Econometrica Supplementary Material

SUPPLEMENT TO "CONTINUOUS IMPLEMENTATION"

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BELOW WE PROVIDE THE PROOF of Theorem 4, which is omitted in the main text.

PROOF OF THE "IF PART" OF THEOREM 4: Assume that $f:\bar{T}\to A$ is rationalizable implementable by a finite mechanism $\mathcal{M}=(M,g)$, that is, that, for all $\bar{t}\in\bar{T}$, $m\in R(\bar{t}|\mathcal{M},\bar{\mathcal{T}})\Rightarrow g(m)=f(\bar{t})$.

We first recall the following well known lemma.

LEMMA 1—Dekel, Fudenberg, and Morris (2006): *Fix any model $\mathcal{T}=(T,\kappa)$ such that $\bar{\mathcal{T}}\subset\mathcal{T}$ and any finite mechanism $\mathcal{M}$.* (i) *For any $\bar{t}\in\bar{T}$ and any sequence $\{t[n]\}_{n=0}^{\infty}$ in T, if $t[n]\to_P\bar{t}$, then, for n large enough, we have $R(t[n]|\mathcal{M},\mathcal{T})\subset R(\bar{t}|\mathcal{M},\mathcal{T})$.* (ii) *For any type $t\in T$, $R(t|\mathcal{M},\mathcal{T})$ is nonempty.*

Now pick any model $\mathcal{T}=(T,\kappa)$ such that $\bar{\mathcal{T}}\subset\mathcal{T}$. We show that there exists an equilibrium that continuously implements f on $\bar{\mathcal{T}}$. For each player i and each type $\bar{t}_i\in\bar{T}_i$, fix some $m_i(\bar{t}_i)\in R_i(\bar{t}_i|\mathcal{M},\bar{\mathcal{T}})$ and restrict the space of strategies of player i by assuming that $\sigma_i(\bar{t}_i)=m_i(\bar{t}_i)$ for each type $\bar{t}_i\in\bar{T}_i$. Because M is finite and T is countable, standard arguments[1] show that there exists a Bayes Nash equilibrium in $U(\mathcal{M},\mathcal{T})$. Let us first establish that σ is a Bayes Nash equilibrium in $U(\mathcal{M},\mathcal{T})$. It is clear by construction that, for each $i\in\mathcal{I}$ and $t_i\notin\bar{T}_i$,

$$m_i\in\operatorname{Supp}(\sigma_i(t_i))\quad\Rightarrow\quad m_i\in BR_i(\pi_i(\cdot|t_i,\sigma_{-i})|\mathcal{M}).$$

Now fix a player $i\in\mathcal{I}$ and a type $\bar{t}_i\in\bar{T}_i$. Since $\bar{\mathcal{T}}\subset\mathcal{T}$ is a model (and hence, $\kappa(\bar{t}_i)$ takes its support in $\Theta\times\bar{T}_{-i}$), it is easily checked that, by construction of σ, $\pi_i(m_{-i}|\bar{t}_i,\sigma_{-i})>0\Rightarrow m_{-i}\in R_{-i}(\bar{t}_{-i}|\mathcal{M},\bar{\mathcal{T}})$ for some $\bar{t}_{-i}\in\bar{T}_{-i}$. Hence, by a well known argument, $BR_i(\pi_i(\cdot|\bar{t}_i,\sigma_{-i})|\mathcal{M})\subset R_i(\bar{t}_i|\mathcal{M},\bar{\mathcal{T}})$. Since $g(R(\bar{t}|\mathcal{M},\bar{\mathcal{T}}))=\{f(\bar{t})\}$, we have, for all $\tilde{m}_i\in R_i(\bar{t}_i|\mathcal{M},\bar{\mathcal{T}})$,

$$\begin{aligned}&\sum_{(\theta,m_{-i})\in\Theta\times M_{-i}}\pi_i(\theta,m_{-i}|\bar{t}_i,\sigma_{-i})\big[u_i(g(\tilde{m}_i,m_{-i}),\theta)\big]\\&\quad=\sum_{\theta,\bar{t}_{-i}}\bar{\kappa}(\bar{t}_i)[\theta,\bar{t}_{-i}]u_i(f(\bar{t}_i,\bar{t}_{-i}),\theta),\end{aligned}$$

[1]The existence of a Bayes Nash equilibrium can be proved using Kakutani–Fan–Glicksberg's fixed point theorem. The space of strategy profiles is compact in the product topology. Using the fact that $u_i:A\times\Theta\to\mathbb{R}$ is bounded, all the desired properties of the best-response correspondence (in particular upper hemicontinuity) can be established.

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and so $BR_i(\pi_i(\cdot|\bar{t}_i, \sigma_{-i})|\mathcal{M}) = R_i(\bar{t}_i|\mathcal{M}, \bar{\mathcal{T}})$. Hence we must have $m_i(\bar{t}_i) = \sigma_i(\bar{t}_i) \in BR_i(\pi_i(\cdot|\bar{t}_i, \sigma_{-i})|\mathcal{M})$. Thus, σ is a Bayes Nash equilibrium in $U(\mathcal{M}, \mathcal{T})$ and $\sigma_{|\bar{T}}$ is a pure Nash equilibrium in $U(\mathcal{M}, \bar{\mathcal{T}})$. Now, pick any sequence $\{t[n]\}_{n=0}^{\infty}$ in T, such that $t[n] \to_P \bar{t}$. It is clear that, for each n: $\text{Supp}(\sigma(t[n])) \subset R(t[n] \mid \mathcal{M}, \mathcal{T})$. In addition, for n large enough, we know by Lemma 1 that $R(t[n] \mid \mathcal{M}, \mathcal{T}) \subset R(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}})$. Then, for n large enough, $\text{Supp}(\sigma(t[n])) \subset R(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}})$ and so, $(g \circ \sigma)(t[n]) = f(\bar{t})$ as claimed. *Q.E.D.*

