ONLINE APPENDIX FOR UNCERTAINTY AND UNEMPLOYMENT

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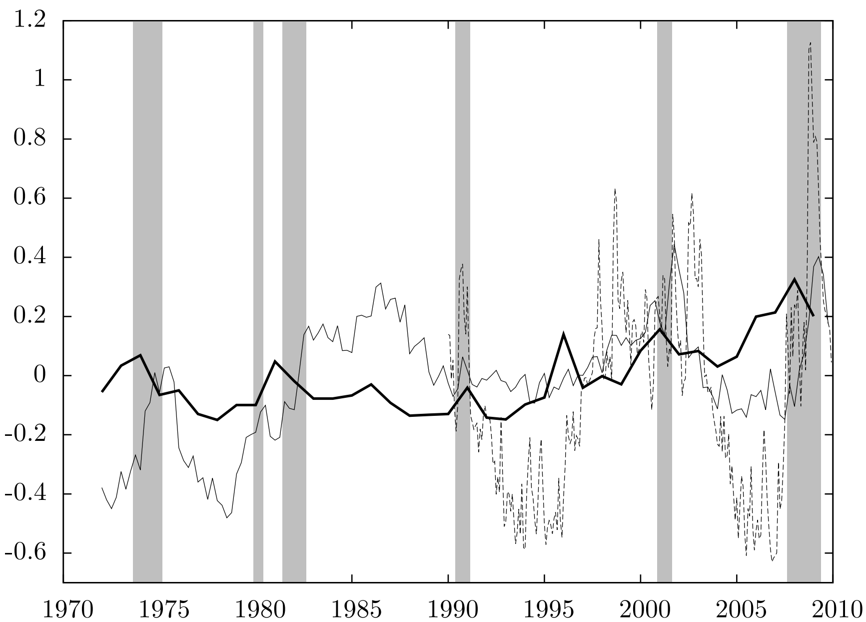
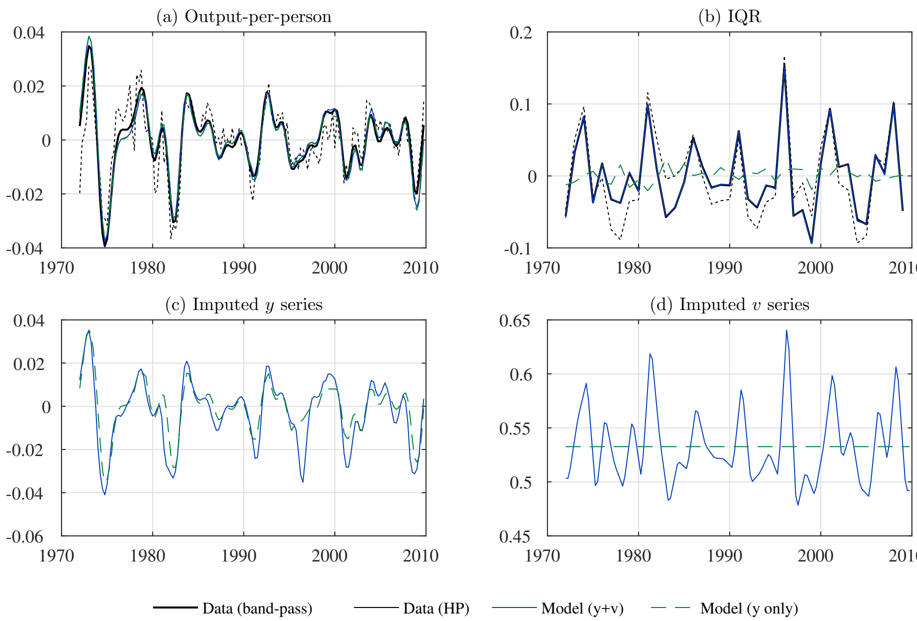
D Additional Figures

Figure 16: Various measures of micro-level risk



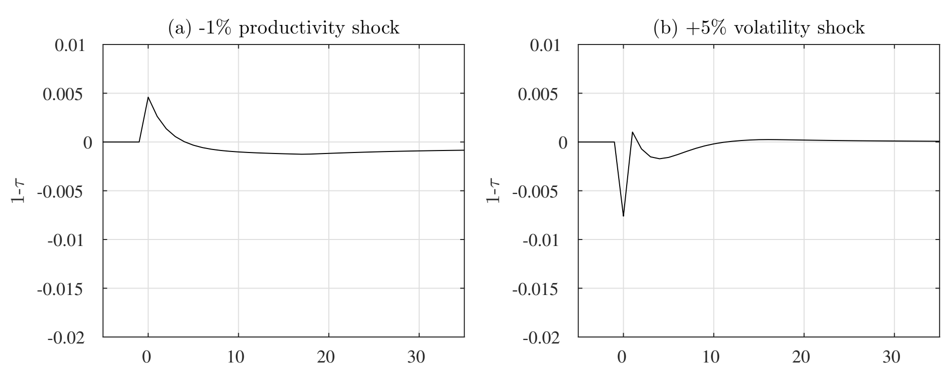
Notes: Data are shown in log deviations from their long-run averages. The thick curve shows the idiosyncratic risk measure from Census data constructed by Bloom et al. (2012); the thin curve shows the cross-sectional dispersion of annual sales growth from Compustat; the dashed line represents the VIX measure constructed by the CBOE. Shaded areas correspond to NBER recessions. See Appendix C for details.

Figure 17: Fit and imputed shocks for the counterfactual exercise



Notes: The black continuous line presents the data detrended using a band-pass filter (6 to 32 quarters for quarterly data, 2 to 8 years for annual data), the dotted black line is the data detrended using an HP filter (smoothing parameter 1600 quarterly, 100 annual), the blue continuous line corresponds to the model with productivity and volatility shocks, the green dashed line is the model with productivity shocks only. Note that the large peak in the IQR series in 1996 is an artifact of the Census data due to the change from SIC87 to NAICS classification in 1997, which biases the measure upward by more than 5% as reported in Bloom et al. (2012).

Figure 18: Response of the labor wedge to aggregate productivity and volatility shocks



Notes: The labor wedge is the ratio between the marginal rate of substitution between consumption and leisure to the marginal productivity of labor. Series presented in log deviation from their steady state values when aggregate productivity and volatility are set to their means. The time period is a month and the shock hits at time $t = 0$.

