Econometrica Supplementary Material

SUPPLEMENT TO "EFFICIENT REPEATED IMPLEMENTATION"

(*Econometrica*, Vol. 79, No. 6, November 2011, 1967–1994)

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WE HERE PRESENT some formal results and proofs omitted from the main paper.

A. TWO-AGENT CASE

PROOF OF THEOREM 3: Consider regime $\widehat{R}$ defined in Section 4.2. We prove the theorem via the following claims.

CLAIM A.1: *Fix any $\sigma \in \Omega^\delta(\widehat{R})$. For any $t > 1$ and $\theta(t)$, if $g^{\theta(t)} = \hat{g}$, then $\pi_i^{\theta(t)} \geq v_i(f)$ for all $i = 1, 2$.*

The proof can be established by reasoning analogous to that behind Lemma 2.

CLAIM A.2: *Fix any $\sigma \in \Omega^\delta(\widehat{R})$. For any t and $\theta(t)$, if $g^{\theta(t)} = \hat{g}$, then $m_1^{\theta(t),\theta^t} = (\cdot, 0)$ and $m_2^{\theta(t),\theta^t} = (\cdot, 0)$ for any θ^t.*

PROOF: Suppose not. Then, for some t, $\theta(t)$, and θ^t, $g^{\theta(t)} = \hat{g}$ and the continuation regime next period at $\mathbf{h}(\theta(t), \theta^t)$ is either D^i or $S^i \neq \Phi^{\tilde{a}^i}$ for some i. By reasoning similar to the three-or-more-player case, it then follows that, for $j \neq i$,

$$\pi_j^{\theta(t),\theta^t} < v_j^j. \tag{A.1}$$

Then, given (A.1), agent j can profitably deviate at $(\mathbf{h}(\theta(t)), \theta^t)$ by announcing the same state as σ_j and an integer higher than i's integer choice at such a history. This is because the deviation does not alter the current outcome (given the definition of ψ of $\hat{g}$) but induces regime D^j in which, by (A.1), j obtains $v_j^j > \pi_j^{\theta(t),\theta^t}$. But this is a contradiction. *Q.E.D.*

CLAIM A.3: *Assume that f is efficient in the range. For any $\sigma \in \Omega^\delta(\widehat{R})$, $\pi_i^{\theta(t)} = v_i(f)$ for any i, $t > 1$, and $\theta(t)$.*

Given Claims A.1 and A.2, and since f is efficient in the range, we can directly apply the proof of Lemma 4.

CLAIM A.4: *$\Omega^\delta(\widehat{R})$ is nonempty if self-selection holds.*

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PROOF: Consider a symmetric Markov strategy profile in which, for any θ, each agent reports $(\theta, 0)$. Given the output function ψ of $\hat{g}$, any unilateral deviation by i at any θ after any history results either in no change in the current period outcome (if he does not change his announced state) or it results in current period outcome belonging to $L_i(\theta)$. Also, given the transition rules of $\widehat{R}$, a deviation does not improve continuation payoff at the next period either. Therefore, given self-selection, it does not pay i to deviate from his strategy. *Q.E.D.*

Finally, given Claims A.3 and A.4, the proof of Theorem 3 follows by exactly the same arguments as those behind Theorem 2. *Q.E.D.*

Alternative Condition to Self-Selection and Condition ω

As mentioned at the end of Section 4.2, the conclusions of Theorem 3 can be obtained using an alternative condition to self-selection and Condition ω if δ is sufficiently large.