PROOF OF THE "ONLY IF PART" OF THEOREM 4: We show that a social choice function $f : \bar{T} \to A$ is continuously implementable by a countable[2] mechanism $\mathcal{M}$ only if it is rationalizable implementable by some mechanism $\mathcal{M}' \subset \mathcal{M}$ (i.e., $M_i' \subset M_i$ for each i and $g' = g_{|M'}$).

Since f is continuously implementable, there exists a mechanism $\mathcal{M} = (M, g)$ such that, for any model $\mathcal{T} = (T, \kappa)$ satisfying $\bar{\mathcal{T}} \subset \mathcal{T}$, there is a Bayes Nash equilibrium σ in the induced game $U(\mathcal{M}, \mathcal{T})$ where, for each $\bar{t} \in \bar{T}$, (i) $\sigma(\bar{t})$ is pure, and (ii) for any sequence $t[n] \to_P \bar{t}$ where, for each n: $t[n] \in T$, we have $(g \circ \sigma)(t[n]) \to f(\bar{t})$. We let C be the set of pure Bayes Nash equilibria of $U(\mathcal{M}, \bar{\mathcal{T}})$. Note that because $\bar{T}$ is finite and M is countable, C is countable. For each $\bar{\sigma} \in C$, we build the set of message profiles $M(\bar{\sigma})$ in the following way.

For each player i and each positive integer ℓ, we define inductively $M_i^{\ell}(\bar{\sigma})$. First, we set $M_i^0(\bar{\sigma}) = \bar{\sigma}_i(\bar{T}_i)$. Then, for each $\ell \geq 1$,

$$M_i^{\ell+1}(\bar{\sigma}) = BR_i\big(\Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}^{\ell}(\bar{\sigma})) \mid \mathcal{M}\big).$$

Recall that in the model $\bar{\mathcal{T}} = (\bar{T}, \bar{\kappa})$, $\text{marg}_{\tilde{\Theta}}\, \bar{\kappa}(\bar{t}_i)[\tilde{\theta}^0] = 1$, for each $i \in \mathcal{I}$ and $\bar{t}_i \in \bar{T}_i$. Since $\bar{\sigma}$ is an equilibrium in $U(\mathcal{M}, \bar{\mathcal{T}})$, $M_i^0(\bar{\sigma}) = \bar{\sigma}_i(\bar{T}_i) \subset BR_i(\Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}^0(\bar{\sigma})) \mid \mathcal{M}) = M_i^1(\bar{\sigma})$. Consequently, it is clear that, for each ℓ, $M_i^{\ell}(\bar{\sigma}) \subset M_i^{\ell+1}(\bar{\sigma})$. Finally, set $M_i(\bar{\sigma}) = \lim_{\ell \to +\infty} M_i^{\ell}(\bar{\sigma}) = \bigcup_{\ell \in \mathbb{N}} M_i^{\ell}(\bar{\sigma})$. In the sequel, for each $\bar{\sigma} \in C$, we will note by $\mathcal{M}(\bar{\sigma})$ the mechanism $(M(\bar{\sigma}), g_{|M(\bar{\sigma})})$.

A first interesting property of the family of sets $\{M(\bar{\sigma})\}_{\bar{\sigma} \in C}$ is that there is a model $\mathcal{T}$, satisfying $\bar{\mathcal{T}} \subset \mathcal{T}$, for which any equilibrium σ in $U(\mathcal{M}, \mathcal{T})$ has full range in $M(\sigma_{|\bar{T}})$, that is, each message profile in $M(\sigma_{|\bar{T}})$ is played under σ at some profile of types in the model $\mathcal{T}$. More precisely, Proposition 1 is the first step of the proof of the only if part of Theorem 4.

PROPOSITION 1: *There exists a model $\mathcal{T} = (T, \kappa)$ such that, for any $\bar{\sigma} \in C$ and $m \in M(\bar{\sigma})$, there exists $t[\bar{\sigma}, m] \in T$ such that $\sigma(t[\bar{\sigma}, m]) = m$ for any equilibrium σ in $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{|\bar{T}} = \bar{\sigma}$.*

[2]As already mentioned, the only if part of the theorem holds beyond finite mechanisms.

PROOF: We build the model $\mathcal{T} = (T, \kappa)$ as follows. For each equilibrium $\bar{\sigma} \in C$, player i, and integer ℓ, we define inductively $t_i[\bar{\sigma}, \ell, m_i]$ for each $m_i \in M_i^\ell(\bar{\sigma})$ and set

$$T_i = \bigcup_{\bar{\sigma} \in C} \bigcup_{\ell=1}^{\infty} \bigcup_{m_i \in M_i^\ell(\bar{\sigma})} t_i[\bar{\sigma}, \ell, m_i] \cup \bar{T}_i.$$

Note that T_i is countable. In the sequel, we fix an arbitrary $\bar{\sigma} \in C$. This equilibrium $\bar{\sigma}$ is sometimes omitted in our notations.