E Numerical Implementation

This section describes the implementation of the model that I use for the quantitative exercises.

E.1 Description of the problem

Under the stochastic processes chosen in section 3, the aggregate state of nature is $s = (y, v)$. Since all the contracting aspects are absent from the joint surplus maximization problem, it is more convenient to solve for the surplus (7) at the beginning of a period in stage A instead of stage B. Define the surplus $\mathbf{V}^A$ in stage A as follows:

$$\begin{aligned}\mathbf{V}^A(y,v,z,n) = \max_{\substack{n_i,\, x_i,\, \tau \\ x,\, d\, \in\, \{0,1\}}} \quad & n\mathbf{U}(y,v)d + (1-d)\Big\{ n\tau\mathbf{U}(y,v) - \kappa\left(y,v\right)n_i \\ & + n\left(1-\tau\right)\lambda p\left(\theta\left(y,v,x\right)\right)x + \mathrm{e}^{y+z}F(n') - k_f \\ & + \beta\mathbb{E}\mathbf{V}^A\left(y',v',z',n'\right)\Big\}\end{aligned} \tag{17}$$

subject to

$$n' = n(1-\tau)\Big(1 - \lambda p\big(\theta\left(y,v,x\right)\big)\Big) + n_i,$$

where n denotes the employment level reached at the end of the previous period.[1] Note that I have used the properties from proposition (1) that $x\left(j\right)$ is uniform across workers, $x\left(j\right) = x, \forall j$, and that the distribution of layoffs across workers is undetermined to impose symmetry in the layoffs rates, $\tau\left(j\right) = \tau, \forall j$. Notice also that I have used the definition of (9) to substitute for the hiring cost $\kappa\left(y,v\right)$. The hiring costs is implicitly defined by the free-entry problem of (12),

$$\begin{aligned}k_e = \sum_{z\in\mathcal{Z}} g_z\left(z\right)\Bigg\{ \max_{\substack{n_e\left(y,v,z\right),\, x_e\left(y,v,z\right), \\ d_e\left(y,v,z\right)\, \in\, \{0,1\}}} \quad & \left(1-d_e\left(y,v,z\right)\right)\Big[e^{y+z}F\left(n_e\left(y,v,z\right)\right) - k_f \\ & -\kappa\left(y,v\right)n_e\left(y,v,z\right) + \beta\mathbb{E}\mathbf{V}^A\left(y',v',z',n_e\left(y,v,z\right)\right)\Big]\Bigg\}, \quad \forall\left(y,v\right).\end{aligned} \tag{18}$$

The equilibrium market tightness implied by the free-entry condition is defined by combining equations (10) and (11), so that

$$\theta\left(y,v,x\right) = \begin{cases} q^{-1}\left(\frac{c}{\kappa(y,v)-x}\right) & \text{for active markets } \left(x \le \kappa\left(y,v\right) - c\right) \\ 0 & \text{for inactive markets } \left(x > \kappa\left(y,v\right) - c\right) \end{cases}$$

Finally, the value of unemployment is defined by (1),

$$\mathbf{U}(y,v) = \max_{x_u(s')} b + \beta\mathbb{E}\left[p\left(\theta(s', x_u(s'))\right)x_u\left(s'\right) + \Big(1 - p\left(\theta\left(s', x_u\left(s'\right)\right)\right)\Big)\mathbf{U}(y',v')\right]. \tag{19}$$

[1] Under this notation, surplus at stage A of a period ($\mathbf{V}^A$) and surplus at stage B ($\mathbf{V}$) are related in the following way:

$$\mathbf{V}(y,v,z,n) \quad = \quad e^{y(s)+z}F\left(n\right) - k_f + \beta\mathbb{E}\mathbf{V}^A\left(y',v',z',n\right),$$

and

$$\begin{aligned}\mathbf{V}^A(y,v,z,n) = \max\, & n\mathbf{U}(y,v)d + (1-d)\left\{n\tau\mathbf{U}(y,v) + n\left(1-\tau\right)\lambda p\left(\theta\left(y,v,x\right)\right)x\right. \\ & \left. -\kappa\left(y,v\right)n_i + \mathbf{V}\left(y,v,z,n'\right)\right\}.\end{aligned}$$

E.2 Algorithm

The problems of (17), (18) and (19) define three nested fixed point problems that we must solve to find a quasi-equilibrium. I describe below the algorithm that I use to solve for them. The value functions are computed on a $ny \times nv \times nz \times nn$ grid (`ny=21; nv=15; nz=15; nn=30` in my baseline calibration).

1. Set $k = 0$. Guess a value function $V^{(0)}(y, v, z, n)$;

2. Using the free-entry condition, solve numerically for $\kappa^{(k)}(y, v)$ such that

$$\begin{aligned} k_e &= \sum_z g_z(z) \left[\max_{n_e(y,v,z)} e^{y+z} F(n_e(y, v, z)) - k_f - \kappa(y, v) n_e(z) \right. \\ &\quad \left. + \beta \mathbb{E} V^{(k)}(y', v', z', n_e(y, v, z)) \right]^+, \ \forall (y, v); \end{aligned}$$

   The RHS of this equation being monotonic in κ, I use a quick bisection method for that step. Save the decision rules $n_e(y, v, z)$ and $d_e(y, v, z)$. Using this new value of $\kappa^{(k)}(y, v)$, compute the equilibrium market tightness from (10) and (11):

$$\theta^{(k)}(y, v, x) = \begin{cases} q^{-1}\left(\frac{c}{\kappa^{(k)}(y,v)-x}\right) & \text{for } x \leq \kappa^{(k)}(y, v) - c \\ 0 & \text{for } x > \kappa^{(k)}(y, v) - c \end{cases}$$

3. By value function iteration, find the fixed point of the mapping,

$$U^{(k)}(y, v) = \max_{x'_u(y',v')} b + \beta \mathbb{E}\left[p(\theta(y', v', x_u)) x_u + \Big(1 - p(\theta(y', v', x_u))\Big) U^{(k)}(y', v') \right],$$

   and save the corresponding decision rule $x_u(y', v')$.

