It is easy to see from our relaxed problem that since the agent has von Neumann–Morgenstern utility and the firm maximizes the expected discounted value of profits, the only link across states comes from the incentive constraints In the proof below, we only consider the relevant incentive constraints

subject to for all $\theta^{T-1}$,

$$(S4) \qquad u(\theta_{T-1} + \tau(\theta^{T-1})) + \delta \int_{\Theta} u(\theta_T + \tau(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T | \theta^{T-1})$$

$$\geq \max_{b, \hat{\theta}_{T-1} \leq \theta_{T-1}} u\left(\hat{\theta}_{T-1} + \tau(\theta^{T-2}, \hat{\theta}_{T-1}) - q_{T-1}b - \frac{1}{a}(\hat{\theta}_{T-1} - \theta_{T-1})\right)$$

$$+ \delta \int_{\Theta} u(\theta_T + b + \tau(\theta^{T-2}, \hat{\theta}_{T-1}, \theta_T)) \, d\Phi(\theta_T|\theta^{T-1})$$

and for all $\theta_{T-1}$, $\theta_T$, and $\hat{\theta}_T \leq \theta_T$,

(S5) $\qquad u(\theta_T + \tau(\theta^{T-1}, \theta_T)) \geq u\left(\hat{\theta}_T + \tau(\theta^{T-1}, \hat{\theta}_T) - \frac{1}{a}(\hat{\theta}_T - \theta_T)\right).$ $\qquad$ Q.E.D.

LEMMA 2: *If the utility function is CARA, the transfer scheme solving problem* (R$'$) *satisfies the following condition: for all* $\theta^{T-1}$, *we have* $\tau(\theta^{T-1}, \theta'_T) - \tau(\theta^{T-1}, \theta''_T) = -(1 - \frac{1}{a})(\theta'_T - \theta''_T)$ *for all* $\theta'_T$, $\theta''_T$. *In particular, the partial derivative* $\frac{\partial}{\partial \theta_T} \tau(\theta^T)$ *exists and equals* $(\frac{1}{a} - 1)$ *for all* $\theta^{T-1}$.

PROOF: It is easy to see from (S5) (by taking the inverse of the $u$ transformation to both sides and applying it to all $\theta$) that $\tau(\theta^{T-1}, \theta'_T) - \tau(\theta^{T-1}, \theta''_T) \geq -(1 - \frac{1}{a})(\theta'_T - \theta''_T)$ for all $\theta'_T$, $\theta''_T$ is a necessary condition for incentive compatibility.[2] Now suppose that for a range of productivities, we have $\tau(\theta^{T-1}, \theta'_T) - \tau(\theta^{T-1}, \theta''_T) > -(1 - \frac{1}{a})(\theta'_T - \theta''_T)$ for all $\theta'_T$, $\theta''_T \in [\theta_T^0 - \varepsilon, \theta_T^0 + \varepsilon]$. We claim that there is a modification to the contract that keeps the same utility to the agent and reduces the net present value of the transfers for the firm. The new scheme is such that $\tilde{\tau}(\theta^{T-1}, \theta'_T) - \tilde{\tau}(\theta^{T-1}, \theta''_T) = -(1 - \frac{1}{a})(\theta'_T - \theta''_T)$ and for each node $\theta^{T-1}$, the new transfer solves $\int_{\theta - \varepsilon}^{\theta + \varepsilon} u(\theta_T + \tilde{\tau}(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T|\theta^{T-1}) = \int_{\theta - \varepsilon}^{\theta + \varepsilon} u(\theta_T + \tau(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T|\theta^{T-1})$. The fact that the new scheme imposes less consumption dispersion to the agent implies that it is potentially able to deliver the same agent's expected utility with lower average transfers. We have to show that this change is incentive feasible. Let us start with condition (S5). The new transfer scheme is incentive compatible in the range $[\theta_T^0 - \varepsilon, \theta_T^0 + \varepsilon]$ by construction. Moreover, it reduces the utility at the top extreme of the range while it increases agent's utility at the bottom of the range. Now, from the specific form of $u$, we have

$$\int_{\Theta} u(\theta_T + b + \tilde{\tau}(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T|\theta^{T-1})$$

$$= \exp\{-\rho b\} \int_{\Theta} u(\theta_T + \tilde{\tau}(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T|\theta^{T-1})$$

$$= \exp\{-\rho b\} \int_{\Theta} u(\theta_T + \tau(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T|\theta^{T-1})$$

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[2]If for a $\theta_T < \infty$, the transfer scheme has slope less than $(1 - \frac{1}{a})$, we can choose $\theta'_T > \theta_T$ and obtain the violation of the incentive compatibility constraint when the agent has shock $\theta'_T$.

$$= \int_{\Theta} u(\theta_T + b + \tau(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T | \theta^{T-1})$$

for all $b$. The equality in the second row is true since the new scheme solves $\int_{\Theta} u(\theta_T + \tilde{\tau}(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T | \theta^{T-1}) = \int_{\Theta} u(\theta_T + \tau(\theta^{T-1}, \theta_T)) \, d\Phi(\theta_T | \theta^{T-1})$. This implies that the change does not affect (S4) or any other incentive constraint (S2) as $V_t((\theta^{t-1}, \theta_t), \hat{\theta}_t, b)$ are unchanged for all $t$ and $b$. Note that this result implies that the transfer $\tau_T$ is partially differentiable in $\theta_T$ with derivative equal to $(1 - \frac{1}{a})$ for all $\theta^{T-1}$ and $\theta_T < \theta_{\max}$.[3] Note that when $\theta_{\max} = \infty$, the function $\tau_T$ is partially differentiable everywhere.                    *Q.E.D.*

To complete the induction argument we need the following lemma.

LEMMA 3: *If a transfer scheme solving* (R′) *is such that for all* $s > t$ $\frac{\partial}{\partial \theta_s} \mathrm{PVT}_s(\theta^s) = \frac{1}{a} - 1$ *for all* $\theta^s$ *and all* $(\theta_{s+1}, \ldots, \theta_T)$, *then* $\frac{\partial}{\partial \theta_t} \mathrm{PVT}_t(\theta^t) = \frac{1}{a} - 1$ *for all* $\theta^t$ *and all* $(\theta_{t+1}, \ldots, \theta_T)$.

PROOF: First note that from the incentive constraint, we have for all $\theta'_t \leq \theta''_t$, $\mathrm{PVT}_t(\theta^{t-1}, \theta''_t) - \mathrm{PVT}_t(\theta^{t-1}, \theta'_t) \geq (1 - \frac{1}{a})(\theta'_t - \theta''_t)$. If this were not true, then the agent with realization $\theta''_t$ would declare $\theta'_t$ and improve welfare. In particular, let $\kappa = \mathrm{PVT}_t(\theta^{t-1}, \theta''_t) - \mathrm{PVT}_t(\theta^{t-1}, \theta'_t)$ and suppose $\kappa < (1 - \frac{1}{a})(\theta'_t - \theta''_t)$. Consider an agent with productivity $\theta''_t$ declaring $\theta'_t$. The agent would (have to) reduce effort so that the argument of the flow utility $u$ in case of zero bond decision would be $\theta'_t + \tau(\theta^{t-1}, \theta'_t) - \frac{1}{a}(\theta'_t - \theta''_t)$ as opposed to $\theta''_t + \tau(\theta^{t-1}, \theta''_t)$ when telling the truth. We now show that there is a plan of bonds $\hat{b}$ such that telling the truth in the future and choosing the constructed bond plan improves agents' welfare. Namely, we show that constraint (S2) is violated at node $\theta^t$. The bond plan $\hat{b}$ is constructed so that the deviating agent gets exactly the same argument in the flow utility $u$ for all nodes but the last period one, where in each of the last period nodes, the agent consumes $\hat{c}_T(\theta^T) = c_T(\theta^T) + \frac{(1 - \frac{1}{a})(\theta'_t - \theta''_t) - \kappa}{\prod_s q_s}$. The bond plan $\hat{b}$ is constructed as follows. Let $\theta^{t+s} := (\theta^{t-1}, \theta'_t, \theta_{t+1}, \ldots, \theta_{t+s})$ be the true history of shocks. Note that the agent expectations are taken according to the distribution implied by this history. Moreover, let $\hat{\theta}^{t+s} := (\theta^{t-1}, \theta'_t, \theta_{t+1}, \ldots, \theta_{t+s})$. To obtain the plan of consumption $\hat{c}_{t+s}(\hat{\theta}^{t+s}) = c_{t+s}(\theta^{t+s}) = \theta_{t+s} + \tau(\theta^{t+s})$ for $s > 1$, the bond plan must solve for all $s > 1$:

$$b_{t+s}(\theta^{t+s-1}) - q_{t+s} b_{t+s+1}(\theta^{t+s}) = \eta_{t+s}(\theta^{t+s-1}) - \eta_{t+s}(\hat{\theta}^{t+s-1}).$$

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[3] More precisely, for each $\theta_T < \theta_{\max}$ choose $\theta'_T > \theta_T$. We have just shown that for all $\theta_T$ such that $\theta_T \leq \theta'_T$, the transfer scheme has constant slope which equals $(1 - \frac{1}{a})$.