THEOREM A.1: *Suppose that $I = 2$ and consider an SCF f such that there exists $\tilde{a} \in A$ such that $v_i(\tilde{a}) < v_i(f)$ for $i = 1, 2$. If f is efficient in the range, there exists a regime R and $\bar{\delta}$ such that, for any $\delta > \bar{\delta}$, $\Omega^\delta(R)$ is nonempty and, for any $\sigma \in \Omega^\delta(R)$, $\pi_i^{\theta(t)}(\sigma, R) = v_i(f)$ for any i, $t \geq 2$, and $\theta(t)$. If f is strictly efficient in the range, then, in addition, $a^{\theta(t),\theta^t}(\sigma, R) = f(\theta^t)$ for any $t \geq 2$, $\theta(t)$, and θ^t.*

PROOF: Following Lemma 1, let S^i be the regime alternating $d(i)$ and $\phi(\tilde{a})$ from which $i = 1, 2$ can obtain payoff exactly equal to $v_i(f)$. For $j \neq i$, let $\pi_j(S^i)$ be the maximum payoff that j can obtain from regime S^i when i behaves rationally in $d(i)$. Since S^i involves $d(i)$, Assumption A implies that $v_j^j > \pi_j(S^i)$. Then, for any i, j, $i \neq j$, there must also exist $\varepsilon > 0$ such that $v_j(\tilde{a}) < v_j(f) - \varepsilon$ and $\pi_j(S^i) < v_j^j - \varepsilon$. Next define $\rho \equiv \max_{i,\theta,a,a'}[u_i(a, \theta) - u_i(a', \theta)]$ and $\bar{\delta} \equiv \frac{\rho}{\rho+\varepsilon}$.

Mechanism $\tilde{g} = (M, \psi)$ is defined such that, for all i, $M_i = \Theta \times \mathbb{Z}_+$ and ψ is such that the following conditions hold:

(i) If $m_i = (\theta, \cdot)$ and $m_j = (\theta, \cdot)$, then $\psi(m) = f(\theta)$.

(ii) If $m_i = (\theta^i, z^i)$, $m_j = (\theta^j, 0)$, and $z^i \neq 0$, then $\psi(m) = f(\theta^j)$.

(iii) For any other m, $\psi(m) = \tilde{a}$.

Let $\widetilde{R}$ denote any regime in which $\widetilde{R}(\emptyset) = \tilde{g}$ and, for any $h = ((g^1, m^1), \ldots, (g^{t-1}, m^{t-1})) \in H^t$ such that $t > 1$, and $g^{t-1} = \tilde{g}$, the following transition rules hold:

RULE A.1: If $m_i^{t-1} = (\theta, 0)$ and $m_j^{t-1} = (\theta, 0)$, then $\widetilde{R}(h) = \tilde{g}$.

RULE A.2: If $m_i^{t-1} = (\theta^i, 0)$, $m_j^{t-1} = (\theta^j, 0)$, and $\theta^i \neq \theta^j$, then $\widetilde{R}(h) = \Phi^{\tilde{a}}$.

RULE A.3: If $m_i^{t-1} = (\theta^i, z^i)$, $m_j^{t-1} = (\theta^j, 0)$, and $z^i \neq 0$, then $\widetilde{R}|h = S^i$.

RULE A.4: If m^{t-1} is of any other type and i is the lowest-indexed agent among those who announce the highest integer, then $\widetilde{R}|h = D^i$.

We next prove the theorem via the following claims.

CLAIM A.5: *Fix any $\sigma \in \Omega^\delta(\widetilde{R})$. For any $t > 1$ and $\theta(t)$, if $g^{\theta(t)} = \tilde{g}$, then $\pi_i^{\theta(t)} \geq v_i(f)$ for all $i = 1, 2$.*

PROOF: Suppose not. Then at some $t > 1$ and $\theta(t)$, $g^{\theta(t)} = \tilde{g}$ but $\pi_i^{\theta(t)} < v_i(f)$ for some i. Let $\theta(t) = (\theta(t-1), \theta^{t-1})$. Given the transition rules, it must be that $g^{\theta(t-1)} = \tilde{g}$ and $m_i^{\theta(t-1), \theta^{t-1}} = m_j^{\theta(t-1), \theta^{t-1}} = (\tilde{\theta}, 0)$ for some $\tilde{\theta}$.