4. Compute one iteration of the mapping:

$$\begin{aligned} V^{(k+1)}(y, v, z, n) &= \max_{\tau, x, n_i} \Big\{ n\tau U^{(k)}(y, v) + n(1 - \tau) \lambda p\left(\theta^{(k)}(y, v, x)\right) x \\ &\quad - \kappa^{(k)} n_i + e^{y+z} F(n') - k_f + \beta \mathbb{E} V^{(k)}(y', v', z', n') \Big\}^+ \\ \text{s.t.} \quad & n' = n(1 - \tau)\Big(1 - \lambda p\big(\theta^{(k)}(y, v, x)\big)\Big) + n_i \end{aligned}$$

   and save the corresponding decision rules $n'(y, v, z, n)$, $n_i(y, v, z, n)$, $x(y, v, z, n)$, $\tau(y, v, z, n)$ and $d(y, v, z, n)$.

5. Stop if $||V^{(k+1)} - V^{(k)}|| \leq \varepsilon$. Otherwise, go back to step 2 with $k \leftarrow k + 1$.

E.3 Additional remarks

A number of remarks are in order:

- For the distribution of entrants g_z, I pick the stationary distribution of z when volatility v is held constant, equal to its mean $\bar{v}$;
- The choice over x_u and x has to be computed very precisely:
  - in step 3, I use the first order condition of the maximization problem and solve for the value of $x_u(y', v')$ using a bisection algorithm;

– in step 4, to simplify the maximization over (x, τ, n_i), I proceed in two steps:

  ∗ for all pairs (n, n') on the $nn \times nn$ grid, compute $r = \frac{n'}{n}$. If $r < 1$, solve the subproblem

$$\begin{aligned} \omega(y, v, r) &= \max_{x,\tau} \tau U^{(k)}(y, v) + (1-\tau)\lambda p\left(\theta^{(k)}(y, v, x)\right)x \\ \text{s.t.} \quad & (1-\tau)\left(1 - \lambda p\left(\theta^{(k)}(y, v, x)\right)\right) = r, \end{aligned}$$

  which yields the optimal mix of layoffs/quits for a given (n, n'). Save the decision rules $x(y, v, r)$, $\tau(y, v, r)$ and the value $\omega(y, v, r)$. If $r \geq 1$, set $\tau(y, v, r) = 0$, $x(y, v, r) = \kappa^{(k)} - c$ and $\omega(y, v, r) = 0$. This problem can be solved quite accurately using its first order conditions;

  ∗ using this optimal mix, the maximization of step 4 can be turned into the simple one-dimensional maximization problem:

$$\begin{aligned} V^{(k+1)}(y, v, z, n) &= \max_{n'} \Big\{ \mathrm{e}^{y+z} F(n') - k_f - \kappa^{(k)}(n' - n)^+ \\ & \quad + n\omega\left(y, v, \frac{n'}{n}\right) + \beta \mathbb{E} V^{(k)}(y', v', z', n') \Big\}^+. \end{aligned}$$

  This procedure provides a very accurate and smooth solution. Because of the reduction of the state-space, it also runs very quickly.

- I use two cubic splines in step 4 to smooth the choice of $n'(y, v, z, n)$ over $[0, n]$ and $[n, \bar{n}]$;
- The whole algorithm takes about 20 minutes to converge for the baseline calibration on my Dell Precision T7600.

E.4 Computing wages

Section F.2 proposes a version of the model without commitment on the worker side in which wages are uniquely determined. This subsection describes how one can easily compute wages from the surplus maximizing allocation. In what follows, it is convenient to use the timing introduced in subsection E.1, expressing value functions and policies at the beginning of a period (stage A).

We start by solving for the incentive constraint (21) described in F.2. For every state (y, v, z, n), compute the promised utility $W'(y, v, z, n)$ such that

$$x(y, v, z, n) = \underset{x}{\operatorname{argmax}}\, p(\theta(y, v, x))(x - W'(y, v, z, n)).$$

Because of the monotonicity of the problem, this can be done efficiently using a bisection method.

It is then useful to write the utility of a worker employed by a firm (z, n) at the beginning of a period (stage A). Define

$$\begin{aligned} \mathbf{W}^A(y, v, z, n) &= d(y, v, z, n)\mathbf{U}(y, v) + (1 - d(y, v, z, n))\Big[\tau(y, v, z, n)\mathbf{U}(y, v) \\ & \quad + (1 - \tau(y, v, z, n))\lambda p(\theta(y, v, x(y, v, z, n)))x(y, v, z, n) \\ & \quad + (1 - \tau(y, v, z, n))[1 - \lambda p(\theta(y, v, x(y, v, z, n)))]W'(y, v, z, n), \end{aligned}$$

where $W'(y, v, z, n)$ is the promised utility at the end of the period. It is now easy to solve for wages. We can use the promise-keeping constraint (6) to derive their wages:

$$w^{incumbent}(y,v,z,n) = W'(y,v,z,n) - \beta\mathbb{E}\left[\mathbf{W}^A(y',v',z',n'(y,v,z,n))\right].$$

Similarly, one can derive the wage of workers hired from unemployment with promised utility $x_u(y,v)$:

$$w^{unemp}(y,v,z,n) = x_u(y,v) - \beta\mathbb{E}\left[\mathbf{W}^A(y',v',z',n'(y,v,z,n))\right].$$

Finally, a worker successfully moving from a firm with state $(\tilde{z},\tilde{n})$ to a firm with state (z,n), hired with promised utility $x(y,v,\tilde{z},\tilde{n})$ receives the wage

$$w^{j2j}(y,v,z,n;\tilde{z},\tilde{n}) = x(y,v,\tilde{z},\tilde{n}) - \beta\mathbb{E}\left[\mathbf{W}^A(y',v',z',n'(y,v,z,n))\right].$$

F Additional Theoretical Results

F.1 Properties of the optimal contracts

This section characterizes various properties of the equilibrium contracts and, in particular, how different elements of the contracts (layoff probability τ, market for on-the-job search x, etc.) vary across workers within a single firm.