For each $\ell \geq 1$ and $m_i \in M_i^\ell(\bar{\sigma})$, we know that there exists $\pi_i^{\ell, m_i} \in \Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}^{\ell-1}(\bar{\sigma}))$ such that $m_i \in BR_i(\pi_i^{\ell, m_i} \mid \mathcal{M})$. Thus we can build $\hat{\pi}_i^{\ell, m_i} \in \Delta(\Theta \times \tilde{\Theta} \times M_{-i}^{\ell-1}(\bar{\sigma}))$ such that

$$\operatorname{marg}_{\Theta \times M_{-i}^{\ell-1}(\bar{\sigma})} \hat{\pi}_i^{\ell, m_i} = \operatorname{marg}_{\Theta \times M_{-i}^{\ell-1}(\bar{\sigma})} \pi_i^{\ell, m_i},$$

while $\operatorname{marg}_{\tilde{\Theta}} \hat{\pi}_i^{\ell, m_i} = \delta_{\tilde{\theta}^{m_i}}$. Note that $BR_i(\hat{\pi}_i^{\ell, m_i} \mid \mathcal{M}) = \{m_i\}$.

In the sequel, for each player i and message $m_i \in M_i^0(\bar{\sigma})$, we pick one type denoted $t_i[\bar{\sigma}, 0, m_i]$ in $\bar{T}_i$ satisfying $\bar{\sigma}_i(t_i[\bar{\sigma}, 0, m_i]) = m_i$. This is well defined because, by construction, $M_i^0(\bar{\sigma}) = \bar{\sigma}_i(\bar{T}_i)$. Now, for each $\ell \geq 1$ and $m_i \in M_i^\ell(\bar{\sigma})$, we define inductively $t_i[\bar{\sigma}, \ell, m_i]$ by[3]

$$\begin{aligned} &\kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}] \\ &= \begin{cases} 0, & \text{if } t_{-i} \neq t_{-i}[\bar{\sigma}, \ell-1, m_{-i}] \\ & \text{for each } m_{-i} \in M_{-i}^{\ell-1}(\bar{\sigma}), \\ \hat{\pi}_i^{\ell, m_i}(\theta, \tilde{\theta}, m_{-i}), & \text{if } t_{-i} = t_{-i}[\bar{\sigma}, \ell-1, m_{-i}] \\ & \text{for some } m_{-i} \in M_{-i}^{\ell-1}(\bar{\sigma}). \end{cases} \end{aligned}$$

This probability measure is well defined since $\hat{\pi}_i^{\ell, m_i}(\Theta \times \tilde{\Theta} \times M_{-i}^{\ell-1}(\bar{\sigma})) = 1$.

To complete the proof, we show that, for any equilibrium σ of $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{|\bar{T}} = \bar{\sigma}$, we have

$$\text{(S1)} \qquad \sigma_i(t_i[\bar{\sigma}, \ell, m_i]) = m_i$$

for each player i, integer ℓ, and message $m_i \in M_i^\ell(\bar{\sigma})$. The proof proceeds by induction on ℓ.

First note that, by construction of $t_i[\bar{\sigma}, 0, m_i]$, we must have, for any equilibrium σ of $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{|\bar{T}} = \bar{\sigma}$,

$$\sigma_i(t_i[\bar{\sigma}, 0, m_i]) = m_i,$$

[3]Here again, we abuse notation and write $t_{-i}[\bar{\sigma}, 0, m_{-i}]$ for $(t_j[\bar{\sigma}, 0, m_j])_{j \neq i}$. Similarly, $t[\bar{\sigma}, 0, m]$ stands for $(t_i[\bar{\sigma}, 0, m_i])_{i \in \mathcal{I}}$. Similar abuses will be used throughout this proof.

for each player i and message $m_i \in M_i^0(\bar{\sigma})$. Now, assume that Equation (S1) is satisfied at rank $\ell - 1$ and let us prove that it is also satisfied at rank ℓ. Fix any $m_i \in M_i^\ell(\bar{\sigma})$ and any equilibrium σ of $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{|\bar{T}} = \bar{\sigma}$. Note that $\text{Supp}(\sigma_i(t_i[\bar{\sigma}, \ell, m_i])) \subset BR_i(\pi_i \mid \mathcal{M})$, where $\pi_i \in \Delta(\Theta \times \tilde{\Theta} \times M_{-i})$ is such that

$$\pi_i(\theta, \tilde{\theta}, m_{-i}) = \sum_{t_{-i}} \kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}]\sigma_{-i}(m_{-i} \mid t_{-i}).$$

In addition, by the inductive hypothesis and the fact that σ is an equilibrium of $U(\mathcal{M}, \mathcal{T})$ satisfying $\sigma_{|\bar{T}} = \bar{\sigma}$, we have $\sigma_{-i}(m_{-i} \mid t_{-i}[\bar{\sigma}, \ell - 1, m_{-i}]) = 1$ for any $m_{-i} \in M_{-i}^{\ell-1}(\bar{\sigma})$. Hence, by construction of $\kappa(t_i[\bar{\sigma}, \ell, m_i])$, we have

$$\begin{aligned}\pi_i(\theta, \tilde{\theta}, m_{-i}) &= \sum_{t_{-i}} \kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}]\sigma_{-i}(m_{-i} \mid t_{-i}) \\ &= \kappa(t_i[\bar{\sigma}, \ell, m_i])[\theta, \tilde{\theta}, t_{-i}[\bar{\sigma}, \ell - 1, m_{-i}]] \\ &= \hat{\pi}_i^{\ell, m_i}(\theta, \tilde{\theta}, m_{-i}).\end{aligned}$$

