Econometrica Supplementary Material

SUPPLEMENT TO "A THEORY OF INTERGENERATIONAL ALTRUISM"

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This supplement contains the proofs omitted from the main text of the paper. For simplicity, this supplement uses appendix and equation numbering that continue from the main text of the paper.

APPENDIX B: PROOF OF THEOREM 1

THE PROOF FOLLOWS AND GENERALIZES that of Diamond (1965), and is based on the following lemmas.

LEMMA 18—Debreu (1954): *Let C be a completely ordered set and $Z = (z_0, z_1, \dots)$ be a countable subset of C. If for every $c, c' \in C$ such that $c \prec c'$, there is $z \in Z$ such that $c \precsim z \precsim c'$, then there exists on C a real, order-preserving function, continuous in any natural topology.*[45]

LEMMA 19: *For any $c \in C$, there exists $x \in X$ such that $c \sim (c_0, x, x, \dots)$.*

PROOF: Given c, let $D_c = \{(c_0, y, y, \dots) : y \in X\}$, $A = \{d \in D_c : d \precsim c\}$, and $B = \{d \in D_c : d \succsim c\}$. By Axiom 1, $A \cup B = D_c$; by Axiom 2, A and B are closed; by Axiom 3, A and B are nonempty. Moreover, D_c is connected. Indeed, for any continuous function $\phi : D_c \to \{0, 1\}$, the function $\bar{\phi} : X \to \{0, 1\}$ defined by $\bar{\phi}(x) = \phi(c_0, x, x, \dots)$ is also continuous. Connectedness of X implies that $\bar{\phi}$ is constant and, hence, that ϕ is constant, showing connectedness of D_c. This implies that $A \cap B \neq \emptyset$. *Q.E.D.*

To conclude the proof of Theorem 1, let Z_0 be a countable dense subset of X, which exists since X is separable, and let Z be the subset of C consisting of streams $(x, y, y, \dots)$ with $x, y \in Z_0$. Lemma 19 implies that Z satisfies the hypothesis of Lemma 18, which yields the result. Indeed, by Lemma 19 there are $x, y \in X$ such that $(c_0, x, x, \dots) \sim c \prec c' \sim (c'_0, y, y, \dots)$. Consider the set $E \subset X^2$ consisting of (z, w) such that $(c_0, x, x, \dots) \prec (z, w, w, \dots) \prec (c'_0, y, y, \dots)$. E is nonempty by connectedness of X and open by Axiom 2. Since Z is dense in X^2, E must contain an element of Z.

APPENDIX C: PROOF OF COROLLARY 1

By Theorem 3, $\succ$ can be represented by

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G\big(U({}_t c)\big).$$

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[45] A natural topology is one under which Axiom 2 holds for that topology.

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Since $(x, c) \succ (y, c)$, $u(x) = u(y) + \overline{u}$ for some $\overline{u} > 0$. Hence, for any $t \geq 1$, $U(c^x) - U(c^y)$ equals $\overline{u} - \sum_{s=1}^{t} \alpha^s \Delta G_s$, where ΔG_s is defined recursively as follows: for $s = t$,

$$\Delta G_t = G\big(U\big({}_t c^y\big)\big) - G\big(U\big({}_t c^y\big) - \overline{u}\big),$$

otherwise

$$\Delta G_s = G\big(U_s\big({}_s c^y\big)\big) - G\left(U_s\big({}_s c^y\big) - \sum_{k=1}^{t-s} \alpha^k \Delta G_{s+k}\right).$$

By Proposition 1, $\Delta G_t < \frac{1-\alpha}{\alpha}\overline{u}$ and

$$\begin{aligned}
\Delta G_{t-1} &= G\big(U_{t-1}\big({}_{t-1}c^y\big)\big) - G\big(U_{t-1}\big({}_{t-1}c^y\big) - \alpha \Delta G_t\big) \\
&< (1-\alpha)\Delta G_t \\
&< \frac{(1-\alpha)^2}{\alpha}\overline{u}.
\end{aligned}$$

Now, suppose that, for all k such that $s < k \leq t-1$, $\Delta G_k < \frac{(1-\alpha)^2}{\alpha}\overline{u}$. It follows that

$$\begin{aligned}
\Delta G_s &< \frac{1-\alpha}{\alpha}\left[\sum_{\tau=1}^{t-s} \alpha^\tau \Delta G_{s+\tau}\right] \\
&< \frac{1-\alpha}{\alpha}\left[\sum_{\tau=1}^{t-s-1} \alpha^\tau \frac{(1-\alpha)^2}{\alpha} + \alpha^{t-s}\frac{(1-\alpha)}{\alpha}\right]\overline{u} \\
&= \frac{(1-\alpha)^2}{\alpha}\left[\sum_{\tau=0}^{t-s-2} \alpha^\tau (1-\alpha) + \alpha^{t-s-1}\right]\overline{u} \\
&= \frac{(1-\alpha)^2}{\alpha}\overline{u}.
\end{aligned}$$

Therefore,

$$\sum_{s=1}^{t} \alpha^s \Delta G_s < \overline{u}\left[\alpha^t \frac{1-\alpha}{\alpha} + \sum_{s=1}^{t-1} \alpha^s \frac{(1-\alpha)^2}{\alpha}\right] = \overline{u}(1-\alpha).$$

We conclude that $U(c^x) - U(c^y) > \alpha\overline{u} > 0$.

