

SUPPLEMENT TO “REPUTATION IN CONTINUOUS-TIME GAMES”:
PUBLIC RANDOMIZATION

(*Econometrica*, Vol. 79, No. 3, May 2011, 773–876)

BY EDUARDO FAINGOLD AND YULIY SANNIKOV

We show that, in the reputation games in the main paper (Section 7), if a continuous-time public randomization device is available, then a Markov perfect equilibrium in publicly randomized strategies exists under Conditions 1 and 4.

AS EXPLAINED IN REMARK 7 in Section 7 of the main paper, when Condition 1 holds but Condition 2 fails, some reputation games may fail to have a sequential equilibrium in continuous time (even when the action sets are finite and mixed strategies are allowed). In this supplement, we enlarge the model of Section 7 to allow the players to condition their behavior on the outcomes of a *continuous-time public randomization device*. In this augmented game, an equilibrium is guaranteed to exist under Conditions 1 and 4.

Before turning to the formalism of continuous-time public randomization, we draw an analogy with discrete time so as to attach a meaning to our continuous-time mixtures. In a discrete-time game, there is a well defined order of events within each period: first, the outcome of the randomization device is realized and publicly observed, then players take their actions simultaneously, then the signals are realized and publicly observed, and, finally, the players update their beliefs. Heuristically, in our continuous-time game it is also possible to capture a similar order of events as follows: at any time t , suppose the current value of the small players' posterior on the behavioral type is ϕ . Then the following order of events occurs:

(i) The outcome of the randomization device is realized and publicly observed.

(ii) The normal type chooses an action a and, simultaneously, each small player i chooses an action b^i , which gives rise to a distribution \bar{b} .

(iii) The public signal $dX = \mu(a, \bar{b}) dt + \sigma(\bar{b}) dZ$ is realized.

(iv) Players receive their flow payoffs $rg(a, \bar{b}) dt$ and $rh(a, b^i, \bar{b}) dt$.

(v) The posterior is updated to $\phi + d\phi$.

Thus, calculated at the stage prior to the realization of the randomization device, the expected flow payoff of the normal type is

$$rg(\pi) dt,$$

where $\pi \in \Delta(A \times \Delta(B))$ designates the distribution over action profiles induced by the randomization device.¹ Also, from the viewpoint of the normal

¹As usual, the function g is linearly extended to $\Delta(A \times \Delta(B))$.

type, the change in posterior beliefs, $d\phi$, has mean and variance given by

$$-\frac{\Gamma^2(\pi, \phi)}{1 - \phi} dt \quad \text{and} \quad \Gamma^2(\pi, \phi) dt,$$

respectively, where

$$\Gamma^2(\pi, \phi) \stackrel{\text{def}}{=} \int_{A \times \Delta(B)} |\gamma(a, \bar{b}, \phi)|^2 \pi(da \times d\bar{b}).$$

Then, following the logic of Sections 6 and 7, in a sequential equilibrium in which the players use a publicly randomized strategy $\pi_t = \pi(\phi_t) \in \Delta(A \times \Delta(B))$ in which the actions are determined by the small players' posterior belief after every history, we expect the value function $U : (0, 1) \rightarrow \mathbb{R}$ of the normal type to solve the differential equation

$$(S1) \quad U''(\phi) = \frac{2U'(\phi)}{1 - \phi} + \frac{2r(U(\phi) - g(\pi(\phi)))}{\Gamma^2(\pi(\phi), \phi)},$$

with

$$(S2) \quad (a, \bar{b}) \in \mathcal{N}(\phi, \phi(1 - \phi)U'(\phi)/r) \quad \forall (a, \bar{b}) \in \text{supp}\pi(\phi),$$

since in equilibrium the incentive constraints must hold following each realization of the randomization device.

This motivates the following definition.

DEFINITION: A *Markov perfect equilibrium with public randomization* is a measurable function $\pi : (0, 1) \rightarrow \Delta(A \times \Delta(B))$ for which there exists a bounded differentiable function $U : (0, 1) \rightarrow \mathbb{R}$ that has an absolutely continuous derivative and satisfies (S1) and (S2) almost everywhere. Function U is the *value function of the normal type* and, for each prior $p \in (0, 1)$ on the behavioral type, $U(p)$ is the *expected discounted payoff of the normal type* under the equilibrium π and the prior p .

Although in this definition we have ultimately assumed (S1) and (S2), we do believe it is possible to derive these conditions, as we did in the setting without public randomization of Section 7. However, this would require a rich mathematical framework in which it were possible to formulate an exact law of large numbers for a continuum of independent and identically distributed random variables. Various such frameworks have been provided in the large games literature and we believe it is possible to extend such frameworks to deal with our dynamic setting in continuous time as well.² Providing this extension, however, is beyond the scope of our paper.

²See Sun (1998) and Khan and Sun (1999) for an approach based on hyperfinite Loeb spaces, and Al-Najjar (2009) for an approach based on finitely additive probability measures.