Consider i deviating at $(\mathbf{h}(\theta(t-1)), \theta^{t-1})$ such that he reports $\tilde{\theta}$ and a positive integer. Given the output function ψ of mechanism $\tilde{g}$, the deviation does not alter the current outcome, but, by Rule A.3 of regime $\widetilde{R}$, can yield continuation payoff $v_i(f)$. Hence, the deviation is profitable, implying a contradiction. *Q.E.D.*

CLAIM A.6: *Fix any $\delta \in (\bar{\delta}, 1)$ and $\sigma \in \Omega^\delta(\widetilde{R})$. For any t and $\theta(t)$, if $g^{\theta(t)} = \tilde{g}$, then $m_1^{\theta(t), \theta^t} = m_2^{\theta(t), \theta^t} = (\theta, 0)$ for any θ^t.*

PROOF: Suppose not. Then for some t, $\theta(t)$, and θ^t, $g^{\theta(t)} = \tilde{g}$ but $m^{\theta(t), \theta^t}$ is not as in the claim. There are three cases to consider.

Case 1—$m_i^{\theta(t), \theta^t} = (\cdot, z^i)$ and $m_j^{\theta(t), \theta^t} = (\cdot, z^j)$ with $z^i \geq z^j > 0$: In this case, given the definition of ψ of $\tilde{g}$, $\tilde{a}$ is implemented in the current period and, by Rule A.4, a dictatorship by, say, i follows forever thereafter. But then, by Assumption A, j can profitably deviate by announcing an integer higher than z^i at such a history; the deviation does not alter the current outcome from $\tilde{a}$ but switches dictatorship to himself as of the next period.

Case 2—$m_i^{\theta(t), \theta^t} = (\cdot, z^i)$ and $m_j^{\theta(t), \theta^t} = (\theta^j, 0)$ with $z^i > 0$: In this case, given ψ, $f(\theta^j)$ is implemented in the current period and, by Rule A.3, continuation regime S^i follows thereafter. Consider j deviating to another strategy identical to σ_j everywhere except at $(\mathbf{h}(\theta(t)), \theta^t)$ it announces an integer higher than z^i.

Given ψ (condition (iii)) and Rule A.4, this deviation yields a continuation payoff $(1-\delta)u_j(\tilde{a}, \theta^t) + \delta v_j^j$, while the corresponding equilibrium payoff does not exceed $(1-\delta)u_j(f(\theta^j), \theta^t) + \delta \pi_j(S^i)$. But since $v_j^j > \pi_j(S^i) + \varepsilon$ and $\delta > \bar{\delta}$, the former exceeds the latter and the deviation is profitable.

Case 3—$m_i^{\theta(t), \theta^t} = (\theta^i, 0)$ and $m_j^{\theta(t), \theta^t} = (\theta^j, 0)$ with $\theta^i \neq \theta^j$: In this case, given ψ, $\tilde{a}$ is implemented in the current period and, by Rule A.2, in every period thereafter. Consider any agent i deviating by announcing a positive integer at $(\mathbf{h}(\theta(t)), \theta^t)$. Given ψ (condition (ii)) and Rule A.3, such a deviation yields continuation payoff $(1-\delta)u_i(f(\theta^j), \theta^t) + \delta v_i(f)$, while the corresponding equilibrium payoff is $(1-\delta)u_i(\tilde{a}, \theta^t) + \delta v_i(\tilde{a})$. But since $v_i(f) > v_i(\tilde{a}) + \varepsilon$ and $\delta > \bar{\delta}$, the former exceeds the latter and the deviation is profitable. *Q.E.D.*

CLAIM A.7: *For any* $\delta \in (\bar{\delta}, 1)$ *and* $\sigma \in \Omega^\delta(\widetilde{R})$, $\pi_i^{\theta(t)} = v_i(f)$ *for any* i, $t > 1$, *and* $\theta(t)$.

Given Claims A.5 and A.6, and since f is efficient in the range, to prove the claim, we can directly apply the proofs of Lemmas 3 and 4.

CLAIM A.8: *For any* $\delta \in (\bar{\delta}, 1)$, $\Omega^\delta(\widetilde{R})$ *is nonempty*.