Moreover, in period $t$ we have

$$q_t b_{t+1}(\theta^t) = [\theta_t'' + \tau(\theta^{t-1}, \theta_t'')] - \left[\theta_t' + \tau(\theta^{t-1}, \theta_t') - \frac{1}{a}(\theta_t' - \theta_t'')\right].$$

It is east to see—by straightforward calculations—that the plan satisfies two key properties. First, it delivers the same consumption plan to the agent at all nodes but the last, as claimed; this is so because of our inductive hypothesis. Second, the plan is budget feasible if $\hat{c}_T(\theta^T) = c_T(\theta^T) + \frac{(1-\frac{1}{a})(\theta_t' - \theta_t'') - \kappa}{\prod_s q_s} > c_T(\theta^T)$, as claimed.

We now have to show that it cannot be the case that the inequality is strict. Suppose some range of skills $[\theta_t^0 - \varepsilon, \theta_t^0 + \varepsilon]$, and consider the modification to the contract that makes it an equality and delivers the same expected utility to the agent over this range. We now show that this change is incentive compatible. The argument is a generalization of the last part of the proof of Lemma 2. *Q.E.D.*

ASSUMPTION 2: *The stochastic process for skills follows $\theta_t - \theta_{t-1} = \beta(L)v_t$, where $\beta(\cdot)$ is a polynomial of order $p$ in the lag operator $L$ and the innovation $v_t$ is a white noise (serially uncorrelated) process assumed to be normally distributed with zero mean and variance $\sigma_v^2$. The moving average process is invertible, that is, the roots of the polynomial $\beta(L)$ lie outside the unit circle (we normalize $\beta_0 = 1$).[4] Moreover, assume that $q_t = q$ for all $t$.*

It should be clear from the proof, that the next proposition—with the appropriate adjustments in notation—can be shown with slightly more general processes for $\theta_t$, as long as linearity in the law and the assumption of Gaussian shocks are maintained. Moreover, constant $q$ is assumed only for notational simplicity. The obtained expressions are those in the main text.

PROPOSITION 3: *Admit Assumptions 1 and 2. For all $t$, $T$ such that $T - t - 1 \geq p$, the consumption process follows*

$$c_{t+1}^* - c_t^* = \frac{\ln(\delta/q)}{\rho} + \frac{\rho}{2a^2}[\beta(q)]^2 \sigma_v^2 + \frac{1}{a}\beta(q)v_{t+1}.$$

*In particular, if the productivity process follows $\theta_t = \theta_{t-1} + v_t$, we have $c_{t+1}^* - c_t^* = \Gamma_t + \frac{1}{a}(y_{t+1}^* - y_t^*) = \frac{\ln(\delta/q_t)}{\rho} + \frac{\rho}{2a^2}\sigma_v^2 + \frac{1}{a}v_{t+1}$ no matter what are the time horizon and the sequence of bond prices.*

[4]Obviously, we assume the following initial conditions $\theta_0 = v_0 = v_{-1} = \cdots = v_{-p} = 0$ for the process, where $p$ is the maximum number of lags in the MA component of the process.

PROOF: First, from $c_t = y_t + \tau_t$ at all nodes,[5] we have that both

$$(S6) \qquad \mathbf{E}_t \sum_{n=0}^{T-t-1} q^n c_{t+1+n}^* = \mathbf{E}_t \sum_{n=0}^{T-t-1} q^n (y_{t+1+n}^* + \tau_{t+1+n}^*(y^{t+1+n})),$$

$$\mathbf{E}_{t+1} \sum_{n=0}^{T-t-1} q^n c_{t+1+n}^* = \mathbf{E}_{t+1} \sum_{n=0}^{T-t-1} q^n (y_{t+1+n}^* + \tau_{t+1+n}^*(y^{t+1+n})).$$

Using the Euler equation

$$\exp\{-\rho(c_t^*)\} = \left(\frac{\delta}{q}\right)^s \mathbf{E}_t\big[\exp\{-\rho(c_{t+s}^*)\}\big]$$

and the properties of the normal distribution, we have, for $s \geq 1$,

$$\mathbf{E}_t c_{t+s}^* = c_t^* + s \frac{\ln \dfrac{\delta}{q}}{\rho} - \frac{\rho}{2} \sum_{n=1}^{s} \sigma_{c_{t+n}}^2,$$

$$\mathbf{E}_{t+1} c_{t+s}^* = c_{t+1}^* + (s-1) \frac{\ln \dfrac{\delta}{q}}{\rho} - \frac{\rho}{2} \sum_{n=1}^{s-1} \sigma_{c_{t+1+n}}^2,$$

where $\sigma_{c_t}^2$ is the variance of consumption growth in period $t$. This implies

$$\sum_{s=1}^{T-t-1} (\mathbf{E}_{t+1} - \mathbf{E}_t) q^s c_{t+s}^* = \frac{1-q^{T-t}}{1-q}\left[ c_{t+1}^* - c_t^* + \frac{\ln \dfrac{q}{\delta}}{\rho} - \frac{\rho}{2}\sigma_{c_{t+1}}^2 \right]$$

and, using (S6),

$$c_{t+1}^* - c_t^* = \Gamma_t + \frac{1-q}{1-q^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)$$

$$\times \left[ \sum_{n=0}^{T-t-1} q^n (y_{t+1+n}^* + \tau_{t+1+n}^*(y^{t+1+n})) \right],$$

with $\Gamma_t = \frac{\ln(\delta/q)}{\rho} + \frac{\rho}{2}\sigma_{c_t}^2$.

---

[5]One would obtain the same result for any process for bonds using the standard rearrangements in the permanent income literature (e.g., Deaton (1992)).

Second, if we apply Lemma 3—in particular, see equation (S3)—together with $\theta_{t+n} = y^*_{t+n}$ for all $t, n$, since $(\mathbf{E}_{t+1} - \mathbf{E}_t)\eta_{t+1}(y^t) = 0$, we obtain

$$
(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n \tau^*_{t+1+n}(y^{t+1+n})\right]
$$

$$
= \left(\frac{1}{a} - 1\right)(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n y^*_{t+1+n}\right].
$$

If we now combine the two last expressions, we obtain

$$
\text{(S7)} \qquad c^*_{t+1} - c^*_t = \Gamma_t + \frac{1}{a}\frac{1-q}{1-q^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n y^*_{t+1+n}\right];
$$

hence $\Gamma_t = \frac{\ln\frac{\delta}{q}}{\rho} + \frac{\rho}{2}\sigma^2_{c_t}$.

From Assumption 2, for $T - t - 1 \geq p$, equation (S7) becomes[6]

$$
c^*_{t+1} - c^*_t = \frac{\ln\dfrac{\delta}{q}}{\rho} + \frac{\rho}{2}\sigma^2_{c_t} + \frac{1}{a}\frac{1-q}{1-q^{T-t}}\sum_{n=0}^{T-t-1} q^n \beta(q)v_{t+1}
$$

$$
= \frac{\ln\dfrac{\delta}{q}}{\rho} + \frac{\rho}{2}\sigma^2_{c_t} + \frac{1}{a}\beta(q)v_{t+1}.
$$

Finally, since the above expression implies that $\sigma^2_{c_t} := \mathrm{var}_t(\Delta c^*_{t+1}) = \frac{[\beta(q)]^2}{a^2}\sigma^2_v$, we obtain the claimed expression for consumption growth.

[6]Recall that $y_t$ follows

$$
y^*_{t+1} = y^*_t + \beta(L)v_{t+1},
$$

with $\beta(\cdot)$ of order $p$. We hence have

$$
(\mathbf{E}_{t+1} - \mathbf{E}_t)y^*_{t+1} = v_{t+1},
$$

$$
(\mathbf{E}_{t+1} - \mathbf{E}_t)qy^*_{t+2} = q(1 + \beta_1)v_{t+1},
$$

$$
(\mathbf{E}_{t+1} - \mathbf{E}_t)q^2 y^*_{t+3} = q^2(1 + \beta_1 + \beta_2)v_{t+1},
$$

$$
\vdots
$$

$$
(\mathbf{E}_{t+1} - \mathbf{E}_t)q^n y^*_{t+1+n} = q^n(1 + \beta_1 + \cdots + \beta_p)v_{t+1} \text{ for } n \geq p.
$$

As long as $T - t - 1 \geq p$, collecting terms vertically, the expression takes the stable form we indicate in the main text.