Proposition 1. *Under the conditions of proposition 2, in a quasi-equilibrium with surplus maximizing policy* $\{\{\tau_j, x_j\}_{j\in[0,n]}, d, n_i, x_i\}$ *the following is true:*

(i) If workers can commit, wages are not uniquely determined. In particular, the transformation $\{w_j + a\Delta, \tau_j, x_j, W'_j - \Delta, d\}$ *leaves worker* j *and the firm indifferent, with* $a = \beta\mathbb{E}(1-d)(1-\tau_j)(1-\lambda p(\theta(s',x_j)))]$ *and* $\Delta\in\mathbb{R}$*;*

(ii) The market for on-the-job search x *is identical for all workers in the same firm;*

(iii) Only the total number of layoffs $\int\tau_j \mathrm{d}j$ *is uniquely determined; the distribution of layoffs* $\{\tau_j\}_{j\in[0,n]}$ *over workers is not.*

Proposition 1 first establishes that wages, w, and continuation values, W', are not unique. There are two reasons behind this result: i) workers and firms are risk neutral and ii) there is commitment from both workers and firms. Under these two conditions, the timing of wages is irrelevant. Only the total discounted value of future wages upon hiring is determined in equilibrium. This result shows the flexibility of the setup proposed in this paper as it can accomodate various profiles of wages over the life-cycle. I propose one particular way to determine wages in section F.2 by relaxing the commitment assumption on the worker side. In that case, the incentive problem uniquely pins down wages and I explore the quantitative properties of that particular assumption in section G.

Second, this proposition shows that all workers within a firm search on the same labor market segment. This result is due to the strict concavity of the search problem. Finally, as was suggested in proposition 1, the distribution of layoff probabilities across workers of a given firm is not uniquely determined. As is evident from the definition of the joint surplus, any permutation or convex combination of these probabilities across workers leaves the surplus unchanged. However, the total number of layoffs at the firm level is uniquely determined.

F.2 Relaxing commitment and completeness

I present in this section an extension of the model in which I relax the assumption of commitment on the worker side and the completeness of contracts. These assumptions may seem, indeed, somewhat unrealistic. First, I show in this subsection that commitment on the worker side is not required because firms have enough instruments to write incentive-compatible contracts that implement the efficient allocation. Second, I prove that firms may write down contracts that only specify $\{w, \tau(s', z'), d(s', z'), W'(s', z')\}$. In particular, this means that firms do not have to specify the labor market segment $x(s', z')$ in which their workers should be searching on the job—arguably the most unrealistic feature of the form of contracts assumed so far. Under the incentive-compatible contracts, firms can balance the current wage vs. continuation utility in such a way that workers choose to search in the optimal submarket.

Notice, however, that commitment on the firm side cannot be relaxed without losing block recursivity. Indeed, as discussed in the main text, it is key for block recursivity to obtain that firms stick to the contracts they advertise. Without commitment, firms would pay wages to workers that make them indifferent between a new job and their current situation. In particular, a firm would need to know the distribution of workers across firms before making its hiring decision, thereby breaking our main tractability results.

If we relax the assumption of commitment on the worker side, two additional constraints arise in the design of the contract. When workers are employed, the firm is worried about two things: 1) either the worker does not want to stay in the firm and decides to return to unemployment at the time when separations take place, or 2) the worker would like to search on a different submarket than the one specified in the contract. When designing a contract $(w, \tau(s', z'), x(s', z'), W'(s', z'), d(s', z'))$, the firm must take into consideration a participation constraint,

$$\lambda p(\theta(s', x))x + (1 - \lambda p(\theta(s', x)))W'(s', z') \geq \mathbf{U}(s'), \forall s' \tag{20}$$

which makes sure that the worker does not prefer to return to unemployment, and we have the following incentive constraint,

$$\begin{aligned} x(s', z') &= \underset{\tilde{x}}{\operatorname{argmax}}\ \lambda p(\theta(s', \tilde{x}))\tilde{x} + \big(1 - \lambda p(\theta(s', \tilde{x}))\big)W'(s', z') \\ \Leftrightarrow x(s', z') &= \underset{\tilde{x}}{\operatorname{argmax}}\ p(\theta(s', \tilde{x}))(\tilde{x} - W'(s', z')), \end{aligned} \tag{21}$$

which verifies that the submarket x specified in the contract coincides with the one chosen by the worker. I now show that for any given contract (w, τ, x, W', d), there is a unique equivalent contract with wage w_{IC} and future utility W'_{IC} that satisfies the above incentive and participation constraints and delivers the same promised utility to the worker.

Proposition 2. *For any optimal contract $\omega = \{w, \tau, x, W', d\}$, there exists a unique equivalent incentive-compatible contract $\omega_{IC} = \{w_{IC}, \tau_{IC}, W'_{IC}, d_{IC}\}$ such that $\forall(s', z')$:*

1. $\tau_{IC}(s', z') = \tau(s', z')$ *and* $d_{IC}(s', z') = d(s', z')$,
2. $\lambda p(\theta(s', x(s', z')))x(s', z') + (1 - \lambda p(\theta(s', x'(s', z'))))W'_{IC}(s', z') \geq \mathbf{U}(s')$,
3. $x(s', z') = \underset{\tilde{x}}{\mathit{argmax}}\ p(\theta(s', \tilde{x}))(\tilde{x} - W'_{IC}(s', z'))$,
4. $\mathbf{W}(s, z, \omega) = \mathbf{W}(s, z, \omega_{IC})$.

Proposition 2 tells us that the allocation that maximizes the worker-firm joint surplus can be implemented by an incentive-compatible contract. In particular, the layoff and exit probabilities are the same: $\tau_{IC} = \tau$, $d_{IC} = d$, and the submarket x chosen by the worker coincides with the efficient one. The wage and future utility (w_{IC}, W'_{IC}) are the only elements that adjust to ensure that the two additional constraints (21) and (20) are satisfied. In addition to being more realistic than complete contracts with full commitment, these contracts offer the advantage of uniquely pinning down wages. They thus offer an alternative to other wage determination procedures. Appendix E.4 presents to numerically implement this procedure. Appendix G shows that the wages this procedure implies match a number of empirical facts, such as a realistic wage dispersion and size-wage differential.

G Wage Predictions

The use of optimal dynamic contracts in search models provides an alternative to the standard assumptions of Nash or Stole and Zwiebel bargaining. However, as shown in proposition 1, wages are not uniquely pinned down if workers can commit to stay in the firm and search on the optimal labor market while employed. In section F.2 of the Appendix, I show how relaxing this commitment assumption yields a unique characterization of wages and contracts, as employers have to design contracts that give the right incentives for workers to stay/leave the firm and apply to the right labor market. Under this specification, wages could in principle vary substantially across workers belonging to the same firm. I explore in this section the quantitative implications of this wage setting mechanism. Because of a rich incentive structure, the model is able to predict an important wage dispersion for observationally equivalent workers and accounts for larger fraction of the empirical variation than standard search model. It also predicts a quantitatively accurate size-wage differential.