PROOF: Consider a symmetric Markov strategy profile in which the true state and zero integer are always reported. At any history, each agent i can deviate in one of the following three ways:

(i) Announce the true state but a positive integer. Given ψ (condition (i)) and Rule A.3, such a deviation is not profitable.

(ii) Announce a false state and a positive integer. Given ψ (condition (ii)) and Rule A.3, such a deviation is not profitable.

(iii) Announce zero integer but a false state. In this case, by ψ (condition (iii)), $\tilde{a}$ is implemented in the current period and, by Rule A.2, in every period thereafter. The gain from such a deviation cannot exceed $(1-\delta)\max_{a,\theta}[u_i(\tilde{a}, \theta) - u_i(a, \theta)] - \delta\varepsilon < 0$, where the inequality holds since $\delta > \bar{\delta}$. Thus, the deviation is not profitable. *Q.E.D.*

B. PERIOD 1: COMPLEXITY CONSIDERATIONS

Here, we introduce players with preference for less complex strategies to the main sufficiency analysis of our paper with pure strategies and show that if players have an aversion to complexity at the very margin, an SCF that satisfies efficiency in the range and Condition ω can be implemented from period 1.

Fix an SCF f and consider the canonical regime with $I \geq 3$, R^*. (Corresponding results for the two-agent case can be similarly derived and, hence, are omitted.) Consider *any* measure of complexity of a strategy under which taking the same action at every history with an identical state is simpler than one that takes different actions at different dates. Formally, we introduce a very weak partial order on the set of strategies that satisfies the following definition.[1]

[1]This partial order on the strategies is similar to the measure of complexity that we used in Lee and Sabourian (2011) on finite mechanisms. The result in this section also holds if we replace Definition B.1 with any measure of complexity that stipulates that Markov strategies are less complex than non-Markov ones.

DEFINITION B.1: For any player i, strategy σ_i' is said to be less complex than strategy σ_i if they are identical everywhere except that there exists $\theta' \in \Theta$ such that σ_i' always takes the same action after observing θ' and σ_i' does not. More formally, for any i, σ_i' is less complex that σ_i if the following conditions hold:

(i) $\sigma_i'(\mathbf{h}, \theta) = \sigma_i(\mathbf{h}, \theta)$ for all $\mathbf{h}$ and all $\theta \neq \theta'$.

(ii) $\sigma_i'(\mathbf{h}, \theta') = \sigma_i'(\mathbf{h}', \theta')$ for all $\mathbf{h}, \mathbf{h}' \in \mathbf{H}^\infty$.

(iii) $\sigma_i(\mathbf{h}, \theta') \neq \sigma_i(\mathbf{h}', \theta')$ for some $\mathbf{h}, \mathbf{h}' \in \mathbf{H}^\infty$.[2]

Next, consider the following refinement of Nash equilibrium of regime R^*: a strategy profile $\sigma = (\sigma_1, \dots, \sigma_I)$ constitutes a Nash equilibrium with complexity cost (NEC) of regime R if, for all i, (i) σ_i is a best response to σ_{-i} and (ii) there exists no σ_i' such that σ_i' is a best response to σ_{-i} and σ_i' is less complex than σ_i. Then, since a NEC is also a Nash equilibrium, Lemmas 3 and 4 hold for any NEC. In addition, we derive the following result.

LEMMA B.1: *Every NEC, σ, of R^* is Markov: for all i, $\sigma_i(\mathbf{h}', \theta) = \sigma_i(\mathbf{h}'', \theta)$ for all $\mathbf{h}', \mathbf{h}'' \in \mathbf{H}^\infty$ and all θ.*

PROOF: Suppose not. Then there exists some NEC, σ, of R^* such that $\sigma_i(\mathbf{h}', \theta') \neq \sigma_i(\mathbf{h}'', \theta')$ for some $i, \theta', \mathbf{h}'$, and $\mathbf{h}''$. Let $\widehat{\theta}$ be the state announced by σ_i in period 1 after observing θ'. Next, consider i deviating to another strategy σ_i' that is identical to σ_i except that at state θ', irrespective of the past history, it always announces state $\widehat{\theta}$ and integer 1; thus, $\sigma_i'(\mathbf{h}, \theta) = \sigma_i(\mathbf{h}, \theta)$ for all $\mathbf{h}$ and all $\theta \neq \theta'$, and $\sigma_i'(\mathbf{h}, \theta') = (\widehat{\theta}, 1)$ for all $\mathbf{h}$.