We get that $\text{Supp}(\sigma_i(t_i[\bar{\sigma}, \ell, m_i])) \subset BR_i(\pi_i \mid \mathcal{M}) = BR_i(\hat{\pi}_i^{\ell, m_i} \mid \mathcal{M}) = \{m_i\}$ as claimed. *Q.E.D.*

We now give a first insight on the second step of the proof. First notice that, by construction, each $M(\bar{\sigma})$ satisfies the following closure property: taking any belief $\pi_i \in \Delta(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}(\bar{\sigma}))$ such that $BR_i(\pi_i \mid \mathcal{M}) \neq \emptyset$, we must have $BR_i(\pi_i \mid \mathcal{M}) \subset M_i(\bar{\sigma})$ and hence, $BR_i(\pi_i \mid \mathcal{M}) = BR_i(\pi_i \mid \mathcal{M}(\bar{\sigma}))$.

Now pick a type $\bar{t}_i \in \bar{T}_i$ and a message $m_i \in R_i^1(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$; it is possible to add a type $t_i^{m_i}$ to the model $\mathcal{T}$ defined in Proposition 1 satisfying the following two properties.[4] First, $h_i^1(t_i^{m_i})$ is arbitrarily close to $h_i^1(\bar{t}_i)$; second, for any equilibrium σ with $\sigma_{|\bar{T}} = \bar{\sigma}$, $\sigma_i(t_i^{m_i}) = m_i$. Indeed, by definition of $R_i^1(\bar{t}_i | \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$, there exists a belief $\pi_i^{m_i} \in \Delta(\Theta^* \times T_{-i} \times M_{-i}(\bar{\sigma}))$, where $\text{marg}_{\Theta^*} \pi_i^{m_i} = \text{marg}_{\Theta^*} \bar{\kappa}(\bar{t}_i)$ and such that $m_i \in BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} \mid \mathcal{M}(\bar{\sigma}))$. Using our assumption on cost of messages, we can slightly perturb $\pi_i^{m_i}$ so that m_i becomes a unique best reply. So let us assume for simplicity that $\{m_i\} = BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} \mid \mathcal{M}(\bar{\sigma}))$. We can define the type $t_i^{m_i}$ assigning probability $\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i}(\theta^*, m_{-i})$ to $(\theta^*, t_{-i}[\bar{\sigma}, m_{-i}])$, where $t_{-i}[\bar{\sigma}, m_{-i}]$ is defined as in Proposition 1 (i.e., $t_{-i}[\bar{\sigma}, m_{-i}]$ plays m_{-i} under any equilibrium σ in $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{|\bar{T}} = \bar{\sigma}$). Now pick any equilibrium σ in $U(\mathcal{M}, \mathcal{T} \cup \{t_i^{m_i}\})$

[4]In this section, for any mechanism $\mathcal{M}$, we use the standard notation where $R_i^\ell(\bar{t}_i \mid \mathcal{M}, \bar{\mathcal{T}})$ stands for the ℓth round of elimination at type $\bar{t}_i$ of messages that are not best responses (see, for instance, Dekel, Fudenberg, and Morris (2007)). Recall that, for any ℓ and $\bar{t}_i$, we have $R_i(\bar{t}_i \mid \mathcal{M}, \bar{\mathcal{T}}) \subset R_i^\ell(\bar{t}_i \mid \mathcal{M}, \bar{\mathcal{T}})$ (for additional details on the relationship between $R_i(\bar{t}_i \mid \mathcal{M}, \bar{\mathcal{T}})$ and $R_i^\ell(\bar{t}_i \mid \mathcal{M}, \bar{\mathcal{T}})$ when the set of messages is countably infinite, see Lipman (1994)).

such that $\sigma_{|\bar{T}} = \bar{\sigma}$. By construction, $\text{Supp}(\sigma_i(t_i^{m_i})) \subset BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_i^{m_i} \mid \mathcal{M})$ and so $BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} \mid \mathcal{M}) \neq \emptyset$. By the closure property described above, $BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_i^{m_i} \mid \mathcal{M}) = BR_i(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_i^{m_i} \mid \mathcal{M}(\bar{\sigma}))$ and so we get that type $t_i^{m_i}$ plays m_i under the equilibrium σ and satisfies the desired property. Using a similar reasoning, we show inductively the following "contagion" result.

PROPOSITION 2: *There exists a model $\hat{\mathcal{T}} = (\hat{T}, \hat{\kappa})$ such that, for each equilibrium $\bar{\sigma} \in C$ and each player i, the following statement holds: For all $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$, there exists a sequence of types $\{\hat{t}_i[n]\}_{n=0}^{\infty}$ in $\hat{T}_i$ such that* (i) *$\hat{t}_i[n] \to_P \bar{t}_i$, and* (ii) *$\sigma_i(\hat{t}_i[n]) = m_i$ for each integer n and equilibrium σ of $U(\mathcal{M}, \hat{\mathcal{T}})$ satisfying $\sigma_{|\bar{T}} = \bar{\sigma}$.*

PROOF: We again define the set $\mathcal{E}$ by

$$\mathcal{E} := \bigcup_{q \in \mathbb{N}^*} \left\{ \frac{1}{q} \right\} \cup \{0\}.$$