APPENDIX D: PROOF OF COROLLARY 2

By representation (5), U clearly depends on c_0 only through $u_0 = u(c_0)$. This implies that $U({}_1c)$—and hence also $U(c)$ (from (5))—depends on c_1 only through $u_1 = u(c_1)$. By induction, $U(c)$ depends on $(c_0, \ldots, c_t)$ only through $(u_0, \ldots, u_t)$, for each t. There remains to establish the result at infinity: If c and $\tilde{c}$ are two streams such that $u(c_t) = u(\tilde{c}_t)$ for all t, we need to show that $U(c) = U(\tilde{c})$. From the previous step, assume without loss of generality that $c_t = \tilde{c}_t$ for all $t \leq T$, where T is any large, finite constant. Since U is H-continuous, we can choose T so that $|U(c') - U(\tilde{c}')| < \varepsilon$ for all $c', \tilde{c}'$ that coincide up to T. Since c and $\tilde{c}$ satisfy this property, $|U(c) - U(\tilde{c})| < \varepsilon$, and since ε was arbitrary, $U(c) = U(\tilde{c})$. This shows that the sequence $\{u_t = u(c_t)\}_{t=0}^{\infty}$ of period-utility levels entirely determines the value of $U(c)$, proving the result.

APPENDIX E: PROOF OF PROPOSITION 4

Consider representation (5) in Theorem 3. For every $c \in C$, we have sequences $\{u_s\}_{s=0}^{\infty}$ and $\{U_s\}_{s=0}^{\infty}$, where $u_s = u(c_s)$ and $U_s = \hat{U}(u_s, u_{s+1}, \ldots)$. Since u is continuous and X is connected, the range of u is a connected interval $\mathcal{I}_u \subset \mathbb{R}$. Recall that the range of U is also a connected interval $\mathcal{U} \subset \mathbb{R}$. Using the notation,

$$d(t, c) = \frac{\partial U_0/\partial u_t}{\partial U_0/\partial u_0}.$$

Note that $\frac{\partial U_s}{\partial u_s} = 1$ for all $s \geq 0$. Since G is differentiable, we have

$$\frac{\partial U_0}{\partial u_t} = \sum_{\tau=0}^{t-1} \alpha^{t-\tau} G'(U_{t-\tau}) \frac{\partial U_{t-\tau}}{\partial u_t}.$$

More generally, for $1 \leq \tau \leq t$,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \sum_{s=0}^{\tau-1} \alpha^{\tau-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t}.$$

So, for $\tau = 1$, $\frac{\partial U_{t-1}}{\partial u_t} = \alpha G'(U_t)$. More generally, for $2 \leq \tau \leq t$,

$$\begin{aligned}
\frac{\partial U_{t-\tau}}{\partial u_t} &= \alpha \sum_{s=0}^{(\tau-1)-1} \alpha^{(\tau-1)-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t} + \alpha G'(U_{t-(\tau-1)}) \frac{\partial U_{t-(\tau-1)}}{\partial u_t} \\
&= \frac{\partial U_{t-(\tau-1)}}{\partial u_t} \alpha\big(1 + G'(U_{t-(\tau-1)})\big).
\end{aligned}$$

So,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \alpha^{\tau} G'(U_t) \prod_{s=1}^{\tau-1} \big(1 + G'(U_{t-s})\big).$$

Let $\prod_{s=1}^{\tau-1}(1 + G'(U_{t-s})) = 1$ if $\tau = 1$. Then,

$$\begin{aligned}
\frac{\partial U_0}{\partial u_t} &= \alpha^t G'(U_t) + G'(U_t) \sum_{\tau=1}^{t-1} \alpha^t G'(U_{t-\tau}) \prod_{s=1}^{\tau-1} \big(1 + G'(U_{t-s})\big) \\
&= \alpha^t G'(U_t) \left[1 + \sum_{\tau=1}^{t-1} G'(U_{t-\tau}) \prod_{s=1}^{\tau-1} \big(1 + G'(U_{t-s})\big) \right].
\end{aligned}$$

APPENDIX F: PROOF OF PROPOSITION 4

Recall that by assumption $\succ^t = \succ^0$ for all $t \geq 0$ and each $\succ^t$ is represented by the function $U({}_tc) = V(c_t, U({}_{t+1}c), U({}_{t+2}c), \ldots)$.

