Supplementary Appendix for "Temptation and Revealed Preference"

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1 Proof of Thm 4.1: Necessity of Menu-Reversal

To establish the necessity of Menu-Reversal(ii), take μ, η such that $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$. By the representation, $U(\{\mu\}^{+t}) + V(\{\mu\}^{+t}) > U(\{\mu, \eta\}^{+t}) + V(\{\mu, \eta\}^{+t})$. Using (6.3) in the paper, deduce that $W(\{\mu\}) > W(\{\mu, \eta\})$. Given that W satisfies Stationarity (defined by (5.1) in the paper), we have $U(\{\mu\}^{+t}) > U(\{\mu, \eta\}^{+t})$. Since U and $U+V$ are continuous functions, there is an open neighborhood $A \subset \Delta \times \Delta$ of $(\{\mu\}^{+t}, \{\mu, \eta\}^{+t})$ s.t. for all $(m, n) \in A$, $U(m) > U(n)$ and $U(m) + V(m) > U(n) + V(n)$. It follows by (6.2) and $D_t \searrow 0$ that $\tau(A) = 0$.

In light of the comment in footnote 9 in the paper, in order to establish the necessity for Menu-Reversal(i) we need only show that $\mu > \eta$ and $\tau(\mu, \eta) = 0$ implies $\tau(\{\mu\}^{+1}, \{\mu, \eta\}^{+1}) = 0$. Since $\mu^{+t} > \eta^{+t}$ for all t, Set-Betweenness implies $\{\mu\}^{+t} \gtrsim \{\mu, \eta\}^{+t}$ for all t. If $\{\mu\}^{+1} > \{\mu, \eta\}^{+1}$ then the no-reversal result follows from Menu-Reversal(ii). Let $\{\mu\}^{+1} \approx \{\mu, \eta\}^{+1}$, and suppose by way of contradiction that $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ for all large t. As in the proof for Menu-Reversal(ii), $W(\{\mu\}) > W(\{\mu, \eta\})$ must hold; this implies

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$V(\mu) < V(\eta)$, and moreover, since $\mu > \eta$, we also have $W(\{\mu, \eta\}) = U(\mu) + V(\mu) - V(\eta)$. But then $\{\mu\}^{+1} \approx \{\mu, \eta\}^{+1}$ implies

$$U(\{\mu\}^{+1}) + V(\{\mu\}^{+1}) = U(\{\mu, \eta\}^{+1}) + V(\{\mu, \eta\}^{+1})$$
$$\implies \delta W(\{\mu\}) + \widehat{V}(\{\mu\}) = \delta W(\{\mu, \eta\}) + \widehat{V}(\{\mu, \eta\})$$
$$\implies (\delta + \beta)U(\mu) + \gamma V(\mu) = (\delta + \beta)W(\{\mu, \eta\}) + \gamma V(\eta)$$
$$\implies (\delta + \beta)[U(\mu) - W(\{\mu, \eta\})] = \gamma[V(\eta) - V(\mu)]$$
$$\implies (\delta + \beta)[-V(\mu) + V(\eta)] = \gamma[V(\eta) - V(\mu)]$$
$$\implies \beta = \gamma - \delta, \text{ contradicting the requirement that } \beta > \gamma - \delta. \text{ This}$$

completes the proof.

2 Proof of Thm 4.1: Lemmas for Proof of Sufficiency

Let $\Delta_s \subset \Delta$ be the set of lotteries on $C \times Z$ with finite support and $\Delta_s(Z)$ the set of lotteries on Z with finite support. Let δ_z denote the lottery degenerate at menu z. Define $\varphi : \Delta_s(Z) \longrightarrow Z$ by $\varphi(\sum p(x)\delta_x) = \sum p(x)x$ for all $\sum p(x)\delta_x \in \Delta_s(Z)$.

The first lemma adds to Lemma D.1 in the paper by establishing properties of $\succsim$. Note that part (d) establishes a "Strong Indifference to Timing" property.[1]

Lemma 1 *(a) There exists μ, η s.t. $\{\mu, \eta\}^{+t} > \{\eta\}^{+t}$ for all $t > 0$.*

(b) There exists μ, η s.t. $\{\mu\}^{+t} > \{\mu, \eta\}^{+t}$ for all large $t > 0$.

(c) $x^{+t} > y^{+t} \implies (\alpha x + (1 - \alpha)z)^{+t} > (\alpha y + (1 - \alpha)z)^{+t}$.

(d) (Strong Indifference to Timing) For all $\mu, \eta, \pi, \nu \in \Delta_s$, if $\mu^1 = \pi^1, \eta^1 = \nu^1, \varphi(\mu^2) = \varphi(\pi^2)$ and $\varphi(\eta^2) = \varphi(\nu^2)$, then $\{\mu, \eta\}^{+t} \approx \{\pi, \nu\}^{+t}$.