Clearly, the case with purely permanent shocks corresponds to the case where $\beta_i = 0$ for $i \geq 1$; hence the result is trivial. It is also easy to show that in this case, $b_t^* \equiv 0$ is consistent with

$$\frac{\partial \tau_t(y^t)}{\partial y_t} = \frac{1}{a} - 1 \quad \text{for all } t$$

and

$$\frac{\partial \tau_t(y^t)}{\partial y_{t-s}} = 0 \quad \text{for all } t, s > 0.$$

It is hence easy to see that

$$\text{(S8)} \qquad \Delta c_{t+1}^* = \frac{\ln \frac{\delta}{q}}{\rho} + \frac{\rho}{2a^2} \sigma_v^2 + \frac{1}{a}(y_{t+1}^* - y_t^*)$$

for all $T < \infty$ and all $\{q_t\}_{t=1}^{T-1}$. *Q.E.D.*

We now use the fact that the tax scheme is linear to show the following lemma that concludes the proof.

PROPOSITION 4: *If the agent has CARA preferences, when facing the above tax, the agent's problem is concave, so the derived tax scheme is optimal.*

PROOF: Note that so far we have shown that the transfer scheme is differentiable. Moreover, the agent's necessary conditions for $e_t^*(\theta^t) = 0$ to be an optimal choice is

$$\frac{\partial}{\partial \theta_t} \sum_{n=0}^{T-t} q^n \tau(\theta^{t+n}) \in \left[ \frac{1}{a} - 1, \frac{1}{b} - 1 \right].$$

Since we have shown that $\frac{\partial}{\partial \theta_t} \sum_{n=0}^{T-t} q^n \tau(\theta^{t+n}) = \frac{1}{a} - 1$, the condition is met.

Now note that since $e_t^*(\theta^t) \equiv 0$, at all nodes we have $y_t(\theta^t) = \theta^t$. We can hence invert the identity map and write the transfer scheme as a function of income histories $y^t$. We have to show that, when facing the optimal tax scheme, the agent's problem is jointly concave in $\{e_t(\theta^t)\}_{t=0}^T$ and $\{b_{t+1}(\theta^t)\}_{t=0}^T$. Consider two contingent plans $e^1, b^1, c^1$ and $e^2, b^2, c^2$. Now consider the plan $e^\alpha, b^\alpha, c^\alpha$, where for all $y^t$ and $\alpha \in [0, 1]$, we have $e_t^\alpha(\theta^t) := \alpha e_t^1(\theta^t) + (1 - \alpha)e_t^2(\theta^t)$, and similarly for $b_t^\alpha$ and $c_t^\alpha$. First of all, since assets enter linearly in the agent's budget constraint and effort enters linearly in the production function, the concavity of the agent's utility in $c - v(e)$, and the additive separability over time and states imply that if we show that $c_t^\alpha - v(e_t^\alpha) \geq \alpha[c_t^1 - v(e_t^1)] + (1-\alpha)[c_t^2 - v(e_t^2)]$,

we are done. If we set $k_t$ to denote the constant of integration of $\tau_t$, then an agent who chooses plan $e^\alpha$ of effort at node $\theta^t$ gets

$$
c_t^\alpha - v(e_t^\alpha) = y_t^\alpha + \sum_{i=0}^{t-1} \tau_t^{(t-i)} y_{t-i}^\alpha + k_t - v(e_t^\alpha)
$$

$$
= \theta_t + e_t^\alpha + \sum_{i=0}^{t-1} \tau_t^{(t-i)} [\theta_{t-i} + e_{t-i}^\alpha] + k_t - v(e_t^\alpha)
$$

$$
\geq [\alpha(\theta_t + e_t^1) + (1-\alpha)(\theta_t + e_t^2)]
$$

$$
+ \sum_{i=0}^{t-1} \tau_t^{(t-i)} [\theta_{t-i} + e_{t-i}^\alpha]
$$

$$
\times [\alpha(\theta_{t-i} + e_{t-i}^1) + (1-\alpha)(\theta_{t-i} + e_{t-i}^2)]
$$

$$
+ k_t - \alpha[v(e_t^1)] + (1-\alpha)[v(e_t^2)]
$$

$$
= \alpha[c_t^1 - v(e_t^1)] + (1-\alpha)[c_t^2 - v(e_t^2)],
$$

where the inequality in the penultimate row comes from the concavity of $v$ in $e$. The last line uses the agent's budget constraint $c_t(y^t) = y_t + \tau(y^t)$.     Q.E.D.

A final remark: Although the proof of the closed form uses finite time, we conjecture that by adapting the Proof of Proposition 7 in Cole and Kocherlakota (2001), we are able to show that the same closed-form solution for $T = \infty$ is unbounded below, despite $u$.

## B.2. *Quadratic Utility*

We now maintain the same assumptions on the cost function $v$ (or the production function $f$) as in (S1). Moreover, we keep the linearity assumption for the process $\theta_t$, that is, $\Delta \theta_t = \beta(L) v_t$, but we do not assume any parametric distribution for the i.i.d. shocks $v_t$ (of course, we need to be able to take expectations). In fact, we now need to assume that $\Theta$ is bounded above by $\theta_{\max} < \infty$ and that agent's preferences are quadratic:

$$
\text{(S9)} \qquad u(c - v(e)) := -\frac{1}{2} \big( \bar{B} - (c - v(e)) \big)^2 \quad \text{with } \bar{B} \gg T \theta_{\max}.
$$

Finally, we are able to derive the closed form only within the class of transfer schemes that admit symmetric cross-partial derivatives. Making assumptions on endogenous variables is of course not desirable, but note that the incentive constraint always imposes some degree of monotonicity on the transfer scheme. Since monotone functions on compact sets are absolutely continuous,

under a few further regularity conditions, we conjecture that one would be able to show at least almost everywhere differentiability of the transfer scheme. Of course, the symmetry of the Hessian is an even stronger condition: we did not investigate how to show it from primitives.

We have the following proposition.

PROPOSITION 5: *If the agent has preferences as in* (S9) *and* $\theta_t = \theta_{t-1} + \beta(L)v_t$, *within the class of transfer schemes that admit symmetric cross-derivatives, taxes are linear in income histories. Moreover, if* $\delta = q$, *the expression of marginal taxes is exactly as in the CARA case. In particular, for* $T \geq t + p + 1$, *we have*

$$\Delta c_{t+1}^* = \frac{1}{a}\beta(q)v_{t+1}.$$

PROOF: First of all, from Lemma 1, in equilibrium we get $e_t^* \equiv 0$; hence the transfer scheme is invertible and we can write it in terms of income histories $y^t$. We now need a crucial lemma, which uses differentiability.

LEMMA 4: *Within the class of transfer schemes that admit symmetric cross-derivatives, the discounted value of marginal transfers* $\sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_s}$ *does not depend on* $(y_t, \ldots, y_T)$ *for all* $s$. *They are hence linear functions of* $y_s$ *given* $y^t$.

PROOF: Consider the following relaxed problem of the firm: Maximize expected discounted profits, choosing the transfer scheme subject to the first-order conditions of the agent, namely for all $t \geq 1$ and $t \geq s \geq 0$,

$$(S10) \qquad \frac{1}{b} - 1 \geq \mathbf{E}_{t-s} \sum_{n=0}^{T-t} \delta^n \left[ \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right]$$

$$= \mathbf{E}_{t-s}\mathbf{E}_t \left[ \sum_{n=0}^{T-t} \delta^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right] \geq \frac{1}{a} - 1,$$

and the Euler equations corresponding to $u$ as in (S9),

$$(S11) \qquad -\bar{B} + c_t(y^t) = \left(\frac{\delta}{q}\right)^s \mathbf{E}_t[c_{t+s}(y^{t+s})] - \left(\frac{\delta}{q}\right)^s \bar{B}.$$

The proof is by backward induction. By looking at the last period of the problem, we have

$$\frac{1}{b} \geq 1 + \frac{\partial \tau_T(y^T)}{\partial y_T} \geq \frac{1}{a}.$$

Since the firm aims to insure the agent, the relevant inequality is the second one. Moreover, given that there is no gain in efficiency in changing the implemented level of effort and that (S11) is not affected as long as the average value of transfers does not change, the firm will set $\frac{\partial \tau_T(y^{T-1}, y_T)}{\partial y_T} = \frac{1}{a} - 1$ for all $y^{T-1}$ and $y_T$. This implies a zero cross derivative: $\frac{\partial \tau_T(y^{T-1}, y_T)}{\partial y_T \, \partial y_t} = 0$ for all $t$. Given our assumptions on the class of transfer schemes, by symmetry, it must be that $\frac{\partial \tau_T(y^T)}{\partial y_t}$ is constant in $y_T$ for all $t < T$.