G.1 Wage dispersion and elasticity

Hornstein et al. (2011) report that standard calibrations of search-and-matching models without on-the-job search cannot generate much dispersion in wages. In their basic calibration of a standard random search model, they obtain a mean-min ratio of 1.036 for wages, while their preferred empirical estimate is about 1.70 with a corresponding coefficient of variation of only 1/12th of the variation in the data. Using wage data from the 1990 Census with different sets of controls, they estimate an empirical coefficient of variation of residual wages ranging from 0.35 to 0.49. I estimate the same dispersion measure in my model by simulating over a large number of periods and obtain an average coefficient of variation of 0.20, which explains between 41% and 57% of the observed residual dispersion in wages, outperforming standard search-and-matching models.

Regarding the evolution of wages over the business cycle, the average wage appears highly procyclical. The elasticity of wages with respect to productivity (output per person) is close to 1 in my model, slightly higher than the elasticity of wages for new hires of 0.79 estimated in Haefke et al. (2013) using CPS data. However, without any explicit mechanism for wage stickiness, the model is unable to replicate the elasticity for all the workers in the CPS, estimated at 0.24 by the same authors. An interesting extension would be to introduce risk aversion for workers. Combined with the dynamic contracting framework of the model, this extension would connect search theory to the implicit contract literature and provide us with a theory of endogenous stickiness, in which case this dimension could be significantly improved.

Turning to earnings risk over the business cycle, Guvenen et al. (2014) report, using administrative data, that the distribution of transitory shocks to log earnings are negatively skewed with a skewness ranging between -0.08 and -0.23. Computing annual growth im log earnings in my model, I find an average skewness of -0.04 that can fall as low as -0.27 over long simulations. However, the model is unable to produce the same cyclicality of earnings risk described by the same authors. They find substantial evidence of countercyclical risk in the left-tail of earnings shocks. My model predicts a non-negligible time-variation in earnings risk (8% standard deviation in the dispersion of transitory log earnings shocks). Consistent with their findings, the right-tail risk, measured by the difference between the 90th percentile (P90) and the 50th percentile (P50) in log earnings growth, is procyclical. However, the left-tail risk, measured by P50 - P10 (10th percentile), is not countercyclical, as the authors show, but procylical in my model. The reason behind this failure appears to stem from the feature of the model, shared by most search models, that the value of earnings by unemployed workers, b in my notation, is constant over the cycle. As a result, workers in the model face strong procyclical upside risk due to the many opportunities to climb the job ladder in good times, but face little downside risk in recessions as the value of unemployment bounds earnings losses from below.

G.2 Size-wage differential

A common finding in the literature is that firm size can explain part of the variation in wages. Brown and Medoff (1989) report that, in a variety of datasets, a substantial size-wage differential remains despite various controls for labor quality and institutions: employees working at large firms earn higher wages than employess at small firms. To investigate whether the model can reproduce this finding, I compute the wages in every establishment at the aggregate steady state. I then run the following regression,

$$\log(wage) = \alpha + \beta \log(employment) + \varepsilon,$$

and evaluate by how much the wage of a worker varies with establishment size. I obtain a coefficient $\beta = 0.008$, about half of the estimate of 0.014 reported in that paper. Interestingly, this size-wage differential can be explained by a mechanism due to search frictions quite different from standard explanations based on labor quality or institutions. The mechanism at work in the model is due to the way firms deal with worker incentives. In this economy, firms that want to expand prefer to retain their current workers in order to save on hiring costs. To do so, they must promise them higher continuation utility. Therefore, all other things being equal, firms that grow tend to offer higher wages on average than firms that shrink. Turning back to firm size, large firms are those that received high idiosyncratic shocks and have grown in the recent past. As a result, they inherit high-paying contracts from the previous periods and tend to pay high wages. This mechanism emphasizes establishment growth as a key determinant for wages. Schmieder (2009) finds supporting evidence in German matched employer-employee data that fast growing establishment offer higher wages.

G.3 Relationship to implicit contract literature

The contracting framework used in this paper is reminiscent of the implicit contract literature initiated by Baily (1974) and Azariadis (1975). These articles considered the optimal contractual arrangement between risk-neutral firms and risk-averse workers and determined conditions under which the optimal contract insulated workers from aggregate labor market conditions by offering rigid wages. The question whether wages are set by spot markets or implicit contracts inspired a large empirical literature led by Beaudry and DiNardo (1991) that derived simple testable implications of both theories and applied them on US panel data. In particular, Beaudry and DiNardo (1991) showed that wages determined on spot markets should solely adjust to current labor market conditions, while wages determined by implicit contracts should display history dependence. Using the aggregate unemployment rate as a proxy for labor market conditions, the authors designed a simple empirical test by running panel regressions of log wages on current unemployment (spot market model), unemployment at the start of the job (contract model with low mobility) and the minimum unemployment rate since the start of the job (contract model with high mobility) in addition to a vector of individual characteristics. Their results showed a greater dependence of wages on past rather than current unemployment rates, offering support to the contracting approach.

These results were later criticized by Hagedorn and Manovskii (2013) who argued that such dependence of wages on past unemployment rates could be driven by selection and was consistent with a search model where wages depended solely on current labor market conditions. They showed in particular that past unemployment rates were a proxies for match quality and that using better measures of match quality virtually eradicated the dependence on past unemployment rates.