Proof. (a) By nondegeneracy of $\mathcal{C}$ (part (ii)) there exists μ, η such that $\tau(\mu, \eta) = 0$ and $\{\mu\}^{+1} \approx \{\mu, \eta\}^{+1} > \{\eta\}^{+1}$. By Sophistication, $\mu > \eta$. By

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[1]This was referred to as "Indifference to Timing" in [3].

Menu-Reversal(i), $\{\mu\}^{+t} \approx \{\mu, \eta\}^{+t}$ for all $t > 0$. Hence by transitivity, $\{\mu, \eta\}^{+t} > \{\eta\}^{+t}$ for all $t > 0$.

(b) By nondegeneracy of $\mathcal{C}$ (part (i)) there exists μ, η such that $\mu < \eta$ and $\mu^{+t} > \eta^{+t}$ for large t. By Sophistication and transitivity we see that $\mu^{+t} > \{\mu, \eta\}^{+t} \approx \eta^{+t}$ for all large t.

(c) By Independence, for any x, y, z, $x^{+t} > y^{+t} \implies \alpha x^{+t} + (1 - \alpha)z^{+t} > \alpha y^{+t} + (1 - \alpha)z^{+t}$, and that by Indifference to Timing, $\alpha x^{+t} + (1 - \alpha)z^{+t} \approx (\alpha x + (1 - \alpha)z)^{+t}$ and $\alpha y^{+t} + (1 - \alpha)z^{+t} \approx (\alpha y + (1 - \alpha)z)^{+t}$. The result follows from transitivity.

(d) We provide only an outline. For each $t > 0$, define the ranking $\gtrapprox_t^{+1}$ over Z of period t menus by $x \gtrapprox_t^{+1} y \iff x^{+t} \gtrapprox y^{+t}$. By Lemma D.1 and Set-Betweenness, $\gtrapprox_t^{+1}$ satisfies GP's Order, Continuity and Set-Betweenness axioms. By part (c) of this lemma it also satisfies GP's Set-Independence axiom. Thus for each $t > 0$, $\gtrapprox_t^{+1}$ admits a GP [1, Theorem 1] representation with normative and temptation utility U_t, V_t. Indifference to Timing and Separability imply that U_t is additively separable between C and Z, and linear in Z (argue as in the proof of [2, Theorem 1], for instance). By Sophistication, part (a) of this lemma and Theorem 6.1 in the paper, $\gtrapprox$ is represented by $U_t + V_t$. By the Indifference to Timing and Separability axioms applied to $\gtrapprox$, we see that $U_t + V_t$ is additively separable between C and Z and linear in Z. Conclude that V_t must have these properties as well. The result then follows. ∎

The following lemma establishes some properties of the normative menu-preference $\succsim$. These are used in the proof of Lemma D.5 in the paper.

Lemma 2 $\succsim$ *satisfies:*

*(i) Set-Betweenness**: $x \succsim y \implies x \succsim x \cup y \succsim y$.*

*(ii) Strong Indifference to Timing**: for any $\mu, \eta, \pi, \nu \in \Delta_s$, if $\mu^1 = \pi^1, \eta^1 = \nu^1, \varphi(\mu^2) = \varphi(\pi^2)$ and $\varphi(\eta^2) = \varphi(\nu^2)$, then $\{\mu, \eta\} \backsim \{\pi, \nu\}$.*

*(iii) Nondegeneracy**: there exist μ, η such that $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$.*

Proof. (i) We need to show $x^{+1} \succsim^* y^{+1} \implies x^{+1} \succsim^* (x \cup y)^{+1} \succsim^* y^{+1}$. Consider two cases.

Case (a): $x^{+1} \succ^* y^{+1}$

Write τ for $\tau(x^{+1}, y^{+1})$. Then by Lemma C.3(a), $x^{+1} >_\tau y^{+1}$. By the Set-Betweenness axiom, $x^{+1} \succsim_t (x \cup y)^{+1} \succsim_t y^{+1}$ for all $t \geq \tau$. Hence $x^{+1} \succsim_{\tau(x^{+1}, (x \cup y)^{+1})} (x \cup y)^{+1} \succsim_{\tau((x \cup y)^{+1}, y^{+1})} y^{+1}$, and by Lemma C.3(c), $x^{+1} \succsim^* (x \cup y)^{+1} \succsim^* y^{+1}$.

Case (b): $x^{+1} \backsim^* y^{+1}$

Suppose by way of contradiction that $x^{+1} \backsim^* y^{+1} \succ^* (x \cup y)^{+1}$. Then by Lemma C.3(a), $x^{+1} >_t (x \cup y)^{+1}$ and $y^{+1} >_t (x \cup y)^{+1}$ for all large t. But this violates Set-Betweenness. A similar argument establishes a contradiction for the case where $(x \cup y)^{+1} \succ^* x^{+1} \backsim^* y^{+1}$.