Now consider $\tau_{T-1}$. Since $\frac{\partial \tau_T(y^T)}{\partial y_{T-1}}$ does not depend on $y_T$, the effort incentive compatibility can be written as

$$\frac{\partial \tau_{T-1}(y^{T-1})}{\partial y_{T-1}} + \delta \frac{\tau_T(y^T)}{\partial y_{T-1}} \mathbf{E}_{T-1} \left[ \frac{u'(c_T)}{u'(c_{T-1})} \right]$$

$$= \frac{1}{a} - 1 \quad \text{for all } y^{T-2} \text{ and } y_{T-1}.$$

Since $\mathbf{E}_{T-1}[\frac{u'(c_T)}{u'(c_{T-1})}] = \frac{q}{\delta}$, we have that $\frac{\partial \tau_{T-1}(y^{T-1})}{\partial y_{T-1}} + q \frac{\partial \tau_T(y^T)}{\partial y_{T-1}}$ is a constant for all $y^{T-2}$ and $y_{T-1}$. Again, since the transfer scheme is assumed to have symmetric cross-derivative, this property implies that $\frac{\partial \tau_{T-1}(y^{T-1})}{\partial y_t} + q \frac{\partial \tau_T(y^T)}{\partial y_t}$ is also constant in $y_{T-1}$ (and $y_T$) for all $t$. Going backward, we have our result: $\sum_{n=t}^{T} q^n \frac{\partial \tau_n(y^n)}{\partial y_s}$ is constant in $y_t, \ldots, y_T$ for all $s$. $\hfill$ Q.E.D.

Given the above results we can apply the law of iterated expectations and get, for a generic $\delta$ and $q$,

$$(S12) \quad \mathbf{E}_t \left[ \sum_{n=0}^{T-t} \delta^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right]$$

$$= \mathbf{E}_t \left[ \sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right.$$

$$\left. + \delta^{T-t} \mathbf{E}_{T-1} \frac{\partial \tau_T(y^T)}{\partial y_t} \frac{u'(c_T - e_T)}{u'(c_t - e_t)} \right]$$

$$= \mathbf{E}_t \left[ \sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right.$$

$$\left. + \delta^{T-t} \frac{\partial \tau_T(y^T)}{\partial y_t} \mathbf{E}_{T-1} \frac{u'(c_T - e_T)}{u'(c_{T-1} - e_{T-1})} \frac{u'(c_{T-1} - e_{T-1})}{u'(c_t - e_t)} \right]$$

$$= \mathbf{E}_t \left[ \sum_{n=0}^{T-t-1} \delta^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right.$$
$$\left. + \delta^{T-t-1} q \frac{\partial \tau_T(y^T)}{\partial y_t} \frac{u'(c_{T-1} - e_{T-1})}{u'(c_t - e_t)} \right]$$

$$= \mathbf{E}_t \left[ \sum_{n=0}^{T-t-2} \delta^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right.$$
$$\left. + \delta^{T-t-1} \mathbf{E}_{T-2} \left( \frac{\partial \tau_{T-1}(y^{T-1})}{\partial y_t} + q \frac{\partial \tau_T(y^T)}{\partial y_t} \right) \frac{u'(c_{T-1} - e_{T-1})}{u'(c_t - e_t)} \right]$$

$$= \mathbf{E}_t \left[ \sum_{n=0}^{T-t-2} \delta^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \frac{u'(c_{t+n} - e_{t+n})}{u'(c_t - e_t)} \right.$$
$$\left. + \delta^{T-t-2} q \left( \frac{\partial \tau_{T-1}(y^{T-1})}{\partial y_t} + q \frac{\partial \tau_T(y^T)}{\partial y_t} \right) \frac{u'(c_{T-2} - e_{T-2})}{u'(c_t - e_t)} \right]$$

$$\vdots$$

$$= \mathbf{E}_t \left[ \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}(y^{t+n})}{\partial y_t} \right],$$

where we repeatedly used the linearity of expectations and the Euler equation. We are hence done since, given that the obtained taxes are linear, Proposition 4 implies that this transfer scheme is optimal (now within the class of schemes we consider). Moreover, we are now able to follow the steps for the derivation of the closed form for CARA utility and obtain a very similar closed form.

Since from the incentive compatibility for effort $e_t$, we have $\mathbf{E}_t[\sum_{n=0}^{T-t} q^n \times \frac{\partial \tau_{t+n}^*(y^{t+n})}{\partial y_t}] = \frac{1}{a} - 1$, by using the law of iterated expectations, we obtain

$$\text{(S13)} \quad (\mathbf{E}_{t+1} - \mathbf{E}_t) \left[ \sum_{n=0}^{T-t-1} q^n \tau_{t+1+n}^*(y^{t+1+n}) \right]$$
$$= \left( \frac{1}{a} - 1 \right) (\mathbf{E}_{t+1} - \mathbf{E}_t) \left[ \sum_{n=0}^{T-t-1} q^n y_{t+1+n}^* \right].$$

The expressions for the optimal individual taxes $\tau_t$ can be obtained by working backward. In Attanasio and Pavoni (2007), we considered generic $q$ and $\delta$. When $q \neq \delta$, the expressions can get quite complicated even for the purely temporary shocks. When $\delta = q$, however, from the Euler equation we have

$\mathbf{E}_t c_{t+s}^* = c_t^*$ for all $s$. So following exactly the lines of the proof of Proposition 3 above for CARA, namely using the standard rearrangements of the permanent income literature, we obtain that

$$c_{t+1}^* - c_t^* = \frac{1-q}{1-q^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n(y_{t+1+n}^* + \tau_{t+1+n}^*(y^{t+1+n}))\right]$$

$$= \frac{1}{a}\frac{1-q}{1-q^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[\sum_{n=0}^{T-t-1} q^n y_{t+1+n}^*\right].$$

Again, for $T - t - 1 \geq p$, the expression stabilizes to the claimed one:

$$c_{t+1}^* - c_t^* = \frac{1}{a}\beta(q)v_{t+1}.$$

$Q.E.D.$

### B.3. *Isoelastic Utility: A Closed Form in Logs*

The outcome of this section is an expression for innovation in log consumption of the form analogous to those obtained in Propositions 3 and 4 for the CARA and quadratic agent's utilities; that is,

$$\ln c_{t+1}^* - \ln c_t^* = \frac{\ln \dfrac{\delta}{q}}{\gamma} + \frac{\gamma}{2a^2}[\beta(\lambda q)]^2 \sigma_v^2 + \frac{1}{a}\beta(\lambda q)v_{t+1},$$

where $v_{t+1}$ is the innovation to log of income, $\frac{1}{\gamma}$ is the intertemporal elasticity of substitution of consumption at two consecutive dates, and $\lambda > 0$ is such that $\lambda q \leq \delta$ for $\gamma \geq 1$. We also obtain expressions for tax rates at different dates.

### B.3.1. *Model and Derivation of the Permanent Income Equation*

Assume a production function of the form

$$\ln y_t = \ln \theta_t + \ln e_t$$

and the process for skills

$$\ln \theta_t = \ln \theta_{t-1} + \beta(L)v_t.$$

As for the CARA case, an additional assumption, which is crucial for us to get an exact closed form, is that the shocks $v_t$ are *normally distributed* with zero mean and variance $\sigma_v^2$ (note that we slightly abuse in notation here).

Recall our specification for preferences

$$\frac{(c_t \cdot e_t^{-\phi(e_t)})^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma}\exp\{(1-\gamma)(\ln c_t - \phi(e_t)\ln e_t)\},$$

where $\phi(e) = \frac{1}{a}$ for $e \leq 1$ and $\phi(e) = \frac{1}{b}$ for $e \geq 1$. Our aim is to write the problem in logarithms so as to exploit the analogies to the case in levels. Clearly, the objective function of the agent is concave in log decisions whenever $\gamma > 1$ and the assumptions are consistent with empirical findings.[7] Since in equilibrium we have $e_t^* \equiv 1$, the Euler equation is the usual one,

$$(S14) \qquad \mathbf{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\exp\left(-\gamma\frac{\ln c_{t+1}}{\ln c_t}\right)\right] = \exp\left(-\gamma\mu_t + \gamma^2\frac{1}{2}\sigma_t^2\right) = \frac{q}{\delta},$$

where we used the fact that in equilibrium $c_{t+1}$ is log normally distributed,[8] with $\mu_t$ and $\sigma_t^2$ being the conditional mean and conditional variance of $\Delta \ln c_{t+1}$, respectively.