Suppose that V depends only on its first two arguments and is strictly increasing in its second argument. Since V is a function, we have recursively that

$$U({}_tc) = U({}_tc') \quad \Leftrightarrow \quad U(\hat{c}_{t-s}, \ldots, \hat{c}_{t-1}, {}_tc) = U(\hat{c}_{t-s}, \ldots, \hat{c}_{t-1}, {}_tc') \quad \text{for } s \geq 1.$$

And by the monotonicity property of V we have, again recursively, that

$$U({}_tc) > U({}_tc') \quad \Leftrightarrow \quad U(\hat{c}_{t-s}, \ldots, \hat{c}_{t-1}, {}_tc) > U(\hat{c}_{t-s}, \ldots, \hat{c}_{t-1}, {}_tc') \quad \text{for } s \geq 1.$$

Therefore, $\{\succ^t\}_{t=0}^{\infty}$ exhibits time consistency.

Now suppose that $\{\succ^t\}_{t=0}^{\infty}$ exhibits times consistency. In particular, this means that ${}_1c \sim^1 {}_1c'$ if and only if $(c_0, {}_1c) \sim^0 (c_0, {}_1c')$. Therefore, for every $(U({}_1c), U({}_2c), \ldots)$ and $(U({}_1c'), U({}_2c'), \ldots)$ that satisfy $U({}_1c) = U({}_1c')$,

$$V(c_0, U({}_1c), U({}_2c), \ldots) = V(c_0, U({}_1c'), U({}_2c'), \ldots).$$

So V can depend only on its first two arguments. Similarly, ${}_1c \succ^1 {}_1c'$ if and only if $(c_0, {}_1c) \succ^0 (c_0, {}_1c')$. Therefore, $U({}_1c) > U({}_1c')$ implies that $V(c_0, U({}_1c)) > V(c_0, U({}_1c'))$; that is, V must be strictly increasing in its second argument.

APPENDIX G: PROOF OF LEMMA 17

Recall that for any $\nu' > \nu$ in $\mathcal{U}$,

$$G(\nu') - G(\nu) < \frac{1-\alpha}{\alpha}(\nu' - \nu).$$

We will show that, for any $\varepsilon > 0$ small enough, there exists a constant $K < \frac{1-\alpha}{\alpha}$ such that

$$G(\nu') - G(\nu) \leq \max\{K(\nu' - \nu), \varepsilon\} \tag{G.1}$$

for all $\nu' > \nu$ in $\mathcal{U}$.

Case (i): Suppose first that $\mathcal{U}$ is bounded and let $\overline{\mathcal{U}} = \text{cl}(\mathcal{U})$. If necessary, extend G to $\overline{\mathcal{U}}$ by continuity. Since $\overline{\mathcal{U}}$ is compact and G is continuous, it is also uniformly continuous. Hence, for any $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that $|\nu - \nu'| < \eta(\varepsilon)$ implies $|G(\nu) - G(\nu')| < \varepsilon$. Let $\Delta(\varepsilon) = \{(\nu, \nu') \in \overline{\mathcal{U}}^2 \mid \nu \geq \nu' + \eta(\varepsilon)\}$. The function $F(\nu, \nu') = \frac{G(\nu)-G(\nu')}{\nu-\nu'}$ is continuous and strictly less[46] than $\frac{1-\alpha}{\alpha}$ on the compact set $\Delta(\varepsilon)$ and thus has a strictly positive upper bound $K < \frac{1-\alpha}{\alpha}$. By construction, (G.1) holds for any $(\nu, \nu') \in \Delta(\varepsilon)$ and any $(\nu, \nu') \in \overline{\mathcal{U}}^2 \setminus \Delta(\varepsilon)$.

Case (ii): Suppose that $\mathcal{U}$ is unbounded both above and below—the intermediate cases follow by combining the two cases shown here. Let $\underline{G} = \inf_{\nu \in \mathcal{U}} G(\nu)$ and $\overline{G} = \sup_{\nu \in \mathcal{U}} G(\nu)$,

[46]This is true by assumption if ν and ν' belong to $\mathcal{U}$, and it is easy to show that it is still true if either ν or ν' belongs to $\overline{\mathcal{U}} \setminus \mathcal{U}$. For example, if ν' is the infimum of $\mathcal{U}$, one can take any point $\tilde{\nu} \in (\nu', \nu)$. By assumption, $G(\nu) - G(\tilde{\nu}) < (1-\alpha)/\alpha(\nu - \tilde{\nu})$ and, by continuity of G, $G(\tilde{\nu}) - G(\nu') \leq (1-\alpha)/\alpha(\tilde{\nu} - \nu')$. Combining these inequalities yields the result, as is easily seen. (One way of showing this is to use the fact that $a/b < c/d \Rightarrow (a+b)/(c+d) < c/d$ for a, b, c, d strictly positive—see the argument at the end of this proof.)