Clearly, σ_i' is less complex than σ_i. Furthermore, for any $\theta^1 \in \Theta$, by part (ii) of Lemma 3 and the definitions of g^* and R^*, we have $a^{\theta^1}(\sigma_i', \sigma_{-i}, R^*) = a^{\theta^1}(\sigma, R^*)$ and $\pi_i^{\theta^1}(\sigma_i', \sigma_{-i}, R^*) = v_i(f)$. Moreover, we know from Lemma 4 that $\pi_i^{\theta^1}(\sigma, R^*) = v_i(f)$. Thus, the deviation does not alter i's payoff. But since σ_i' is less complex than σ_i, such a deviation makes i better off. This contradicts the assumption that σ is a NEC. *Q.E.D.*

This lemma, together with Lemma 4, shows that for every NEC, each player's continuation payoff at any history on the equilibrium path (including the initial history) is equal to his target payoff. Moreover, since a Markov strategy has minimal complexity (i.e., no other strategy exists that is less complex than the Markov strategy), it also follows that the Markov Nash equilibrium described in Lemma 5 is itself a NEC. Thus, if we use NEC as the solution concept, then the conclusions of Theorem 2 hold from period 1.

THEOREM B.1: *Suppose that $I \geq 3$ and consider an SCF f that satisfies Condition ω. If f is efficient in the range, it is payoff-repeatedly implementable in Nash*

[2]We have suppressed the argument g^* in the definition of strategies here to simplify the exposition.

equilibrium with complexity cost. If f is strictly efficient in the range, it is repeatedly implementable in Nash equilibrium with complexity cost.

Note that the notion of NEC requires that each player's equilibrium strategy has minimal complexity among all strategies that are best responses to the strategies of the other agents. As a result, NEC strategies need only be of sufficient complexity to achieve the highest payoff on-the-equilibrium path; off-the-equilibrium payoffs do not figure into these complexity considerations. However, it may be argued that players adopt complex strategies also to deal with the off-the-equilibrium paths. In Lee and Sabourian (2011), we introduce an alternative equilibrium refinement based on complexity that is robust to this criticism (so as to explore what can be achieved by regimes that employ only finite mechanisms). Specifically, we considered the set of subgame perfect equilibria and required players to adopt minimally complex strategies among the set of strategies that are best responses *at every history*, not merely at the beginning of the game. We say that a strategy profile σ is a *weak* perfect equilibrium with complexity cost (WPEC) of regime R if, for all i, (i) σ is a subgame perfect equilibrium (SPE); and (ii) there exists no σ_i' that is less complex than σ_i and best responds to σ_{-i} at *every* (on-or off-the-equilibrium) information set.

In this equilibrium concept, complexity considerations are given less priority than both on- and off-the-equilibrium payoffs. Nevertheless, the same implementation result from period 1 can also be obtained using this equilibrium notion. For this result, we have to modify the regime R^* slightly. Define $\overline{g} = (M, \psi)$ as the following mechanism: $M_i = \Theta \times \mathbb{Z}_+$ for all i and ψ is such that the following conditions hold:

(i) If $m_i = (\theta, \cdot)$ for at least $I - 1$ agents, then $\psi(m) = f(\theta)$.

(ii) Otherwise, $\psi(m) = f(\theta')$, where θ' is the state announced by the lowest-indexed agent announcing the highest integer.

Let $\overline{R}$ be any regime such that $\overline{R}(\emptyset) = \overline{g}$ and, for any $h = ((g^1, m^1), \dots, (g^{t-1}, m^{t-1})) \in H^t$ such that $t > 1$ and $g^{t-1} = \overline{g}$, the following transition rules hold:

RULE B.1: If $m_i^{t-1} = (\cdot, 0)$ for all i, then $\overline{R}(h) = \overline{g}$.