Since we implement $b_t^* \equiv 0$, the budget constraint in equilibrium implies that $\ln c_t^*(y^t) = \ln y_t^* + \ln \tau_t^*(y^t)$. In what follows, for notational simplicity, we abuse notation and use $y^t$ to denote the history of *log incomes*. Since the logarithmic function is strictly monotone (and $y_t \geq 0$), every function of $y_t$ can be written as a function of $\ln y_t$ and vice versa. The objective function for effort plans hence becomes

$$\mathbf{E}_0 \sum_{t=0} \delta^t \frac{1}{1-\gamma} \exp\{(1-\gamma)(\ln y_t + \ln \tau_t(y^t) - v(\ln e_t))\}.$$

Given our specification for $v$ and $u$, the entire objective function can be expressed in logarithms. It is now easy to see the strong analogy to the case in levels considered above. In particular, we follow the main line of proof we adopted for the quadratic utility with the additional feature of log normality to obtain the precise expression for (deterministic) consumption growth rates as in the CARA case. If *we assume that the transfer scheme $\tau$ is differentiable*, the first-order condition for the log of effort $\ln e_t$ is

$$(S15) \qquad \mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left(\frac{c_{t+n}}{c_t}\right)^{1-\gamma} \frac{\partial \ln \tau_{t+n}(y^{t+n})}{\partial \ln y_t} = \frac{1}{a} - 1.$$

Once again, if *the transfer scheme admits a symmetric cross-derivative*, we can show backward that the conditional expectations can be decomposed since $\frac{\partial \ln \tau_{t+n}(y^{t+n})}{\partial \ln y_t}$ does not depend on (log) $y_{t+n}$.

The strong similarity with the model in levels has one last caveat. Since $c_t$ is log normally distributed, we have

$$\mathbf{E}_t\left[\left(\frac{c_{t+n}}{c_t}\right)^{1-\gamma}\right] = \mathbf{E}_t\left[\exp\left((1-\gamma)\frac{\ln c_{t+1}}{\ln c_t}\right)\right]$$

[7]For the United Kingdom, see Attanasio and Weber (1993).
[8]For a more extensive argument on this, see the very last section in the Appendix of Attanasio and Pavoni (2006).

$$= \exp\left\{(1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right\}.$$

Moreover, from the Euler equation (S14), we obtain

$$(S16) \quad \exp\left\{(1-\gamma)\mu_t + \frac{1}{2}(1-\gamma)^2\sigma_t^2\right\}$$

$$= \exp\left\{-\gamma\mu_t + \gamma^2\frac{1}{2}\sigma_t^2\right\}\exp\left\{\mu_t + \frac{1}{2}(1-2\gamma)\sigma_t^2\right\}$$

$$= \frac{q}{\delta}\exp\left\{\mu_t + \frac{1}{2}\sigma_t^2 - \gamma\sigma_t^2\right\}$$

$$:= \frac{q}{\delta}\lambda_t,$$

where $\lambda_t := \exp(\mu_t + (\frac{1}{2} - \gamma)\sigma_t^2) > 0$. In the log utility case, when $\delta = q$, then $\lambda_t = 1$.[9] Similarly, by the law of iterated expectations, assuming constant $\mu_t$ and $\sigma_2^2$, we get

$$\mathbf{E}_t\left[\left(\frac{c_{t+n}}{c_t}\right)^{-\gamma}\right] = \mathbf{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\mathbf{E}_{t+1}\left(\frac{c_{t+2}}{c_{t+1}}\right)^{-\gamma} \cdot \ldots \cdot \mathbf{E}_{t+n-1}\left(\frac{c_{t+n}}{c_{t+n-1}}\right)^{-\gamma}\right]$$

$$= \left(\frac{q}{\delta}\right)^n$$

and

$$\mathbf{E}_t\left[\exp\left((1-\gamma)\frac{\ln c_{t+n}}{\ln c_t}\right)\right]$$

$$= \mathbf{E}_t\left[\exp\left\{-\gamma\frac{\ln c_{t+1}}{\ln c_t}\right\}\lambda\mathbf{E}_{t+1}\exp\left\{-\gamma\frac{\ln c_{t+2}}{\ln c_{t+1}}\right\}\lambda\cdot\ldots\cdot$$

$$\cdot \mathbf{E}_{t+n-1}\exp\left\{-\gamma\frac{\ln c_{t+n}}{\ln c_{t+n-1}}\right\}\lambda\right]$$

$$= \left(\frac{q\lambda}{\delta}\right)^n.$$

---

[9]Since $\mu_t = \frac{\ln\delta/q}{\gamma} + \frac{\gamma}{2}\sigma_t^2$, then $\lambda_t = \exp(\frac{\ln\delta/q}{\gamma} + \frac{1-\gamma}{2}\sigma_t^2)$, which implies $\lambda_t \leq \frac{\delta}{q}$ since $\gamma \geq 1$. Moreover, when $u$ is logarithmic, we have $\lambda = \frac{\delta}{q}$, which is obviously consistent with

$$\lim_{\gamma\to 1^+}\mathbf{E}_t\left[\exp\left((1-\gamma)\frac{\ln c_{t+n}}{\ln c_t}\right)\right] = 1.$$

Again using the law of iterated expectations in the same way we did to derive equation (S12), the incentive constraint (S15) can be written as

$$(S17) \qquad \mathbf{E}_t \sum_{n=0}^{T-t} (q\lambda)^n \frac{\partial \ln \tau_{t+n}(y^{t+n})}{\partial \ln y_t} = \frac{1}{a} - 1.$$

Since taxes are linear in the log space and the agent's objective function is concave for $\gamma \geq 1$, the so-derived scheme is the optimal one within the class of differentiable schemes with symmetric cross-derivative as it solves the relaxed problem (only subject to the first-order conditions), while being globally incentive compatible.

We can now follow the same steps as for the model in levels to obtain the desired permanent income expressions: from $\ln c_t = \ln y_t + \ln \tau_t$ at all nodes, we get

$$\mathbf{E}_t \sum_{n=0}^{T-t-1} (q\lambda)^n \ln c_{t+1+n}^* = \mathbf{E}_t \sum_{n=0}^{T-t-1} (q\lambda)^n (\ln y_{t+1+n}^* + \ln \tau_{t+1+n}^*(y^{t+1+n})),$$

$$\mathbf{E}_{t+1} \sum_{n=0}^{T-t-1} (q\lambda)^n \ln c_{t+1+n}^* = \mathbf{E}_{t+1} \sum_{n=0}^{T-t-1} (q\lambda)^n (\ln y_{t+1+n}^* + \ln \tau_{t+1+n}^*(y^{t+1+n})).$$

By repeatedly using Euler equation (S14), together with the properties of the normal distribution, we obtain

$$\ln c_{t+1}^* - \ln c_t^*$$
$$= \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2 + \frac{1 - q\delta}{1 - (q\delta)^{T-t}}$$
$$\times (\mathbf{E}_{t+1} - \mathbf{E}_t)\left[ \sum_{n=0}^{T-t-1} (q\lambda)^n (\ln y_{t+1+n}^* + \ln \tau_{t+1+n}^*(y^{t+1+n})) \right].$$

Finally, from the expression for marginal log taxes, we obtain

$$\ln c_{t+1}^* - \ln c_t^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2$$
$$+ \frac{1}{a}\frac{1 - q\delta}{1 - (q\delta)^{T-t}}(\mathbf{E}_{t+1} - \mathbf{E}_t)\left[ \sum_{n=0}^{T-t-1} (q\lambda)^n (\ln y_{t+1+n}^*) \right].$$

It is hence again easy to see that for $T \geq t + 1 + p$, using the properties of the ARIMA($p$) process we postulated above, the previous expression stabilizes

into

$$\text{(S18)} \qquad \Delta \ln c_{t+1}^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2 + \frac{1}{a}\frac{1-q\delta}{1-(q\delta)^{T-t}} \sum_{n=0}^{T-t-1} (q\lambda)^n \beta(q\lambda)v_{t+1}$$

$$= \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\sigma^2 + \frac{1}{a}\beta(q\lambda)v_{t+1}.$$

Since the polynomial is invertible and $0 < q\lambda \le \delta$, all expressions are well defined. Moreover, from (S18), we obtain that $\sigma^2 := \mathrm{var}_t(\Delta \ln c_{t+1}^*) = \frac{[\sum_i((q\lambda)^i\beta_i)]^2}{a^2}\sigma_v^2 = \frac{[\beta(\lambda q)]^2}{a^2}\sigma_v^2$, as claimed above.

### B.3.2. *Expressions for Taxes*

The analysis is tedious but straightforward. Taxes are defined by the two incentive constraints: the effort incentive constraints and the Euler equations. Moreover, since taxes take very complicated expressions for the periods close to $T$, we derive the expressions only for stable values, hence for sufficiently large $T$.