We build the model $\hat{\mathcal{T}} = (\hat{T}, \hat{\kappa})$ as follows. For each $\varepsilon \in \mathcal{E}$, $\ell \in \mathbb{N}^*$, $\bar{\sigma} \in C$, $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i^{\ell}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$, we build inductively $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$ and set

$$\hat{T}_i = \bigcup_{\varepsilon \in \mathcal{E}} \bigcup_{\ell=1}^{\infty} \bigcup_{\bar{\sigma} \in C} \bigcup_{\bar{t}_i \in \bar{T}_i} \bigcup_{m_i \in R_i^{\ell}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})} \hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i] \cup T_i,$$

where T_i is as defined in Proposition 1. Note that $\hat{T}_i$ is countable. In the sequel, we fix an arbitrary $\bar{\sigma} \in C$. This equilibrium $\bar{\sigma}$ is sometimes omitted in our notations.

We know that, for each integer ℓ, player i of type $\bar{t}_i \in \bar{T}_i$, and message $m_i \in R_i^{\ell}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$, there exists $\pi_{\bar{t}_i}^{\ell, m_i} \in \Delta(\Theta \times \bar{T}_{-i} \times M_{-i}(\bar{\sigma}))$ such that

$$\text{marg}_{\Theta \times \bar{T}_{-i}} \pi_{\bar{t}_i}^{\ell, m_i} = \bar{\kappa}(\bar{t}_i),$$
$$\text{marg}_{\bar{T}_{-i} \times M_{-i}(\bar{\sigma})} \pi_{\bar{t}_i}^{\ell, m_i}(\bar{t}_{-i}, m_{-i}) > 0 \quad \Rightarrow \quad m_{-i} \in R_{-i}^{\ell-1}(\bar{t}_{-i} \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}}),$$

and

$$m_i \in BR_i\big(\text{marg}_{\Theta^* \times M_{-i}(\bar{\sigma})} \pi_{\bar{t}_i}^{\ell, m_i} \mid \mathcal{M}(\bar{\sigma})\big).$$

For ease of exposition, we sometimes consider $\pi_{\bar{t}_i}^{\ell, m_i}$ as a measure over $\Theta \times \bar{T}_{-i} \times M_{-i}(\bar{\sigma})$ and sometimes as a measure over $\Theta^* \times \bar{T}_{-i} \times M_{-i}(\bar{\sigma})$ assigning probability 1 to $\{\tilde{\theta}^0\}$. Similar abuses will be used throughout the proof.

First, we let $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$ be such that $\hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i])$ satisfies the two conditions

$$\text{marg}_{\tilde{\Theta}}\, \hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = \varepsilon \delta_{\tilde{\theta}^{m_i}} + (1-\varepsilon)\delta_{\tilde{\theta}^0} \tag{S2}$$

and

$$\text{marg}_{\Theta \times \hat{T}_{-i}}\, \hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = \pi_{\bar{t}_i}^{1,m_i} \circ (\tau_{-i}^{\varepsilon,1})^{-1}, \tag{S3}$$

where $(\tau_{-i}^{\varepsilon,1})^{-1}$ stands for the preimage of the function $\tau_{-i}^{\varepsilon,1} : \Theta \times \bar{T}_{-i} \times M_{-i} \to \Theta \times \hat{T}_{-i}$, defined by $\tau_{-i}^{\varepsilon,1}(\theta, \bar{t}_{-i}, m_{-i}) = (\theta, t_{-i}[\bar{\sigma}, m_{-i}])$, and $t_{-i}[\bar{\sigma}, m_{-i}] \in T_{-i}$ is the type profile defined in Proposition 1. Recall that $\sigma_{-i}(t_{-i}[\bar{\sigma}, m_{-i}]) = m_{-i}$ for any equilibrium σ in $U(\mathcal{M}, \mathcal{T})$ such that $\sigma_{|\bar{T}} = \bar{\sigma}$. Now, for each $\ell \geq 2$, define $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$ inductively by

$$\text{marg}_{\tilde{\Theta}}\, \hat{\kappa}(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = \varepsilon \delta_{\tilde{\theta}^{m_i}} + (1-\varepsilon)\delta_{\tilde{\theta}^0}$$

and

$$\text{marg}_{\Theta \times \hat{T}_{-i}}\, \hat{\kappa}(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = \pi_{\bar{t}_i}^{\ell,m_i} \circ (\tau_{-i}^{\varepsilon,\ell})^{-1},$$

where $(\tau_{-i}^{\varepsilon,\ell})^{-1}$ stands for the preimage of the function $\tau_{-i}^{\varepsilon,\ell} : \Theta \times \bar{T}_{-i} \times M_{-i} \to \Theta \times \hat{T}_{-i}$, defined by $\tau_{-i}^{\varepsilon,\ell}(\theta, \bar{t}_{-i}, m_{-i}) = (\theta, \hat{t}_{-i}[\varepsilon, \ell-1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}])$.

CLAIM 1: *For each $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}}) : \hat{t}_i[\hat{\varepsilon}(\ell), \ell, \bar{\sigma}, \bar{t}_i, m_i] \to_P \bar{t}_i$ as $\ell \to \infty$ for some mapping $\hat{\varepsilon}$ taking values in $\mathcal{E} \setminus \{0\}$.*

PROOF: In the sequel, we will denote by $\bar{h}$ the (continuous) mapping that projects $\bar{T}$ into T^* and, in a similar way, by $\hat{h}$ the (continuous) mapping from $\hat{T}$ to T^*.