In this paper, wages are determined through long-term contracts. However, several features distinguish this framework from the implicit contract literature. First, workers are risk neutral, so that the motive for firms to insure their workers against income risk is absent. Second, the frictions faced at the contracting stage are different: under lack of commitment from workers, as considered in sections F.2 and G, firms use wages to incentivize workers to stay or direct their search on the job to some specific market segments. In the resulting incentive-compatible contract, wages are uniquely determined and solely depend on a firm's state at the beginning of a period $(s', z'; n)$. Wages are, in particular, independent from past unemployment rates. In that sense, this paper is closer to the search model of Hagedorn and Manovskii (2013) in which

Table 1: Results from simulated wage regressions

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Contemporaneous unemployment rate	-4.332***			-3.535***	0.010		
	(0.013)			(0.030)	(0.040)		
Unemployment at start of job		-3.917***		-0.063***		-0.020	
		(0.013)		(0.034)		(0.024)	
Minimum unemployment since start of job			-4.285***	-0.888***			0.013
			(0.014)	(0.041)			(0.030)
Aggregate productivity y_t					1.171***	0.957***	0.986***
					(0.008)	(0.026)	(0.026)
Volatility v_t					0.241***	0.201***	0.211***
					(0.003)	(0.005)	(0.005)
Output-per-person						0.078***	0.071***
						(0.009)	(0.009)
Job tenure							0.002***
							(2.9e-5)
Constant	0.769***	0.753***	0.755***	0.772***	0.445***	0.294***	0.285***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.018)	(0.018)

Notes: Standard errors are in parentheses. The dependent variable is the logarithm of monthly wages simulated from a population of 1000 workers for 1200 periods (100 years). Output-per-person is the aggregate output divided by employment in a given period. The job tenure variable is the number of months less than a year that a worker has spent in the same job.

wages only depend on current conditions, but in which the dynamic matching of workers with firms over the business cycle leads to a dynamic selection of jobs consistent with the above results.

Simulating a population of workers from my model for a large number of periods, I first replicate the results from Beaudry and DiNardo (1991) by running the same regressions on simulated wages in Table 1. Consistent with their findings, I find a large negative, significant impact of current and past unemployment rates on wages in columns 1 to 3. Testing the three specifications at the same time in column 4, current, initial and minimum unemployment rates all preserve their negative, highly significant impact. However, consistent with the findings of Hagedorn and Manovskii (2013), this dependence is largely driven by spurious correlations and selection, to the extent that past unemployment rates correlate with the distribution of existing jobs. Controlling for the aggregate state of the economy as captured by the two shocks y_t and v_t, column 5 shows that the dependence on the current unemployment rate vanishes. Next, controlling for a measure of productivity for current existing jobs in column 6, output-per-person in the present case, cancels out the dependence on the unemployment rate at the start of the job.[2] Similarly, my findings suggest that the minimum unemployment rate also proxies for match quality: a low minimum unemployment rate, distinct from the current rate, proxies for a long tenure in a given job. Long tenures indicate good matches and higher wages. Adding a control for job tenure in column 7 eradicates the dependence on the minimum unemployment rate.

As a conclusion, this model is able to replicate the observation of history dependence of wages from Beaudry and DiNardo (1991), but this dependence is driven by the dynamic selection of jobs, consistent with the recent findings of Hagedorn and Manovskii (2013).

[2]Aggregate conditions in the past, as measured by the unemployment rate at the start of the job, have an impact on the current distribution of jobs through the type and employment of firms that entered/exited in the past. My result suggests that the initial unemployment rate proxies for the general productivity of matches in the pool of of existing jobs.

H Proofs

H.1 Proofs of part 2.6

Proof of proposition 1. Let me first introduce some notation. For a generic firm policy $\gamma = \left\{ \{\omega(j)\}_{j\in[0,n]}, d(s',z'), n_i(s',z'), x_i(s'$ define $\tilde{\mathbf{J}}(s,z,n,\gamma)$ the value of a firm evaluated at that policy in the current period:

$$\begin{aligned}\tilde{\mathbf{J}}(s,z,n,\gamma) &= \mathrm{e}^{y(s)+z}F(n) - k_f - \int_0^n w(j)\,\mathrm{dj} \\ &+\beta\mathbb{E}\left\{(1-d)\left(-n_i\frac{c}{q(\theta(s',x_i))} + \mathbf{J}\left(s',z',n',\left\{\hat{W}'(s',z';j')\right\}_{j'\in[0,n']}\right)\right)\right\},\end{aligned}$$

subject to (4) and (5). Define the corresponding surplus:

$$\begin{aligned}&\tilde{\mathbf{V}}(s,z,n,\gamma) \equiv \mathbf{J}(s,z,n,\gamma) + \int_0^n \mathbf{W}(s,z,\omega(j))\mathrm{d}j \qquad (22)\\ &= \mathrm{e}^{y(s)+z}F(n) - k_f + \beta\mathbb{E}\left\{nd\mathbf{U}(s') + (1-d)\left[\mathbf{U}(s')\int_0^n \tau\mathrm{d}j\right.\right.\\ &\quad + \int_0^n (1-\tau)\lambda p(\theta(s',x))\,x\mathrm{d}j - n_i\frac{c}{q(\theta(s',x_i))}\\ &\quad \left.\left.+\mathbf{J}\left(s',z',n',\left\{\hat{W}'(s',z';j')\right\}_{j'\in[0,n']}\right) + \int(1-\tau)(1-\lambda p(\theta(s',x)))\,W\mathrm{d}j\right]\right\}.\end{aligned}$$

Under this notation, for any optimal policy γ^*, we have $\mathbf{J}(s,z,n,\{W(j)\}_{j\in[0,n]}) = \tilde{\mathbf{J}}(s,z,n,\gamma^*)$. The proof proceeds in the following steps: a) I show that the promise keeping constraint for incumbent workers (6) must bind for any optimal policy γ^*, b) I show the equivalence between the maximization of $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{V}}$, c) I show how the maximization of $\tilde{\mathbf{V}}$ can be equivalently written under the form of equation (7).

a) We can write the firm's problem as

$$\begin{aligned}\mathbf{J}(s,z,n,\{W(j)\}_{j\in[0,n]}) &= \max_{\gamma} \tilde{\mathbf{J}}(s,z,n,\gamma)\\ \text{subject to} \quad & (4),\ (5) \text{ and}\\ & W(j) \leq \mathbf{W}\Big(s,z;\omega(j)\Big),\ \forall j\in[0,n].\end{aligned}$$

The wage $w(j)$ only appears linearly in the term $\int_0^n w(j)\mathrm{d}j$ and in the promise keeping constraint. In particular, it does not affect the incentive structure of the problem. It is therefore optimal to offer the lowest possible wage, so that the promise keeping constraint binds with equality. For a given policy $\gamma = \left\{\{w,\tau,x,W'\}_{j\in[0,n]}, d, n_i, x_i\right\}$, the optimal wage $w(j)$ is such that $W(j) = \mathbf{W}(s,z;\omega(j))$, i.e.,

$$\begin{aligned}w(j) &= W(j) - \beta\mathbb{E}\Big[\big(d(s',z') + (1-d(s',z'))\,\tau(s',z';j)\big)\mathbf{U}(s')\\ &+ (1-d(s',z'))(1-\tau(s',z';j))\lambda p(\theta(s',x(s',z';j)))\,x(s',z';j)\\ &+ (1-d(s',z'))(1-\tau(s',z';j))\big(1-\lambda p(\theta(s',x(s',z';j)))\big)W'(s',z';j)\Big], \forall j. \qquad (23)\end{aligned}$$