Finally, we note that neither the existence problem nor the issue of randomization in continuous time is new to our paper. Public randomization was used by [Harris, Reny, and Robson \(1995\)](#) to establish the existence of subgame perfect equilibria in extensive-form games with continuum of actions. In the experimentation literature, [Bolton and Harris \(1999\)](#) made use of continuous-time randomization to obtain the existence of Markov perfect equilibria. Like us, they define Markov perfect equilibria directly using a differential equation for the value. The heuristic derivation presented above follows that paper closely.

The main result of this supplement is the following proposition:

PROPOSITION: *Under Conditions 1 and 4, a Markov perfect equilibrium with public randomization exists.*

PROOF: It is enough to show that the differential inclusion

$$(S3) \quad U''(\phi) \in \Lambda(\phi, U(\phi), U'(\phi)) \quad \text{a.e.}$$

has a bounded solution, where $\Lambda: (0, 1) \rightrightarrows \mathbb{R}$ is the correspondence defined by

$$\begin{aligned} & \Lambda(\phi, u, u') \\ & \stackrel{\text{def}}{=} \left\{ \frac{2u'}{1-\phi} + \frac{2r(u-g(\pi))}{\Gamma^2(\pi, \phi)} \mid \begin{array}{l} \pi \in \Delta(A \times \Delta(B)) \text{ such that} \\ \text{supp } \pi \subseteq \mathcal{N}(\phi, \phi(1-\phi)u'/r) \end{array} \right\} \\ & \forall (\phi, u, u') \in (0, 1) \times \mathbb{R}^2. \end{aligned}$$

Indeed, if such a solution exists, then for each ϕ , we can find $\pi(\phi)$ satisfying (S1) and (S2), by the definition of \mathcal{N} . Moreover, such $\pi(\phi)$ can be chosen measurably as a function of ϕ , by a standard measurable selection argument.

We proceed with the proof that (S3) has a bounded solution.

STEP 0—Quadratic Growth: *Given any closed interval $[p, q] \subset (0, 1)$, there exists $K > 0$ such that*

$$|\Lambda(\phi, u, u')| \leq K(1 + |u'|^2) \quad \forall (\phi, u, u') \in [p, q] \times [g, \bar{g}] \times \mathbb{R}.$$

Given Conditions 1 and 4, this is immediate from Corollary B.1 of the main paper.

STEP 1—Existence Away From 0 and 1: *Given any closed interval $[p, q] \subset (0, 1)$, the differential inclusion has a C^2 solution on $[p, q]$ that takes values in the set of feasible payoffs $[g, \bar{g}]$.* This follows directly from an existence theorem due to [Bebernes and Kelley \(1973\)](#) for boundary value problems for second-order differential inclusions. [Bebernes and Kelley \(1973\)](#) sufficient condition for existence is that the nonlinearity Λ satisfy the Nagumo condition. This condition is implied by the quadratic growth condition from Step 0.

STEP 2—A priori Bound on the Derivative: *Given any closed interval $[p, q] \subset (0, 1)$, there is a constant $R > 0$ such that every C^2 function $U : [p, q] \rightarrow \mathbb{R}$ that takes values in $[g, \bar{g}]$ and solves the differential inclusion on $[p, q]$ satisfies $|U'| \leq R$. Let $L = (\bar{g} - g)/(q - p)$ and choose $R > 0$ large enough so that*

$$(S4) \quad \frac{1}{2K} \log \frac{1+R^2}{1+L^2} > \bar{g} - g.$$

Let $U : [p, q] \rightarrow [g, \bar{g}]$ be an arbitrary C^2 function that solves the differential inclusion on the interval $[p, q]$. Suppose, toward a contradiction, that $U'(\phi_1) > R$ for some $\phi_1 \in [p, q]$. (The proof for the case $U'(\phi_1) < -R$ is similar.) By the mean value theorem, there is some $\bar{\phi} \in [p, q]$ such that $U'(\bar{\phi}) \leq L$. Hence, we can find an interval $[\phi_0, \phi_1]$ or $[\phi_1, \phi_0]$ such that $U'(\phi_0) = L$ and $U' \geq 0$ on this interval. Thus,

$$\begin{aligned} \frac{1}{2K} \log \frac{1+R^2}{1+L^2} &= \int_L^R \frac{v \, dv}{K(1+v^2)} \leq \int_{U'(\phi_0)}^{U'(\phi_1)} \frac{v \, dv}{K(1+v^2)} \\ &= \int_{\phi_0}^{\phi_1} \frac{U'(\phi)U''(\phi) \, d\phi}{K(1+U'(\phi)^2)} \leq \left| \int_{\phi_0}^{\phi_1} U'(\phi) \, d\phi \right| \\ &\leq |U(\phi_1) - U(\phi_0)| \leq \bar{g} - g, \end{aligned}$$

where the second inequality follows from the differential inclusion and the quadratic growth condition of Step 0. By (S4), the inequalities above yield a contradiction.