RULE B.2: If, for some i, $m_j^{t-1} = (\cdot, 0)$ for all $j \neq i$ and $m_i^{t-1} = (\cdot, z^i)$ with $z^i \neq 0$, then $\overline{R}|h = S^i$ (Lemma 1).

RULE B.3: If m^{t-1} is of any other type and i is the lowest-indexed agent among those who announce the highest integer, then $\overline{R}|h = D^i$.

This regime is identical to R^* except for the output function defined for the one-period mechanism when two or more agents play distinct messages; in such cases, the immediate outcome for the period results from the state announced by the agent announcing the highest integer.

Then, by the same argument as above for NEC, it suffices to show that any WPEC must also be Markov. To see this, assume not. Then there exists some WPEC, σ, of R^* such that $\sigma_i(\mathbf{h}', \theta') \neq \sigma_i(\mathbf{h}'', \theta')$ for some i, θ', $\mathbf{h}'$, and $\mathbf{h}''$.

Next, let $\overline{\theta} \in \arg\max_\theta u_i(f(\theta), \theta')$ and consider i deviating to another strategy σ_i' that is identical to σ_i except that at state θ', irrespective of the past history, it always reports state $\overline{\theta}$ and integer 1; thus, $\sigma_i'(\mathbf{h}, \theta) = \sigma_i(\mathbf{h}, \theta)$ for all $\mathbf{h}$ and all $\theta \neq \theta'$, and $\sigma_i'(\mathbf{h}, \theta') = (\overline{\theta}, 1)$ for all $\mathbf{h}$.

Clearly, σ_i' is less complex than σ_i. Furthermore, by applying the same arguments as in Lemmas 2–4 to the notion of SPE, it can be shown that, at any history beyond period 1 at which $\overline{g}$ is being played, the equilibrium strategies choose integer 0 and each agent's equilibrium continuation payoff at this history is exactly the target payoff.

Thus, since σ_i' chooses 1 at any $\mathbf{h}$ if the realized state is θ', it follows that, at any such history, (i) σ_i' induces S^i in the continuation game and the target utility is achieved, and (ii) either other $I-1$ agents report the same state and the outcome in the current period is not affected, or other players disagree on the state and $f(\overline{\theta})$ is implemented (see the modified outcome function ψ of the mechanism). Therefore, σ_i' induces a payoff no less than σ_i after any history. Since σ_i' is also less complex than σ_i, we have a contradiction to σ being a WPEC.

C. MIXED STRATEGIES

We next extend the main analysis of the paper (Section 4.2) to incorporate mixed/behavioral strategies (also see Section 5). Let $b_i : \mathbf{H}^\infty \times G \times \Theta \to \triangle(\bigcup_{g\in G} M_i^g)$ denote a mixed (behavioral) strategy of agent i, with b denoting a mixed strategy profile. With some abuse of notation, given regime R and any history $\mathbf{h}^t \in \mathbf{H}^t$, let $g^{\mathbf{h}^t}(b, R) \equiv (M^{\mathbf{h}^t}(b, R), \psi^{\mathbf{h}^t}(b, R))$ be the mechanism played at $\mathbf{h}^t$, let $a^{\mathbf{h}^t, m^t}(b, R) \in A$ be the outcome implemented at $\mathbf{h}^t$ when the current message profile is m^t, and let $\pi_i^{\mathbf{h}^t}(b, R)$ be agent i's expected continuation payoff at $\mathbf{h}^t$ if the strategy profile b is adopted. We write $\pi_i(b, R) \equiv \pi_i^{\mathbf{h}^1}(b, R)$.

Also, for any strategy profile b and regime R, let $\mathbf{H}^t(\theta(t), b, R)$ be the set of $t-1$ period histories that occur with positive probability given state realizations $\theta(t)$ and let $M^{\mathbf{h}^t, \theta^t}(b, R)$ be the set of message profiles that occur with positive probability at any history $\mathbf{h}^t$ after observing θ^t. The arguments in the above variables will be suppressed when the meaning is clear.