The firm's problem is thus equivalent to

$$\begin{aligned}\mathbf{J}(s,z,n,\{W(j)\}_{j\in[0,n]}) &= \max_{\substack{\{\omega(j)\}_{j\in[0,n]},\, d(s',z'),\\ n_i(s',z'),\, x_i(s',z')}} \tilde{\mathbf{J}}\left(s,z,n,\left\{\{\omega(j)\}_{j\in[0,n]},d,x_i,n_i\right\}\right)\\ \text{subject to} &\quad (4),\ (5) \text{ and } (23).\end{aligned}$$

b) Let me now define the surplus maximization problem

$$\begin{aligned}\mathbf{V}(s,z,n) &= \max_{\gamma=\left\{\{\omega(j)\}_{j\in[0,n]},\ d,n_i,x_i\right\}} \tilde{\mathbf{V}}(s,z,n,\gamma)\\ &\quad \text{subject to (4) and (5).}\end{aligned}$$

The surplus is invariant with the wage, so for any decision rules $\{\{\tau,x,W'\}_{j\in[0,n]},d,\ n_i,\ x_i\}$, it is always possible to set the wage $w(j)$ according to (23). In that case, from the definition of the surplus (22), :

$$\tilde{\mathbf{V}}(s,z,n,\gamma)=\tilde{\mathbf{J}}(s,z,n,\gamma)+\int_0^n W(j)dj.$$

In this equation, $\int_0^n W(j)dj$ is a predetermined constant. Therefore, it is absolutely equivalent to maximize the left hand side under constraints (4) and (5), as to maximize the right hand side under the same constraints with the addition of (23), which corresponds to the firm's problem according to step (i). We therefore conclude that

$$\mathbf{V}(s,z,n)=\mathbf{J}\left(s,z,n,\{W(j)\}_{j\in[0,n]}\right)+\int_0^n W(j)\,dj.$$

Any policy that solves the firm's problem must maximize the joint surplus. On the other hand, for any policy $\gamma=\left\{\{\tau,x,W'\}_{j\in[0,n]},d,n_i,x_i\right\}$ that maximizes the joint surplus, there exists a wage (set according to (23)) that maximizes the firm's profits.

c) Because of the above equivalence, we may now write the surplus maximization problem as

$$\begin{aligned}\mathbf{V}(s,z,n)= &\max_{\substack{\{\tau,x,W'\}_{j\in[0,n]},\\ d,n_i,x_i}} \mathrm{e}^{y(s)+z}F(n)-k_f+\beta\mathbb{E}\Bigg\{n\mathbf{U}(s')d\\ &+(1-d)\left[\mathbf{U}(s')\int_0^n\tau\mathrm{d}j+\int(1-\tau)\lambda p\left(\theta\left(s',x\right)\right)x\mathrm{d}j-n_i\frac{c}{q\left(\theta\left(s',x_i\right)\right)}\right.\\ &\left.\underbrace{+\mathbf{J}(s',z',n',\left\{\hat{W}'\left(s',z';j'\right)\right\}_{j'\in[0,n']})+\int(1-\tau)\left(1-\lambda p\left(\theta\left(s',x\right)\right)\right)W'\mathrm{d}j}_{=\mathbf{V}(s',z',n')-n_ix_i}\right]\Bigg\}.\\ = &\max_{\substack{\{\tau,x,W'\}_{j\in[0,n]},\\ d,n_i,x_i}} \mathrm{e}^{y(s)+z}F(n)-k_f+\beta\mathbb{E}\Bigg\{n\mathbf{U}(s')d+(1-d)\left[\mathbf{U}(s')\int_0^n\tau\mathrm{d}j\right.\\ &\left.+\int(1-\tau)\lambda p\left(\theta\left(s',x\right)\right)x\mathrm{d}j-n_i\left(\frac{c}{q\left(\theta\left(s',x_i\right)\right)}+x_i\right)+\mathbf{V}(s',z',n')\right]\Bigg\},\end{aligned}\tag{24}$$

subject to (4).

This expression shows that the distribution of continuation utilities, $\{W'(s',z';j)\}_{j\in[0,n]}$, is irrelevant for the joint surplus. The joint surplus maximization problem may be equivalently written as

$$\begin{aligned}\mathbf{V}(s,z,n)=\max_{\substack{d(s',z'),n_i(s',z'),x_i(s',z'),\\ \{\tau(s',z';j),x(s',z';j)\}_{j\in[0,n]}}} & \mathrm{e}^{y(s)+z}F(n)-k_f+\beta\mathbb{E}\Bigg\{nd\mathbf{U}(s') \\ & +(1-d)\left[\mathbf{U}(s')\int_0^n\tau\mathrm{dj}+\int_0^n(1-\tau)\lambda p\left(\theta\left(s',x\right)\right)x\mathrm{dj}\right. \\ & \left.-\left(\frac{c}{q\left(\theta\left(s',x_i\right)\right)}+x_i\right)n_i+\mathbf{V}\left(s',z',n'\right)\right]\Bigg\}\end{aligned}$$

subject to (4) which is the definition of the surplus in equation (7). Because the joint surplus does not depend on the distribution of contracts, we can conclude in particular that any combination of wages and continuation utilities, $\{w(j),W'(s'z';j)\}$ that satisfy (6) with equality implement the allocation and maximize profits. In practice, any profile of future promised utilities $\{W'(s',z';j)\}_{j\in[0,n]}$ may be implemented as long as wages are set according to (23). There is thus a multiplicity of contracts that implement the allocation and these contracts can easily be solved. $\square$

H.2 Proofs of part 2.9

This section demonstrates all the proofs of existence, efficiency and uniqueness.

H.2.1 Proposition 2.(i): Existence

Let me first introduce a number of assumptions and definitions required to show the existence of a solution to the free-entry condition and joint surplus maximization problem. Denote $\mathcal{Z}=\{\underline{z}<...<\overline{z}\}$, $\overline{y}=\max_{s\in\mathcal{S}}y(s)$ and $\underline{y}=\min_{s\in\mathcal{S}}y(s)$.