For any $\bar{t}_i \in \bar{T}_i$, since[5] for all $\ell \geq 1$ and all $m_i \in R_i^{\ell}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}}) : \hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i] \to \hat{t}_i[0, \ell, \bar{\sigma}, \bar{t}_i, m_i]$ as $\varepsilon \to 0$, by Lemma 2 in the main text, for all $\ell \geq 1$, for all $\ell' \geq 1$, and all $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}}) : \hat{h}_i^{\ell}(\hat{t}_i[\varepsilon, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \to \hat{h}_i^{\ell}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i])$ as $\varepsilon \to 0$.

[5]A type in $\hat{T}_i$ is either in T_i—which is endowed with the discrete topology, say τ_{T_i}—or in $\hat{T}_i \setminus T_i$. Any point in $\hat{T}_i \setminus T_i$ is identified with an element of the set $\mathcal{E} \times \mathbb{N} \times C \times \bar{T}_i \times M_i$, where $\mathbb{N}$, C, $\bar{T}_i$, M_i are all endowed with the discrete topology, while $\mathcal{E}$ is endowed with the usual topology on $\mathbb{R}$ induced on $\mathcal{E}$. Finally, $\mathcal{E} \times \mathbb{N} \times C \times \bar{T}_i \times M_i$ is endowed with the product topology; call this topology $\tau_{\hat{T}_i \setminus T_i}$. The topology over $\hat{T}_i$ is the coarsest topology that contains $\tau_{T_i} \cup \tau_{\hat{T}_i \setminus T_i}$. It can easily be checked that under such a topology, $\hat{\mathcal{T}}$ satisfies the conditions of Lemma 2 in the main text.

Let us now show that, for all $\ell \geq 1$ and $\ell' \geq \ell$: $\hat{h}_i^{\ell}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \bar{h}_i^{\ell}(\bar{t}_i)$ for all $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$. First notice that the first-order beliefs are equal, that is, for all $\ell' \geq 1$, $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$,

$$\begin{aligned}\hat{h}_i^1(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) &= \operatorname{marg}_{\Theta} \hat{\kappa}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \\ &= \operatorname{marg}_{\Theta} \pi_{\bar{t}_i}^{\ell', m_i} \circ \left(\tau_{-i}^{0,\ell'}\right)^{-1} \\ &= \operatorname{marg}_{\Theta} \pi_{\bar{t}_i}^{\ell', m_i} = \operatorname{marg}_{\Theta} \bar{\kappa}(\bar{t}_i) = \bar{h}_i^1(\bar{t}_i),\end{aligned}$$

where the third and the fourth equalities are by definition of $\tau_{-i}^{0,\ell'}$ and $\pi_{\bar{t}_i}^{\ell', m_i}$, respectively. Now fix some $\ell \geq 2$ and let L be the set of all belief profiles of players other than i at order $\ell - 1$. Toward an induction, assume that, for all $\ell' \geq \ell - 1$: $\hat{h}_j^{\ell-1}(\hat{t}_j[0, \ell', \bar{\sigma}, \bar{t}_j, m_j]) = \bar{h}_j^{\ell-1}(\bar{t}_j)$ for each j, $\bar{t}_j \in \bar{T}_j$ and $m_j \in R_j^{\ell'}(\bar{t}_j \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$. Then for all $\ell' \geq \ell$: $\operatorname{proj}_{\Theta \times L} \circ (\operatorname{id}_{\Theta} \times \hat{h}_{-i}) \circ \tau_{-i}^{0,\ell'} = \overline{\operatorname{proj}}_{\Theta \times L} \circ (\operatorname{id}_{\Theta} \times \bar{h}_{-i} \times \operatorname{id}_{M_{-i}(\bar{\sigma})})$, where $\operatorname{id}_{\Theta}$ (resp. $\operatorname{id}_{M_{-i}(\bar{\sigma})}$) is the identity mapping from Θ to Θ (resp. from $M_{-i}(\bar{\sigma})$ to $M_{-i}(\bar{\sigma})$), while $\operatorname{proj}_{\Theta \times L}$ (resp. $\overline{\operatorname{proj}}_{\Theta \times L}$) is the projection mapping from $\Theta \times T^*$ to $\Theta \times L$ (resp. from $\Theta \times T^* \times M_{-i}(\bar{\sigma})$ to $\Theta \times L$); hence, for all $\ell' \geq \ell$, $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$,