We denote by $B^\delta(R)$ denote the set of mixed strategy Nash equilibria of regime R with discount factor δ. We modify the notion of Nash repeated implementation to incorporate mixed strategies as follows.

DEFINITION C.1: An SCF f is payoff-repeatedly implementable in mixed strategy Nash equilibrium from period τ if there exists a regime R such that

(i) $B^\delta(R)$ is nonempty and (ii) every $b \in B^\delta(R)$ is such that $\pi_i^{\mathbf{h}^t}(b,R) = v_i(f)$ for any i, $t \geq \tau$, $\theta(t)$, and $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R)$. An SCF f is repeatedly implementable in mixed strategy Nash equilibrium from period τ if, in addition, every $b \in B^\delta(R)$ is such that $a^{\mathbf{h}^t, m^t}(R) = f(\theta^t)$ for any $t \geq \tau$, $\theta(t)$, θ^t, $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R)$, and $m^t \in M^{\mathbf{h}^t, \theta^t}(b, R)$.

We now state and prove the result for the case of three or more agents. The two-agent case can be analogously dealt with and, hence, is omitted to avoid repetition.

THEOREM C.1: *Suppose that $I \geq 3$ and consider an SCF f that satisfies Condition ω. If f is efficient, it is payoff-repeatedly implementable in mixed strategy Nash equilibrium from period 2. If f is strictly efficient, it is repeatedly implementable in mixed strategy Nash equilibrium from period* 2.

PROOF: Consider the canonical regime R^* in the main paper. Fix any $b \in B^\delta(R^*)$, and also fix any t, $\theta(t)$, and $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R^*)$ such that $g^{\mathbf{h}^t} = g^*$. Also, suppose that θ^t is observed in the current period t.

Let $r_i(m_i)$ denote player i's randomization probability of announcing message $m_i = (\theta^i, z^i)$ at this history $(\mathbf{h}^t, \theta^t)$ with $r(m) = r_1(m_1) \times \cdots \times r_I(m_I)$. Also, denote the marginals by $r_i(\theta^i) = \sum_{z^i} r_i(\theta^i, z^i)$ and $r_i(z^i) = \sum_{\theta^i} r_i(\theta^i, z^i)$.

We write agent i's continuation payoff at the given history, after observing $(\mathbf{h}^t, \theta^t)$, as

$$\pi_i^{\mathbf{h}^t,\theta^t}(b, R^*) = \sum_{m \in [\Theta \times \mathbb{Z}_+]^I} r(m)\big[(1-\delta)u_i\big(a^{\mathbf{h}^t,m}(b,R^*), \theta^t\big) + \delta \pi_i^{\mathbf{h}^t,\theta^t,m}(b, R^*)\big].$$

Then we can also write i's continuation payoff at $\mathbf{h}^t$ prior to observing a state as

$$\pi_i^{\mathbf{h}^t}(b, R^*) = \sum_{\theta^t \in \Theta} p(\theta^t) \pi_i^{\mathbf{h}^t,\theta^t}(b, R^*).$$

We proceed by establishing the following claims. First, at the given history, we obtain a lower bound on each agent's *expected* equilibrium continuation payoff *at the next period*.

CLAIM C.1: $\sum_{m \in \Theta \times \mathbb{Z}_+} r(m) \pi_i^{\mathbf{h}^t,\theta^t,m} \geq v_i(f)$ *for all* i.

PROOF: Suppose not. Then, for some i, there exists $\varepsilon > 0$ such that $\sum_m r(m) \pi_i^{\mathbf{h}^t,\theta^t,m} < v_i(f) - \varepsilon$. Let $\underline{u} = \min_{i,a,\theta} u_i(a, \theta)$ and fix any $\varepsilon' > 0$ such that $\varepsilon'(v_i(f) - \underline{u}) < \varepsilon$. Also, fix any integer $\overline{z}$ such that, given b, at $(\mathbf{h}^t, \theta^t)$ the probability that an agent other than i announces an integer greater than $\overline{z}$ is less than ε' (since the set of integers is infinite, it is always feasible to find such an integer).