(ii) Observe that by Lemma 1 above, $\{\mu, \eta\} \approx_t \{\pi, \nu\}$ for all $t \geq 1$. Thus, $\tau(\{\mu, \eta\}^{+1}, \{\pi, \nu\}^{+1}) = 0$ and $\{\mu, \eta\}^{+1} \approx \{\pi, \nu\}^{+1}$. From Lemma C.3(c), we see that $\{\mu, \eta\}^{+1} \backsim^* \{\pi, \nu\}^{+1}$, and so $\{\mu, \eta\} \backsim \{\pi, \nu\}$.

(iii) By parts (i) and (iii), $\succsim$ admits a GP representation with some U and V. Since, by Lemma D.4(ii), there exists μ, η s.t. $\{\mu\} \succ \{\mu, \eta\}$ we see that U and V are nonconstant and V is not a positive affine transformation of U. Also, there exists μ, η s.t. $\{\mu\} \sim \{\mu, \eta\} \succ \{\eta\}$ and so we see that V is not a negative affine transformation of U either. Thus U and V are nonconstant and affinely independent. In particular, $U + V$ and V are nonconstant and affinely independent. Given linearity and nonconstancy, it can be shown that there is μ, η such that $U(\mu) + V(\mu) > U(\eta) + V(\eta)$ and $V(\mu) < V(\eta)$. The result follows from Lemma A.1(c). ∎

3 Proof of Theorem 4.2

Arguing as in the proof of GP [2, Thm 2] yields ordinal equivalence if and only if there exist $a > 0, b_u, b_v \in \mathbb{R}$ such that $U' = aU + b_u$ and $V' = aV + b_v$ if and only if $\delta = \delta'$ and there exist $a > 0, b_u, b_v \in \mathbb{R}$ such that $u' = au + b_u$ and $V' = aV + b_v$. Note that $W' = aW + b_u$ and $\widehat{V}' = \alpha \widehat{V} + \beta b_u + \gamma b_v$, and

for any c and x,

$$V'(c,x) = aV(c,x) + b_v$$
$$\iff v'(c) + \widehat{V}'(x) = a[v(c) + \widehat{V}(x)] + b_v$$
$$\iff v'(c) + \alpha\widehat{V}(x) + \beta b_u + \gamma b_v = av(c) + a\widehat{V}(x) + b_v$$
$$\iff v'(c) + \beta b_u + \gamma b_v = av(c) + b_v$$
$$\iff v'(c) = av(c) - \beta b_u + (1-\gamma)b_v.$$

Moreover, for any x,

$$\widehat{V}'(c,x) = \alpha\widehat{V}(c,x) + \beta b_u + \gamma b_v$$
$$\iff \beta'U'(c,x) + \gamma'V'(c,x) = \alpha\beta U(c,x) + \alpha\gamma V(c,x) + \beta b_u + \gamma b_v$$
$$\iff \beta'U'(c,x) + \gamma'V'(c,x) = \beta[\alpha U(c,x) + b_u] + \gamma[\alpha V(c,x) + b_v]$$
$$\iff \beta'U'(c,x) + \gamma'V'(c,x) = \beta U'(c,x) + \gamma V'(c,x)$$
$$\iff (\beta' - \beta)U'(c,x) = (\gamma - \gamma')V'(c,x).$$

Since U', V' are affinely independent (by nondegeneracy), it follows that $\gamma - \gamma' = \beta' - \beta = 0$. This completes the proof.

4 Proof of Theorems 4.4-4.6

Lemma 3 *If Weak Menu-Temptation Stationarity holds, then $\beta \geq 0$.*

Proof. We show that the following pareto property holds

$$W(x) > W(y) \text{ and } \widehat{V}(x) \geq \widehat{V}(y) \implies \beta\delta W(x) + \gamma\widehat{V}(x) \geq \beta\delta W(y) + \gamma\widehat{V}(y).$$

The result then follows. Show first that if $x \succ y$ then

$$\{x^{+1}\} \sim \{x^{+1}, y^{+1}\} \implies \{x^{+2}\} \sim \{x^{+2}, y^{+2}\}.$$

Consider the contrapositive. Take any x, y such that $x \succ y$ and $\{x^{+2}\} \not\sim \{x^{+2}, y^{+2}\}$. By Lemma 2(ii)-(iii) above, it must be that $\{x^{+2}\} \succ \{x^{+2}, y^{+2}\}$. By Lemma C.3(c), $x^{+t} > y^{+t}$ and $\{x^{+2}\}^{+t} > \{x^{+2}, y^{+2}\}^{+t}$ for all large t. Set-Betweenness, Reversal and the Weak Menu-Temptation Stationarity imply $\{x^{+1}\}^{+t} > \{x^{+1}, y^{+1}\}^{+t}$ for all large t. By Lemma D.4(i), it follows that $\{x^{+1}\} \succ \{x^{+1}, y^{+1}\}$, as desired. The representation then yields the above pareto property. ■