which are finite and distinct because G is bounded and strictly increasing. Fix any $\varepsilon < \overline{G} - \underline{G}$. Let $\underline{\nu}(\varepsilon) = G^{-1}(\underline{G} + \varepsilon)$ and $\overline{\nu}(\varepsilon) = G^{-1}(\overline{G} - \varepsilon)$. If either $\nu \leq \underline{\nu}(\varepsilon)$ and $\nu' \leq \underline{\nu}(\varepsilon)$, or $\nu \geq \overline{\nu}(\varepsilon)$ and $\nu' \geq \overline{\nu}(\varepsilon)$, then (G.1) holds by construction. Now take any $\overline{\upsilon}, \underline{\upsilon} \in \mathcal{U}$ with $\overline{\upsilon} > \overline{\nu}(\varepsilon) + 2(\frac{\alpha\varepsilon}{1-\alpha} + 1)$ and $\underline{\upsilon} < \underline{\nu}(\varepsilon) - 2(\frac{\alpha\varepsilon}{1-\alpha} + 1)$. On the compact set $[\underline{\upsilon}, \overline{\upsilon}]$, the continuous function G is uniformly continuous, so there exist $\eta > 0$ and $\eta(\varepsilon) = \min\{\eta, \frac{1}{2}(\overline{\upsilon} - \overline{\nu}(\varepsilon)), \frac{1}{2}(\underline{\nu}(\varepsilon) - \underline{\upsilon})\}$ such that $|\nu - \nu'| < \eta(\varepsilon)$ implies $|G(\nu) - G(\nu')| < \varepsilon$. Let $\Delta'(\varepsilon) = \{(\nu, \nu') \in [\underline{\upsilon}, \overline{\upsilon}]^2 \mid \nu \geq \nu' + \eta(\varepsilon)\}$. By the same argument as before, the function $F(\nu, \nu') = \frac{G(\nu)-G(\nu')}{\nu-\nu'}$ has a strictly positive upper bound $K_1 < \frac{1-\alpha}{\alpha}$ on the set $\Delta'(\varepsilon)$.

Define $\overline{\upsilon}_m = \frac{1}{2}(\overline{\upsilon} + \overline{\nu}(\varepsilon))$ and $\underline{\upsilon}_m = \frac{1}{2}(\underline{\upsilon} + \underline{\nu}(\varepsilon))$. The only difficulty is to show the claim when $\nu' < \overline{\nu}(\varepsilon) \leq \overline{\upsilon} < \nu$ or $\nu' < \underline{\upsilon} \leq \underline{\nu}(\varepsilon) < \nu$. We focus on the first case. If $\nu' < \overline{\upsilon}(\varepsilon)$, by construction $\overline{\upsilon}_m - \nu' \geq \eta(\varepsilon)$ and hence

$$\frac{G(\overline{\upsilon}_m) - G(\nu')}{\overline{\upsilon}_m - \nu'} < K_1. \tag{G.2}$$

Now note that

$$\nu - \overline{\upsilon}_m > \overline{\upsilon} - \overline{\upsilon}_m = \frac{1}{2}\big(\overline{\upsilon} - \overline{\nu}(\varepsilon)\big) > \frac{\alpha\varepsilon}{1-\alpha} + 1.$$

Hence, there exists a strictly positive $K_2 < \frac{1-\alpha}{\alpha}$ such that, for all $\nu > \overline{\upsilon}$, we have $\nu - \overline{\upsilon}_m > \varepsilon / K_2$. Since $\nu > \overline{\nu}(\varepsilon)$ and $\overline{\upsilon}_m > \overline{\nu}(\varepsilon)$, it follows that

$$\frac{G(\nu) - G(\overline{\upsilon}_m)}{\nu - \overline{\upsilon}_m} \leq \frac{\varepsilon}{\nu - \overline{\upsilon}_m} < K_2. \tag{G.3}$$

For any strictly positive a, b, c, d, $(a + c)/(b + d) \leq \max\{a/b, c/d\}$. Combining this inequality to (G.2) and (G.3), we conclude that

$$\frac{G(\nu) - G(\nu')}{\nu - \nu'} \leq \max\{K_1, K_2\}.$$

By a similar argument, for all $\nu' < \underline{\upsilon} \leq \underline{\nu}(\varepsilon) < \nu$,

$$\frac{G(\nu) - G(\nu')}{\nu - \nu'} \leq \max\{K_1, K_3\}$$

for some strictly positive $K_3 < \frac{1-\alpha}{\alpha}$. Letting $K = \max\{K_1, K_2, K_3\}$ then proves the claim of the lemma.

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