Consider agent i deviating to another strategy which is identical to the equilibrium strategy b_i except that at $(\mathbf{h}^t, \theta^t)$ it reports $\overline{z}+1$. Note from the definition of mechanism g^* and the transition rules of R^* that such a deviation at $(\mathbf{h}^t, \theta^t)$ does not alter the current period t's outcomes and expected utility, while the continuation regime at the next period is S^i or D^i with probability at least $1-\varepsilon'$. The latter implies that the expected continuation payoff as of the next period $t+1$ from the deviation is at least

$$(1-\varepsilon')v_i(f)+\varepsilon'\underline{u}. \tag{C.1}$$

Also, by assumption, the corresponding equilibrium expected continuation payoff as of $t+1$ is at most $v_i(f)-\varepsilon$, which, since $\varepsilon'(v_i(f)-\underline{u})<\varepsilon$, is less than (C.1). Recall that the deviation does not affect the current period t's outcomes/payoffs. Therefore, the deviation is profitable—a contradiction. *Q.E.D.*

CLAIM C.2: $\sum_m r(m)\pi_i^{\mathbf{h}^t,\theta^t,m}=v_i(f)$ *for all* i.

Given efficiency of f, this follows immediately from the previous claim.

CLAIM C.3: $\sum_\theta r_i(\theta,0)=1$ *for all* i.

PROOF: Suppose otherwise. Then there exists a message profile m' which occurs with a positive probability at $(\mathbf{h}^t, \theta^t)$ such that, for some i, $m'_i=(\cdot, z^i)$ with $z^i>0$. Since f is efficient, by similar arguments as for Claim 2 in the proof of Lemma 3, there must exist $j\neq i$ such that $\pi_j^{\mathbf{h}^t,\theta^t,m'}<v_j^j$. Then, given Claim C.2, it immediately follows that there exists $\varepsilon>0$ such that

$$v_j^j>v_j(f)+\varepsilon. \tag{C.2}$$

Next, fix any $\varepsilon'\in(0,1)$ such that

$$\varepsilon'(v_j(f)-\underline{u})<\varepsilon r(m'). \tag{C.3}$$

Also fix any integer $\overline{z}>z^i$ such that, given b, at $(\mathbf{h}^t, \theta^t)$ the probability that an agent other than j announces an integer greater than $\overline{z}$ is less than ε'.

Consider j deviating to another strategy which is identical to the equilibrium strategy b_j except that it reports $\overline{z}+1$ at the given history $(\mathbf{h}^t, \theta^t)$. Again, this deviation does not alter the expected outcomes in period t, but, with probability $(1-\varepsilon')$, the continuation regime at the next period is either S^j or D^j (Rules B.2 and B.3). Furthermore, since $\overline{z}>z^i$, the continuation regime is D^j

with probability $\frac{r(m')}{1-\varepsilon'}$. Thus, at $(\mathbf{h}^t, \theta^t)$ the expected continuation payoff at the next period $t+1$ resulting from this deviation is at least

$$\frac{r(m')}{1-\varepsilon'}v_j^j + \left(1 - \varepsilon' - \frac{r(m')}{1-\varepsilon'}\right)v_j(f) + \varepsilon' \underline{u}.$$

We know from Claim C.2 that the corresponding equilibrium expected continuation payoff at $t+1$ is $v_j(f)$. By (C.2) and (C.3), and since the deviation does not alter the current period outcomes, the deviation is profitable—a contradiction. *Q.E.D.*

It follows from Claims C.1–C.3 that g^* must always be played on the equilibrium path. Therefore, by applying similar arguments to Lemma 2 and using the efficiency of f, it must be that $\pi_i^{\mathbf{h}^t} = v_i(f)$ for all i, $t > 1$, $\theta(t)$, and $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R^*)$. The remainder of the proof follows arguments analogous to those for the corresponding results with pure strategies in Section 4.2. *Q.E.D.*

REFERENCE

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Manuscript received October, 2009; final revision received April, 2011.