$$\begin{aligned}&\operatorname{marg}_{\Theta \times L} \hat{\kappa}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \circ (\operatorname{id}_{\Theta} \times \hat{h}_{-i})^{-1} \\ &\quad = \operatorname{marg}_{\Theta \times L} \pi_{\bar{t}_i}^{\ell', m_i} \circ \left(\tau_{-i}^{0,\ell'}\right)^{-1} \circ (\operatorname{id}_{\Theta} \times \hat{h}_{-i})^{-1} \\ &\quad = \pi_{\bar{t}_i}^{\ell', m_i} \circ \left(\tau_{-i}^{0,\ell'}\right)^{-1} \circ (\operatorname{id}_{\Theta} \times \hat{h}_{-i})^{-1} \circ (\operatorname{proj}_{\Theta \times L})^{-1} \\ &\quad = \pi_{\bar{t}_i}^{\ell', m_i} \circ \left(\operatorname{id}_{\Theta} \times \bar{h}_{-i} \times \operatorname{id}_{M_{-i}(\bar{\sigma})}\right)^{-1} \circ (\overline{\operatorname{proj}}_{\Theta \times L})^{-1} \\ &\quad = \operatorname{marg}_{\Theta \times L} \pi_{\bar{t}_i}^{\ell', m_i} \circ \left(\operatorname{id}_{\Theta} \times \bar{h}_{-i} \times \operatorname{id}_{M_{-i}(\bar{\sigma})}\right)^{-1} \\ &\quad = \operatorname{marg}_{\Theta \times L} \bar{\kappa}(\bar{t}_i) \circ (\operatorname{id}_{\Theta} \times \bar{h}_{-i})^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\hat{h}_i^{\ell}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \\ &\quad = \delta_{\hat{h}_i^{\ell-1}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i])} \times \operatorname{marg}_{\Theta \times L} \hat{\kappa}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) \circ (\operatorname{id}_{\Theta} \times \hat{h}_{-i})^{-1} \\ &\quad = \delta_{\bar{h}_i^{\ell-1}(\bar{t}_i)} \times \operatorname{marg}_{\Theta \times L} \bar{\kappa}(\bar{t}_i) \circ (\operatorname{id}_{\Theta} \times \bar{h}_{-i})^{-1} = \bar{h}_i^{\ell}(\bar{t}_i),\end{aligned}$$

showing that $\hat{h}_i^{\ell}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \bar{h}_i^{\ell}(\bar{t}_i)$. Thus, we have proved that, for all $\ell \geq 1$, all $\ell' \geq \ell$: $\hat{h}_i^{\ell}(\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]) = \bar{h}_i^{\ell}(\bar{t}_i)$ for any $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i^{\ell'}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$, that is, $\hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i] \to_P \bar{t}_i$ as $\ell' \to \infty$ for any $\bar{t}_i \in \bar{T}_i$

and $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$. In addition, we know that, for all $\ell' \geq 1$ and all $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}}) : \hat{t}_i[\varepsilon, \ell', \bar{\sigma}, \bar{t}_i, m_i] \to_P \hat{t}_i[0, \ell', \bar{\sigma}, \bar{t}_i, m_i]$ as $\varepsilon \to 0$. Since T^* is a metrizable space, $\hat{t}_i[\hat{\varepsilon}(\ell'), \ell', \bar{\sigma}, \bar{t}_i, m_i] \to_P \bar{t}_i$ as $\ell' \to \infty$ for some function $\hat{\varepsilon} : \mathbb{N}^* \to \mathcal{E} \setminus \{0\}$ satisfying $\lim_{\ell' \to \infty} \hat{\varepsilon}(\ell') = 0$. *Q.E.D.*

CLAIM 2: *For each $\varepsilon \in \mathcal{E} \setminus \{0\}$, ℓ, $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$, we have $\sigma_i(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$ for any equilibrium σ of $U(\mathcal{M}, \hat{\mathcal{T}})$ satisfying $\sigma_{|\bar{T}} = \bar{\sigma}$.*

PROOF: Fix a type $\bar{t}_i \in \bar{T}_i$ and an equilibrium σ of $U(\mathcal{M}, \hat{\mathcal{T}})$ satisfying $\sigma_{|\bar{T}} = \bar{\sigma}$. We will show by induction on ℓ that, for all $\varepsilon \in \mathcal{E} \setminus \{0\}$ and $\ell \geq 1$: $\sigma_i(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$ for all messages $m_i \in R_i^\ell(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$.

Recall that, by construction, for all $m_i \in M_i(\bar{\sigma})$, $t_i[\bar{\sigma}, m_i] \in T_i$ is the type in Proposition 1 such that $\sigma_i(t_i[\bar{\sigma}, m_i]) = m_i$. First, fix $\varepsilon \in \mathcal{E} \setminus \{0\}$ and $m_i \in R_i^1(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$ and let us prove that $\sigma_i(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$. For each $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$, define the belief

$$\pi_i^{\varepsilon,1} = \hat{\kappa}(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) \circ \gamma^{-1} \in \Delta(\Theta^* \times \hat{T}_{-i} \times M_{-i}),$$

where $\gamma : (\theta^*, t_{-i}[\bar{\sigma}, m_{-i}]) \mapsto (\theta^*, t_{-i}[\bar{\sigma}, m_{-i}], m_{-i})$. Note that by construction, $\pi_i^{\varepsilon,1}$ is the belief of type $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$ on $\Theta^* \times \hat{T}_{-i} \times M_{-i}$ when he believes that m_{-i} is played at each $(\theta^*, t_{-i}[\bar{\sigma}, m_{-i}])$. Hence, for each $\varepsilon \geq 0$, $\pi_i^{\varepsilon,1}$ corresponds to beliefs of type $\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]$ when the equilibrium σ is played. Now, by Equations (S2) and (S3), the belief $\pi_i^{0,1}$ of type $\hat{t}_i[0, 1, \bar{\sigma}, \bar{t}_i, m_i]$ satisfies

$$\begin{aligned}
\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_i^{0,1} &= \operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i} \circ (\tau_{-i}^{0,1})^{-1} \circ (\gamma_\Theta)^{-1} \\
&= \operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i},
\end{aligned}$$

where $\gamma_\Theta : (\theta, t_{-i}[\bar{\sigma}, m_{-i}]) \mapsto (\theta, \tilde{\theta}^0, t_{-i}[\bar{\sigma}, m_{-i}], m_{-i})$. Since $\operatorname{Supp}(\sigma_i(\hat{t}_i[0, 1, \bar{\sigma}, \bar{t}_i, m_i])) \subset BR_i(\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_i^{0,1} \mid \mathcal{M})$, we have $BR_i(\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i} \mid \mathcal{M}) \neq \emptyset$. In addition, since $\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i}(\Theta \times \{\tilde{\theta}^0\} \times M_{-i}(\bar{\sigma})) = 1$, by construction of $M_i(\bar{\sigma})$ we have $BR_i(\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i} \mid \mathcal{M}) \subset M_i(\bar{\sigma})$. Thus,

$$BR_i\big(\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i} \mid \mathcal{M}(\bar{\sigma})\big) = BR_i\big(\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i} \mid \mathcal{M}\big).$$