The whole analysis is considerably simplified if we describe the transfer scheme in terms of the histories of the shocks $v_t$. To simplify the notation, we keep all symbols as above (although this is an abuse in notation of course). Let $v^t = (v_1, \ldots, v_t)$ be a given history of shocks. By repeatedly applying the law of motion for $\ln \theta_t$, we have

$$\text{(S19)} \qquad \ln \theta_t = \ln \theta_0 + \sum_{s=0}^{t-1} \beta(L)v_{t-s}.$$

We normalized $\theta_0 = 1$ and that lagged terms in the MA expressions $v_{-s}$, $s = 0, \ldots, p$, are set to zero as well by the other initial conditions. It is easy to see that in the last period, we have

$$\text{(S20)} \qquad \frac{\partial \ln \tau_T(v^T)}{\partial v_T} = \frac{1}{a} - 1;$$

this is so since, given $v^{T-1}$, the agent can lie over $v_T$ exactly in the same way as he would lie over $\theta_t$, with exactly the same marginal net costs/returns, as $\beta_0 = 1$. As before, it is easy to show that taxes are linear in $v^t$. Note, however, that a lie over $v_t$ today affects future income not only through the transfer scheme, but also via the persistence pattern of the process for $\theta_t$. In particular, consider an agent who lies over $v_t$ and then tells the truth over future $v_{t+s}$. That agent will have to lie (implicitly) over all future $\theta_{t+s}$ precisely by the amount of the future effect of $v_t$ over $\theta_{t+s}$. Of course, the agent will then be forced to make income levels appear to be consistent with the lie, namely $\hat{y}_{t+s} = \hat{\theta}_{t+s}$. For $t \le T - p$, we

hence have (note that we can eliminate the conditional expectation because of the linearity of taxes)

$$(S21) \quad \sum_{s=0}^{T-t}(q\lambda)^s \frac{\partial \ln \tau_{t+s}(v^{t+s})}{\partial v_t}$$

$$= \left(\frac{1}{a} - 1\right)\left[1 + (q\lambda)(1 + \beta_1) + (q\lambda)^2(1 + \beta_1 + \beta_2) + \cdots \right.$$

$$\left. + (q\lambda)^p \frac{\beta(1)}{1 - q\lambda}\right].$$

Consider now the Euler equation between periods $t$ and $t+1$ for $t \le T - p$. For $b_t^* \equiv 0$ to be incentive compatible at each node, we have

$$(S22) \quad \exp\{-\rho(\ln \theta_t + \ln \tau_t(v^t))\} = \frac{\delta}{q}\mathbf{E}_t\big[\exp\{-\rho(\ln \theta_{t+1} + \ln \tau_{t+1}(v^{t+1}))\}\big].$$

As we saw, the incentive compatibility constraint together with the symmetric partial derivative assumption implies that there is a function $\hat{\eta}_{t+1}$ such that $\ln \tau_{t+1}(v^{t+1}) = \hat{\eta}_{t+1}(v^t) + \tau_{t+1}^{(t+1)}v_{t+1}$. (Note that the functions $\hat{\eta}$ are not the same as the function $\eta$ in Proposition 3 but very similar in nature, namely for all $s \ge 0$, $\frac{\partial \hat{\eta}_{t+1}(v^t)}{\partial v_{t-s}} = \frac{\partial \ln \tau_{t+1}(v^{t+1})}{\partial v_{t-s}}$.) Since $\theta_{t+1}$ is normally distributed, taking the log operator on both sides and using the properties of the normal distribution, since $\mathbf{E}_t v_{t+1} = 0$, (S22) becomes

$$(S23) \quad \ln \theta_t + \ln \tau_t(v^t) = \Gamma_t^{t+1} + \mathbf{E}_t \ln \theta_{t+1} + \hat{\eta}_{t+1}(v^t)$$

$$= \Gamma_t^{t+1} + \ln \theta_t + \sum_{i=1}^{p} \beta_i v_{t+1-i} + \hat{\eta}_{t+1}(v^t),$$

where we used the projection result $\mathbf{E}_t \theta_{t+1} = \theta_t + \sum_{i=1}^{p} \beta_i v_{t+1-i}$. We also used the linearity of the tax on $v_{t+1}$ together with $\mathbf{E}_t v_{t+1} = 0$. More in general, for all $t, s \ge 1$, we have

$$(S24) \quad \ln \theta_t + \ln \tau_t(v^t) = \Gamma_t^{t+s} + \ln \theta_t + \sum_{n=1}^{\min\{s,p\}} \sum_{i=n}^{p} \beta_i v_{t+n-i} + \hat{\eta}_{t+s}(v^t).$$

Now, so that (S23) holds true for all $v_t$ given $v^{t-1}$, it must be that

$$\frac{\partial \ln \tau_t(v^t)}{\partial v_t} = \frac{\partial \hat{\eta}_{t+1}(v^t)}{\partial v_t} + \beta_1 = \frac{\partial \ln \tau_{t+1}(v^{t+1})}{\partial v_t} + \beta_1.$$

In general, the Euler equation between periods $t$ and $t + s$, $s \geq 1$, implies

$$(S25) \qquad \frac{\partial \ln \tau_t(v^t)}{\partial v_t} = \frac{\partial \ln \tau_{t+s}(v^{t+s})}{\partial v_t} + \sum_{i=1}^{\min\{s,p\}} \beta_i.$$

Hence, for $s \geq p$, marginal taxes become constant. Now, so that both the Euler equations and the incentive constraint (S21) hold simultaneously, by repeatedly using (S25), we have

$$\sum_{s=0}^{T-t} (q\lambda)^s \frac{\partial \ln \tau_{t+s}(v^{t+s})}{\partial v_t}$$

$$= \frac{\partial \ln \tau_t(v^t)}{\partial v_t} \Big[ 1 + q\lambda(1 - \beta_1) + (q\lambda)^2(1 - \beta_1 - \beta_2) + \cdots$$

$$+ (q\lambda)^p \frac{2 - \beta(1)}{1 - q\lambda} \Big]$$

$$= \Big( \frac{1}{a} - 1 \Big) \Big[ 1 + q\lambda(1 + \beta_1) + (q\lambda)^2(1 + \beta_1 + \beta_2) + \cdots$$

$$+ (q\lambda)^p \frac{\beta(1)}{1 - q\lambda} \Big].$$

It is hence easy to see that

$$\frac{\partial \ln \tau_t(v^t)}{\partial v_t}$$

$$= \Big( \frac{1}{a} - 1 \Big) \Big[ 1 + q\lambda(1 + \beta_1) + (q\lambda)^2(1 + \beta_1 + \beta_2) + \cdots$$

$$+ (q\lambda)^p \frac{\beta(1)}{1 - q\lambda} \Big]$$

$$\Big/ \Big[ 1 + q\lambda(1 - \beta_1) + (q\lambda)^2(1 - \beta_1 - \beta_2) + \cdots$$

$$+ (q\lambda)^p \frac{2 - \beta(1)}{1 - q\lambda} \Big]$$

$$:= \Big( \frac{1}{a} - 1 \Big) \kappa$$

and, of course, all other taxes can be obtained from this expression using (S25). Note that $\kappa > 0$ and, for future reference, that when $\beta_i = 0$ for all $i > 0$, $\kappa = 1$ so $\frac{\partial \ln \tau_t(v^t)}{\partial v_t} = \frac{\partial \ln \tau_{t+s}(v^{t+s})}{\partial v_t} = \frac{1}{a} - 1$.

Finally, we derive the expression that relates the change in the cross-sectional variance of consumption with the change in the cross-sectional variance of income. Again, so as to have stable formulas, we assume $t \geq p$ and $t \leq T - p$, so that all the above expressions apply fully. We have

$$
\text{(S26)} \quad \ln c_t^*(\theta^t) = \ln y_t^* + \ln \tau_t^*(v^t) + \ln \theta_t + \ln \tau_t^*(v^t)
$$
$$
= \theta_t + \tau_t^{(0)} v_t + \tau_t^{(-1)} v_{t-1} + \cdots + \tau_t^{(-p)} v_{t-p}
$$
$$
+ \tau_t^{(-p)} v_{t-p-1} + \cdots + \tau_t^{(-p)} v_1 + \tau_0 + t\Gamma,
$$

where the constant of integration $\tau_0$ is chosen to satisfy the planner's budget constraint and, as we showed above, for all $t$, $\tau_t^{(-s)} = (\frac{1}{a} - 1)\kappa - \beta_1 - \beta_2 - \cdots - \beta_s$. Similarly, for $t+1$, we have

$$
\ln c_{t+1}^*(\theta^{t+1}) = \theta_{t+1} + \tau_{t+1}^{(0)} v_{t+1} + \tau_{t+1}^{(-1)} v_t + \cdots + \tau_{t+1}^{(-p)} v_{t+1-p}
$$
$$
+ \tau_t^{(-p)} v_{t-p} + \cdots + \tau_t^{(-p)} v_1 + \tau_0 + (t+1)\Gamma,
$$

where for all $n$, we have $\tau_{t+1}^{(-n)} = \tau_t^{(-n)}$. Recall that we are interested in computing the unconditional variance of both the above term and that in (S26), and then taking the difference. This difference in variances can be stated as

$$
\Delta \text{var}(\ln c_{t+1}^*) := \text{var}(\ln c_{t+1}^*) - \text{var}(\ln c_t^*)
$$
$$
= \text{var}(\theta_{t+1}) - \text{var}(\theta_t) + \text{var}(\tau_{t+1}^*(v^{t+1})) - \text{var}(\tau_t^*(v^t))
$$
$$
+ 2\big[\text{cov}(\theta_{t+1}, \tau_{t+1}^*(v^{t+1})) - \text{cov}(\theta_t, \tau_t^*(v^t))\big].
$$