STEP 3—Extension to $(0, 1)$: For each n , let $U_n : [1/n, 1 - 1/n] \rightarrow \mathbb{R}$ be a C^2 solution of the differential inclusion on $[1/n, 1 - 1/n]$ and let $R_n > 0$ denote the corresponding a priori bound on the derivative from Step 2. Fix n and consider $m \geq n$. The restriction of U_m to $[1/n, 1 - 1/n]$ is a solution of the differential inclusion that takes values in the set of feasible payoffs on $[1/n, 1 - 1/n]$. Hence Step 2 applies and we have $|U'_m(\phi)| < R_n$ for all $\phi \in [1/n, 1 - 1/n]$, which implies a uniform bound on the second derivative U''_m because of the quadratic growth condition. By the Arzelà–Ascoli theorem, for every n , the sequence $(U_m)_{m \geq n}$ has a subsequence that converges in the C^1 topology on $[1/n, 1 - 1/n]$. Hence, by a standard diagonalization argument, $(U_n)_{n \geq 1}$ has a subsequence $(U_{n_k})_{k \geq 1}$ that converges pointwise to some function $U : (0, 1) \rightarrow [g, \bar{g}]$. Moreover, on any closed subinterval J of $(0, 1)$, the convergence takes place in $C^1(J)$, that is, U is C^1 and $(U_{n_k}, U'_{n_k}) \rightarrow (U, U')$ uniformly on J .

STEP 4—Absolute Continuity of U' : First, note that (a) $U'_{n_k} \rightarrow U'$ pointwise, and (b) for every $[p, q] \subset (0, 1)$, U'_{n_k} is Lipschitz continuous on $[p, q]$ uniformly over all k with $n_k > \max\{1/p, 1/(1-q)\}$. This is because by Step 2 there

exists $R > 0$ such that for all k with $1/n_k < \min\{p, 1 - q\}$, we have $|U'_{n_k}| \leq R$ on $[p, q]$, which implies a uniform bound on U''_{n_k} by the quadratic growth condition of Step 0.

It follows that U' is locally Lipschitz, and hence absolutely continuous.

STEP 5— U Is a Solution: Fix an arbitrary $\varepsilon > 0$ and a ϕ_0 at which U'' exists. Since Λ is upper hemicontinuous and $(U_{n_k}, U'_{n_k}) \rightarrow (U, U')$ uniformly on a neighborhood of ϕ_0 (by Step 3), there exist $\delta > 0$ and $K \geq 1$ such that for all $k \geq K$ and all $\phi \in (\phi_0 - \delta, \phi_0 + \delta)$ we have

$$\Lambda(\phi, U_{n_k}(\phi), U'_{n_k}(\phi)) \subset \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon].$$

Thus, for almost all $\phi \in (\phi_0 - \delta, \phi_0 + \delta)$ and all $k \geq K$,

$$(S5) \quad U''_{n_k}(\phi) \in \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon].$$

On the other hand, for all $h > 0$ and $k \geq 1$, we have

$$\frac{U'_{n_k}(\phi_0 + h) - U'_{n_k}(\phi_0)}{h} = \frac{1}{h} \int_{\phi_0}^{\phi_0+h} U''_{n_k}(x) dx.$$

This implies that for each $k \geq K$ and $|h| < \delta$,

$$\frac{U'_{n_k}(\phi_0 + h) - U'_{n_k}(\phi_0)}{h} \in \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon]$$

by the differential inclusion (S5) and the fact that Λ has closed convex values. Thus, letting $k \rightarrow \infty$ first and then $h \rightarrow 0$ yields

$$U''(\phi_0) \in \Lambda(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon].$$

Finally, taking ε to zero yields the desired result.

Q.E.D.

REFERENCES

- AL-NAJJAR, N. I. (2009): "Large Games and the Law of Large Numbers," *Games and Economic Behavior*, 64, 1–34. [2]
- BEBERNES, J., AND W. KELLEY (1973): "Some Boundary Value Problems for Generalized Differential Equations," *SIAM Journal of Applied Mathematics*, 25, 16–23. [3]
- BOLTON, P., AND C. HARRIS (1999): "Strategic Experimentation," *Econometrica*, 67, 349–374. [3]
- HARRIS, C., P. RENY, AND A. ROBSON (1995): "The Existence of Subgame-Perfect Equilibrium in Continuous Games With Almost Perfect Information: A Case for Public Randomization," *Econometrica*, 63, 507–544. [3]
- KHAN, M., AND Y. SUN (1999): "Non-Cooperative Games on Hyperfinite Loeb Spaces," *Journal of Mathematical Economics*, 31, 455–492. [2]

SUN, Y. (1998): "A Theory of Hyperfinite Processes: The Complete Removal of Individual Uncertainty via Exact LLN," *Journal of Mathematical Economics*, 29, 419–503. [2]

Dept. of Economics, Yale University, Box 208281, New Haven, CT 06520-8281, U.S.A.; eduardo.faingold@yale.edu

and

Dept. of Economics, Princeton University, Princeton, NJ 08544-1021, U.S.A.; sannikov@princeton.edu.

Manuscript received August, 2007; final revision received October, 2010.