Recall that, by construction of $\pi_{\bar{t}_i}^{1,m_i}$, $m_i \in BR_i(\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_{\bar{t}_i}^{1,m_i} \mid \mathcal{M}(\bar{\sigma}))$. Consequently,

$$m_i \in BR_i\big(\operatorname{marg}_{\Theta^* \times M_{-i}} \pi_i^{0,1} \mid \mathcal{M}\big).$$

In addition, we have

$$\operatorname{marg}_{\Theta \times M_{-i}} \pi_i^{\varepsilon,1} = \operatorname{marg}_{\Theta \times M_{-i}} \pi_i^{0,1}.$$

Hence, for $\varepsilon \in \mathcal{E} \setminus \{0\}$, by construction of $\pi_i^{\varepsilon,1}$, $\{m_i\} = BR_i(\text{marg}_{\Theta^* \times M_{-i}} \pi_i^{\varepsilon,1} \mid \mathcal{M})$ and $\sigma_i(\hat{t}_i[\varepsilon, 1, \bar{\sigma}, \bar{t}_i, m_i]) = m_i$.

Now, for each $\ell \geq 2$, proceed by induction and assume that $\sigma_{-i}(\hat{t}_{-i}[\varepsilon, \ell - 1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}]) = m_{-i}$ for any $\bar{t}_{-i} \in \bar{T}_{-i}$, $m_{-i} \in R_{-i}^{\ell-1}(\bar{t}_{-i} \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$, and $\varepsilon \in \mathcal{E} \setminus \{0\}$. Fix $\varepsilon \in \mathcal{E} \setminus \{0\}$ and $m_i \in R_i^{\ell}(\bar{t}_i \mid \mathcal{M}(\bar{\sigma}), \bar{\mathcal{T}})$. For each $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$, define the belief

$$\pi_i^{\varepsilon,\ell} = \hat{\kappa}(\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]) \circ \gamma_\ell^{-1} \in \Delta(\Theta^* \times \hat{T}_{-i} \times M_{-i}),$$

where $\gamma_\ell : (\theta^*, \hat{t}_{-i}[\varepsilon, \ell - 1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}]) \mapsto (\theta^*, \hat{t}_{-i}[\varepsilon, \ell - 1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}], m_{-i})$.

Note that, by construction, $\pi_i^{\varepsilon,\ell}$ is the belief of type $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$ on $\Theta^* \times \hat{T}_{-i} \times M_{-i}$ when he believes that m_{-i} is played at each $(\theta^*, \hat{t}_{-i}[\varepsilon, \ell - 1, \bar{\sigma}, \bar{t}_{-i}, m_{-i}])$. Hence, by the induction hypothesis, for each $\varepsilon \geq 0$, $\pi_i^{\varepsilon,\ell}$ corresponds to beliefs of type $\hat{t}_i[\varepsilon, \ell, \bar{\sigma}, \bar{t}_i, m_i]$ when the equilibrium σ is played. The end of the proof mimics the case $\ell = 1$. *Q.E.D.*

This completes the proof of Proposition 2. *Q.E.D.*

COMPLETION OF THE PROOF OF THE "ONLY IF PART" OF THEOREM 4: Pick $\hat{\mathcal{T}} = (\hat{T}, \hat{\kappa})$ as defined in Proposition 2. By definition of continuous implementation, there exists an equilibrium σ in $U(\mathcal{M}, \hat{\mathcal{T}})$ that continuously implements f, and point (i) in this definition ensures that $\sigma_{|\bar{T}}$ is a pure equilibrium. Now pick any $\bar{t} \in \bar{T}$ and $m \in R(\bar{t} \mid \mathcal{M}(\sigma_{|\bar{T}}), \bar{\mathcal{T}})$; we show that $g_{|M(\sigma_{|\bar{T}})}(m) = f(\bar{t})$, proving that the mechanism $\mathcal{M}(\sigma_{|\bar{T}})$ implements f in rationalizable messages. Applying Proposition 2, we know that there exists a sequence of types $\{\hat{t}[n]\}_{n=0}^{\infty}$ in $\hat{T}$ such that (i) $\hat{t}[n] \to_P \bar{t}$ and (ii) $\sigma(\hat{t}[n]) = m$ for all n. By (i) and the fact that σ continuously implements f, we have $(g \circ \sigma)(\hat{t}[n]) \to f(\bar{t})$, while by (ii), we have $(g \circ \sigma)(\hat{t}[n]) = g(m)$ for all n. Hence, we must have $g(m) = f(\bar{t})$ and so $g_{|M(\sigma_{|\bar{T}})}(m) = f(\bar{t})$, as claimed. *Q.E.D.*

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