Now note that

$$
\text{var}(\ln \tau_t^*(v^t)) = \Big(\big[\tau_t^{(0)}\big]^2 + \cdots + (1 + t - p)\big[\tau_t^{(-p)}\big]^2\Big)\sigma_v^2,
$$

while

$$
\text{var}(\ln \tau_{t+1}^*(v^{t+1})) = \Big(\big[\tau_{t+1}^{(0)}\big]^2 + \cdots + (2 + t - p)\big[\tau_{t+1}^{(-p)}\big]^2\Big)\sigma_v^2.
$$

Moreover, for $t \geq p$, $\tau_{t+1}^{(-n)} = \tau_t^{(-n)}$, we have

$$
\text{var}(\ln \tau_{t+1}^*(v^{t+1})) - \text{var}(\tau_t^*(v^t)) = \big[\tau_t^{(-p)}\big]^2 \sigma_v^2,
$$
$$
\text{cov}\big(\theta_t, \tau_t^{(-s)} v_{t-s}\big) = \text{cov}\big(\theta_{t+1}, \tau_{t+1}^{(-s)} v_{t+1-s}\big) \quad \text{for } s \leq p,
$$

and

$$
\text{cov}\big(\theta_t, \tau_t^{(-p)} v_{t-s}\big) = \text{cov}\big(\theta_{t+1}, \tau_{t+1}^{(-p)} v_{t+1-s}\big) = \tau_{t+1}^{(-p)} \beta(1) \sigma_v^2 \quad \text{for } s \geq p.
$$

Given that for $t \geq p$, only the correlation with $v_1$ remains in the $t+1$ terms, we have

$$\text{cov}(\theta_{t+1}, \tau_{t+1}^*(v^{t+1})) - \text{cov}(\theta_t, \tau_t^*(v^t)) = \tau_{t+1}^{(-p)}\beta(1)\sigma_v^2.$$

In summary,

$$\Delta\text{var}(\ln c_{t+1}^*) = \Delta\text{var}(\theta_{t+1}) + \left[\tau_t^{(-p)}\right]^2\sigma_v^2 + 2\tau_{t+1}^{(-p)}\beta(1)\sigma_v^2.$$

Finally, since from the definition of $\theta_t$ in (S19), for $t \geq p$, we have

$$\Delta\text{var}(\ln y_{t+1}^*) = \Delta\text{var}(\ln\theta_{t+1}) = [\beta(1)]^2\sigma_v^2 > 0,$$

which implies

$$\Delta\text{var}(\ln c_{t+1}^*) = \left[\beta(1) + \tau_{t+1}^{(-p)}\right]^2\sigma_v^2 = \frac{[\beta(1) + \tau_t^{(-p)}]^2}{[\beta(1)]^2}\Delta\text{var}(\ln y_{t+1}^*),$$

where we recall that

$$\tau_{t+1}^{(-p)} = \left(\frac{1}{a} - 1\right)\frac{[1 + q\lambda(1+\beta_1) + (q\lambda)^2(1+\beta_1+\beta_2) + \cdots]}{[1 + q\lambda(1-\beta_1) + (q\lambda)^2(1-\beta_1-\beta_2) + \cdots]}$$
$$- \beta(1) + 1;$$

hence $\beta(1) + \tau_t^{(-p)} = 1 + (\frac{1}{a} - 1)\kappa$ and

$$(S27) \qquad \Delta\text{var}(\ln c_{t+1}^*) = \left(\frac{1 + \left(\frac{1}{a} - 1\right)\kappa}{\beta(1)}\right)^2 \Delta\text{var}(\ln y_{t+1}^*).$$

Since both $\beta(1) > 0$ and $\kappa > 0$, the parameter $a$ is identified. Moreover, for $\beta_i = 0$ for all $i$, we have

$$(S28) \qquad \Delta\text{var}(\ln c_{t+1}^*) = \frac{1}{a^2}\Delta\text{var}(\ln y_{t+1}^*).$$

## B.4. *An Extended Model With Two Types of Shocks*

We now briefly present an extension of our model that allows for two types of (independent) shocks to income, with different degrees of persistence. Although we develop the model in levels, very similar expressions can be derived for the log-linear case.

Assume agents have preferences over $c_t$, $l_t$, and $e_t$ as $-\frac{1}{\rho}\exp\{-\rho(c_t - e_t - l_t)\}$. Moreover, assume that individual income can be decomposed into two components, that is, $y_t = x_t + \xi_t$, where $x_t = f(\theta_t^p, e_t)$ and $\xi_t = g(v_t^T, l_t)$. In

this model, $x_t$ represents the permanent component of income as $\theta_t^p = \theta_{t-1}^p + v_t^p$, with $v_t^p$ i.i.d., while $\xi_t$ represents the temporary component, as $v_t^T$ is i.i.d. The production function $f$ is as in (13), and a similar functional form for $g$ is assumed:

$$\xi_t = g(v_t^T, l_t)$$
$$= v_t^T + a^T \min\{l_t, 0\} + b^T \max\{l_t, 0\} \quad \text{with} \quad a^T > 1 > b^T.$$

Since effort is again time constant, in equilibrium, the income process displays the process[10]

(S29)     $y_t = y_{t-1} + v_t^p + \Delta v_t^T.$

We now follow a line of proof very similar to that used for the baseline model and we show that the reaction of consumption to the different shocks for $T \to \infty$ can be written as[11]

(S30)     $\Delta c_t^* = \dfrac{\ln(\delta/q)}{\rho} + \dfrac{\rho}{2}\left[\left(\dfrac{1}{a^p}\right)^2 \sigma_{v^p}^2 + \left(\dfrac{1-q}{a^T}\right)^2 \sigma_{v^T}^2\right] + \dfrac{1}{a^p} v_t^p + \dfrac{1-q}{a^T} v_t^T,$

where, for consistency, we denoted by $a^p$ the slope of $f$ for $e_t \leq 0$.

The closed form for the version of our model with two types of shocks provides a structural interpretation of recent empirical evidence. Using the evolution of the cross-sectional variance and covariance of consumption and income, Blundell, Pistaferri, and Preston (2008) estimated two parameters, $\phi$ and $\psi$, that represent the fraction of permanent and temporary shocks reflected in consumption. Within our model, estimates of these parameters can be interpreted as the severity of informational problems for income shocks of different persistence.

### B.4.1. *Proof of the Closed Form Expression (S30)*

The analysis is performed separately for the two types of shocks. Obviously, $e_t^* = l_t^* = 0$ at all nodes. We can hence equivalently describe the transfer scheme in terms of incomes. In the presence of both permanent and temporary shocks, the firm should obviously condition its transfers on $\xi_t = g(v_t^T, l_t)$ realizations as well. Denote by $h^t = (x^t, \xi^t)$ the combined public history. In the CARA case, by following the same line of proof as for Proposition 3, we can

---

[10]We could easily allow the temporary shock $v_t^T$ to follow a MA($p$) process.

[11]The corresponding expression for the model in logarithms is

$$\Delta c_{t+1}^* = \frac{\ln(\delta/q)}{\gamma} + \frac{\gamma}{2}\left[\left(\frac{1}{a^p}\right)^2 \sigma_{v^p}^2 + \left(\frac{1-\lambda q}{a^T}\right)^2 \sigma_{v^T}^2\right] + \frac{1}{a^p} v_{t+1}^p + \frac{1-\lambda q}{a^T} v_{t+1}^T.$$

show the differentiability of the scheme and the first-order conditions of the agent by solving

$$\mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left[ \frac{\partial \tau_{t+n}(h^{t+n})}{\partial \xi_t} \frac{u'(c_{t+n} - e_{t+n} - l_{t+n})}{u'(c_t - e_t - l_t)} \right] = \frac{1}{a^T} - 1$$

and

$$\mathbf{E}_t \sum_{n=0}^{T-t} \delta^n \left[ \frac{\partial \tau_{t+n}(h^{t+n})}{\partial x_t} \frac{u'(c_{t+n} - e_{t+n} - l_{t+n})}{u'(c_t - e_t - l_t)} \right] = \frac{1}{a^p} - 1,$$

where, for consistency, we denoted by $a^p$ the slope of $f$ for $e_t \le 0$. By the same proposition, the slopes $\frac{\partial \tau_{t+n}(h^{t+n})}{\partial \xi_t}$ do not depend on histories before or after period $t$, so we can use the Euler equation and apply the law of iterated expectations to get, for a generic $\delta$ and a deterministic sequence of bond prices (in the notation below $\zeta$ stays for $x$ or $\xi$),

$$\mathbf{E}_t \left[ \sum_{n=0}^{T-t} \delta^n \frac{\partial \tau_{t+n}(h^{t+n})}{\partial \zeta_t} \frac{u'(c_{t+n} - e_{t+n} - l_{t+n})}{u'(c_t - e_t - l_t)} \right]$$

$$= \mathbf{E}_t \left[ \sum_{n=0}^{T-t} \left( \prod_{s=0}^{n} q_{t+s-1} \right) \frac{\partial \tau_{t+n}(h^{t+n})}{\partial \zeta_t} \right].$$