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Lemma 4 *If Strong Menu-Temptation Stationarity holds, then either QSC or FT.*

Proof. If $\widehat{V}$ is constant then we have the QSC model. If $\widehat{V}$ is an affine transformation of W, then $\widehat{V}(x) = \delta\beta W(x)$, and Weak Menu-Temptation Stationarity implies $\beta \geq 0$, thereby yielding the QSC model again. The remainder of the proof establishes that if $\widehat{V}$ is not constant and not an affine transformation of W, then the FT model holds, that is, $\beta = 0$ and $\gamma > 0$.

The argument in the previous lemma can be used to show that if $x \succ y$ then $[\{x^{+1}\} \sim \{x^{+1}, y^{+1}\} \iff \{x^{+2}\} \sim \{x^{+2}, y^{+2}\}]$, which in turn implies that if $W(x) > W(y)$ then

$$\widehat{V}(x) \geq \widehat{V}(y) \iff \beta\delta W(x) + \gamma\widehat{V}(x) \geq \beta\delta W(y) + \gamma\widehat{V}(y).$$

Since $\widehat{V}$ is not an affine transformation of W, we can find x, y s.t. $W(x) > W(y)$ and $\widehat{V}(x) = \widehat{V}(y)$. It then follows from the displayed equivalence that $\beta\delta W(x) = \beta\delta W(y)$, which implies $\beta = 0$. Since $\widehat{V}$ is nonconstant, by linearity we can find x', y' s.t. $W(x') > W(y')$ and $\widehat{V}(x') > \widehat{V}(y')$. The displayed equivalence implies $\gamma > 0$. ∎

Lemma 5 *Given Axioms 1-9, the following are equivalent:*
(i) Menus Do Not Tempt holds;
(ii) $\tau \leq 1$;
(iii) there exists $\beta \geq 0$ such that wlog $\widehat{V}(x) = \beta W(x)$ for all x.

Proof. The necessity of (i) or (ii) for (iii) is straightforward.

Proof of $(i) \implies (iii)$: By Menus Do Not Tempt and Set-Betweenness, $x \succ y \implies x^{+t} > y^{+t}$ for all large $t \implies \{(c, x)\}^{+t} \approx \{(c, x), (c, y)\}^{+t}$ for all large $t \implies \{(c, x)\} \sim \{(c, x), (c, y)\}$. That is,

$$x \succ y \implies \{(c, x)\} \sim \{(c, x), (c, y)\}.$$

In particular, $W(x) > W(y) \implies \widehat{V}(x) \geq \widehat{V}(y)$ for all x, y. If $\widehat{V}$ is constant, then we get the ct representation with $\beta = 0$ (after applying the uniqueness result [1, Thm 4] of GP's model, if necessary). If not, then nonconstancy, continuity and linearity of W and $\widehat{V}$ implies that $W(x) > W(y) \implies$

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$\widehat{V}(x) > \widehat{V}(y)$ for all x, y (if $W(x) > W(y)$ and $\widehat{V}(x) = \widehat{V}(y)$, then by non-constancy there is $\widehat{V}(w) > \widehat{V}(z)$ and by linearity and continuity, there is α s.t. $W(x\alpha z) > W(y\alpha w)$ and $\widehat{V}(x\alpha z) < \widehat{V}(y\alpha w)$, violating Menus Do Not Tempt). It follows further, by linearity and nonconstancy, that both are in fact ordinally equivalent, and thus cardinally equivalent. In this case we get the QSC representation with $\beta > 0$ (after applying [1, Thm 4] if necessary).

Proof of $(ii) \implies (iii)$: If $\tau \leq 1$, then $x^{+1} \gtrsim y^{+1} \iff x^{+t} \gtrsim y^{+t}$ for all t. Clearly then, $\succsim = \succsim^1$. By Lemma D.6, $\succsim^1$ is represented by $\delta W + \widehat{V}$. Thus W and $\delta W + \widehat{V}$ are ordinally equivalent. Since W and $\widehat{V}$ are linear, it follows that $\widehat{V} = \beta W$ for $\beta \geq 0$, after applying [1, Thm 4] if necessary. The result follows. ∎

References

[1] Gul, F. and W. Pesendorfer (2001): 'Temptation and Self-Control', *Econometrica* 69, pp 1403-1435.

[2] __ (2003): 'Self-Control and the Theory of Consumption', *Econometrica* 72, pp 119-158.

[3] Noor, J. (2006): 'Commitment and Self-Control', *Journal of Economic Theory* 135(1), pp 1-34.