Of course, in the quadratic utility case, exactly the same expression for marginal taxes can be obtained by assuming that the transfer scheme admits symmetric cross-derivatives in all elements of $h^t$. If we write the expressions for a constant $q$, we get

$$(S31) \qquad \mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}(h^{t+n})}{\partial \xi_t} = \frac{1}{a^T} - 1,$$

$$\mathbf{E}_t \sum_{n=0}^{T-t} q^n \frac{\partial \tau_{t+n}(h^{t+n})}{\partial x_t} = \frac{1}{a^p} - 1.$$

Assuming CARA (or quadratic) preferences, for permanent shocks (i.e., $x_t$ follows an ARIMA(0)), the Euler equation implies that only contemporaneous marginal taxes are positive and

$$\frac{\partial \ln \tau_t(y^t)}{\partial \ln x_t} = \frac{1}{a} - 1.$$

In this case, absent temporary shocks, we have

$$\Delta \ln c_{t+1} = \frac{\ln \dfrac{\delta}{q}}{\gamma} + \frac{\gamma}{2}\sigma_c^2 + \frac{1}{a}v_{t+1}.$$

Since from the above expression, the variance of log consumption is $\sigma_c^2 = \frac{1}{a^2}\sigma_v^2$, we have

$$(S32) \qquad \ln \tau_t(y^t) = \left(\frac{1}{a} - 1\right)\ln y_t + t\left[\frac{\ln \dfrac{\delta}{q}}{\gamma} + \frac{\gamma}{2a^2}\sigma_v^2\right] + \ln \tau_0$$

if we add temporary shocks. Since the analysis can be done independently, by comparing Euler equations at different dates, we can easily show that the tax rates for the purely temporary shock are related as

$$(S33) \qquad 1 + \frac{\partial \tau_t(h^t)}{\partial \xi_t} = \frac{\partial \tau_{t+s}(h^{t+s})}{\partial \xi_t} \geq 0 \quad \text{for all } t, s > 0.$$

It is hence easy to see by direct inspection of (S33) and (S31) that, as $T \to \infty$, the expressions for transfers become

$$1 + \tau_x = \frac{1}{a^p} \quad \text{and} \quad 1 + \tau_\xi = \frac{1-q}{a^T},$$

where $1 + \tau_x = 1 + \frac{\partial \tau_t(h^t)}{\partial x_t}$ and $1 + \tau_\xi = 1 + \frac{\partial \tau_t(h^t)}{\partial \xi_t} = \frac{\partial \tau_{t+k}(h^{t+k})}{\partial \xi_t}$ for $k > 0$. Hence tax rates are time-invariant and the agent's consumption reaction to income shocks is given by[12]

$$\Delta c_{t+1} = \Gamma + \frac{1}{a^p}\Delta x_{t+1} + \frac{1-q}{a^T}\Delta \xi_{t+1} = \Gamma + \frac{1}{a^p}v_{t+1}^p + \frac{1-q}{a^T}v_{t+1}^T,$$

where $\Gamma \geq 0$ and $\Gamma = 0$ when $u$ is quadratic and $\delta = q$.

As explained in the proof of Proposition 4, all the above expressions constitute optimal transfer schemes since the agent's problem is concave because all taxes are linear in all arguments.

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[12]As should be clear from the analysis for the isoelastic model, the corresponding equation for the model in logarithms is

$$\Delta \ln c_{t+1} = \Gamma + \frac{1}{a^p}\Delta \ln x_{t+1} + \frac{1-\lambda q}{a^T}\Delta \ln \xi_{t+1} = \Gamma + \frac{1}{a^p}v_{t+1}^p + \frac{1-\lambda q}{a^T}v_{t+1}^T,$$

where $\lambda = \exp\{\frac{\ln \delta/q}{\gamma} + \frac{1-\gamma}{2}\sigma_c^2\}$.

Finally, given the expressions for marginal taxes, we have

$$(S34) \quad c_t^*(h^t) = y_t^* + \tau_t^*(h^t)$$

$$= x_t^* + \xi_t^* + \left(\frac{1}{a^P} - 1\right)x_t^* + \left(\frac{1-q}{a^T} - 1\right)\xi_t^* + \sum_{s=1}^{t-1} \frac{1-q}{a^T}\xi_{t-s}^*$$

$$= \frac{1}{a^P}x_t^* + \frac{1-q}{a^T}\sum_{s=0}^{t-1}\xi_{t-s}^* + t\Gamma + \tau_0.$$

Now,

$$(S35) \quad \Delta \mathrm{var}(c_t^*(h^t)) = \left[\mathrm{var}\left(\frac{1}{a^P}x_t^*\right) - \mathrm{var}\left(\frac{1}{a^P}x_{t-1}^*\right)\right]$$

$$+ \mathrm{var}\left(\frac{1-q}{a^T}\sum_{s=0}^{t-1}\xi_{t-s}^*\right) - \mathrm{var}\left(\frac{1-q}{a^T}\sum_{s=0}^{t-2}\xi_{t-s}^*\right)$$

$$= \left(\frac{1}{a^P}\right)^2 \sigma_{v^P}^2 + \left(\frac{1-q}{a^T}\right)^2 \sigma_{v^T}^2$$

$$= \left(\frac{1}{a^P}\right)^2 \Delta \mathrm{var}(y_t^*) + \psi,$$

where $\psi := (\frac{1-q}{a^T})^2\sigma_{v^T}^2$ is a constant in the regression and the last lines use the fact that $\mathrm{var}(y_t^*) = \mathrm{var}(x_t^*) + \mathrm{var}(\xi_t^*) + 2\mathrm{cov}(x_t^*, \xi_t^*) = \mathrm{var}(x_t^*) + \sigma_{v^T}^2$, hence $\Delta \mathrm{var}(y_t^*) = \Delta \mathrm{var}(x_t^*)$.

## APPENDIX C: BIAS CORRECTION FOR THE VARIANCE BASED TEST

Recall that in Section 4.2 we had the following expression for the changes in the cross-sectional variance:

$$(S36) \quad \Delta \mathrm{Var}(\ln c_t) = \left(\frac{1}{a^P}\right)^2 \Delta \mathrm{Var}(\ln x_t) + \left(\frac{1-\lambda q}{a^T}\right)^2 \mathrm{Var}(\xi_t^i).$$

The observable version of equation (S36) is

$$(S37) \quad \Delta \overline{\mathrm{Var}(c_{gt})} = \frac{1}{a^2}\Delta\overline{\mathrm{Var}(y_{gt})} + \frac{1}{a^2}\Delta\varepsilon_{gt}^y - \Delta\varepsilon_{gt}^c,$$

where $\varepsilon_{gt}^y = \mathrm{Var}(y_{gt}) - \overline{\mathrm{Var}(y_{gt})}$ and $\varepsilon_{gt}^c = \mathrm{Var}(c_{gt}) - \overline{\mathrm{Var}(c_{gt})}$. The variance of the residuals $\varepsilon$ go to zero as the size of the cells in each time period increases. Moreover, information on the within-cell variability can be used to

correct OLS estimates of the coefficients in equation (S37). In particular, a bias correct estimator is given by the expression

$$(S38) \qquad \widehat{\theta} = A^{-1}[\widetilde{\theta} - B],$$

where $\widetilde{\theta} = (Z'Z)^{-1}Z'w$ is the OLS estimator, $B = (Z'Z)^{-1}\{\frac{1}{T-1}\sum_{t=2}^{T}\frac{\sigma_{cygt}}{N_{gt}} + \frac{\sigma_{cygt-1}}{N_{gt-1}}\}$ allows for the possibility of correlation between the $\varepsilon_t^y$ and $\varepsilon_t^c$, and

$$A = \left[ I - (Z'Z)^{-1}\frac{1}{T-1}\sum_{t=2}^{T}\left(\frac{\sigma_{ygt}^2}{N_{gt}} + \frac{\sigma_{ygt-1}^2}{N_{gt-1}}\right) \right].$$

In computing the variance–covariance matrix of this estimator, it is necessary to take into account the MA structure of the residuals as well as the possibility that observations for different groups observed at the same time will be correlated.

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