Assumption 1. *F is bi-Lipschitz continuous, i.e., there exists $(\underline{\alpha}_F,\overline{\alpha}_F)$ such that*

$$\forall(n_1,n_2),\quad \underline{\alpha}_F|n_2-n_1|\leq|F(n_2)-F(n_1)|\leq\overline{\alpha}_F|n_2-n_1|.$$

Assumption 2. *(i) p,q are twice continuously differentiable; (ii) p is strictly increasing and strictly concave; q is strictly decreasing and strictly convex; (iii) $p(0)=0$, $q(0)=1$, (iv) $p\circ q^{-1}$ is strictly concave.*[3]

To prove the existence of a solution to the free-entry problem, I make one additional assumption about the distribution of idiosyncratic productivity of entrants. Denote $g'_z(s,z)$ the cross-sectional distribution of z' one period after entry in state s, i.e., $g_{z'}(s,z')=\sum_{z\in\mathcal{Z}}g_z(z)\pi_z(z'|s,z)$.

Assumption 3. *For all $s\in\mathcal{S}$, the distribution g_z first order stochastically dominates $g_{z'}(s,z')$.*

Assumption 3 is an assumption on the productivity process which guarantees that entrants are (weakly) more productive on average than incumbents. This is a key condition to ensure that a non-zero measure of entrants hire a strictly positive number of workers upon entry, so that the free-entry condition may effectively pin down the value of κ in equilibrium.

To proceed with the proof of proposition 2.(i), I show that there exists a common solution to the joint surplus maximization, free-entry condition and unemployed workers' problem. This establishes the behavior of variables $(\{\tau(s',z';j),x(s',z';j)\}_{j=0}^n$, $d(s',z')$, $n_i(s',z')$, $x_i(s',z'))$ without the need to describe the set of contracts that implement the efficient allocation. Contracts may then be solved following the proof of proposition 1 or using the refinement of subsection F.2 in this appendix when the assumption of commitment on the worker side is relaxed. Let us first define the set where our optimal surplus $\mathbf{V}$

[3] (iv) is a regularity condition ensuring that workers' problem is well defined and concave.

lies and introduce our last assumption on parameters. Let $\overline{n}$ be an arbitrary upper bound on employment chosen sufficiently large so that it does not constrain the equilibrium.

Definition 1. Let $\mathcal{V}$ be the set of value functions $V:(s;z,n)\in\mathcal{S}\times\mathcal{Z}\times[0,\overline{n}]\longrightarrow\mathbb{R}$ (i) strictly increasing in n, (ii) satisfying $\forall s, \sum_z g_z(z)\left[V(s,z,0)\right]^+\leq\beta k_e$, (iii) bounded in $[\underline{V},\overline{V}]$, (iv) bi-Lipschitz continuous in n such that

$$\forall V\in\mathcal{V},\forall(s,z),\forall n^{(1)}\geq n^{(2)},\quad \underline{\alpha}_V(n^{(2)}-n^{(1)})\leq V(s,z,n^{(2)})-V(s,z,n^{(1)})\leq\overline{\alpha}_V(n^{(2)}-n^{(1)}),$$

with

$$\begin{aligned}
\underline{\alpha}_V &= e^{\underline{y}+\underline{z}}\underline{\alpha}_F+\beta(1-\beta)^{-1}b>0,\\
\overline{\alpha}_V &= (1-\beta)^{-1}\left(\mathrm{e}^{\overline{y}+\overline{z}}\overline{\alpha}_F+\beta\left(\lambda\overline{x}+(1-\beta)^{-1}(b+\beta\overline{x})\right)\right)\\
\underline{V} &= -k_f,\\
\overline{V} &= (1-\beta)^{-1}[\mathrm{e}^{\overline{y}+\overline{z}}F(\overline{n})-k_f+\beta\overline{n}\left(\lambda\overline{x}+(1-\beta)^{-1}(b+\beta\overline{x})\right)].
\end{aligned}$$

Assumption 4. *Assume* $\overline{n}>\underline{\alpha}_V^{-1}(k_e+k_f)$.

Assumption 4 is a sufficient condition on parameters that guarantees that there is always a solution to the free-entry problem. We can now establish the existence of a solution to the free-entry problem.

Lemma 1. *Under Assumptions 1-4, for $V\in\mathcal{V}$, $s\in\mathcal{S}$, the free-entry problem (9)-(12) admits a solution. There exists a unique hiring cost per worker $\kappa(s)$, an optimal level of hiring for entering firms $n_e^V(s,z)$ and exit decision $d_e^V(s,z)$ such that*

1. *Submarket x is active* $\Rightarrow\theta^V(s,x)>0\Rightarrow c/q(\theta(s,x))+x=\kappa^V(s)$,
2. *For all* $s\in\mathcal{S}$,

$$k_e \;=\; \max_{n_e(s,z)}\sum_z g_z(z)\left[V(s,z,n_e(s,z))-\kappa^V(s)\,n_e(s,z)\right]^+,$$

3. $\theta^V(s,x)=\begin{cases}q^{-1}\left(\frac{c}{\kappa^V(s)-x}\right), & \text{for } \underline{x}\leq x\leq\kappa^V(s)-c,\\ 0, & \text{for } x\geq\kappa^V(s)-c.\end{cases}$

Proof. For $V\in\mathcal{V}$, $s\in\mathcal{S}$ and $\kappa\in\mathbb{R}$, let us define the following auxiliary function

$$\psi^{s,V}(\kappa)=\max_{0\leq n_e^{s,V}(z)\leq\overline{n}}\sum_z g_z(z)\left[V\left(s,z,n_e^{s,V}(z)\right)-\kappa n_e^{s,V}(z)\right]^+.$$

The objective of this proof is to show that, for all $s\in\mathcal{S}$, there exists a unique $\kappa^V(s)$ such that $k_e=\psi^{s,V}(\kappa(s))$. Because V is continuous in $n\in[0,\overline{n}]$ and z has a finite support, $\psi^{s,V}$ is a well-defined function for $\kappa\in\mathbb{R}$. The Theorem of the Maximum tells us that ψ^V is a continuous function of κ. Notice that V being increasing in n, $\psi^{s,V}(0)=\sum_z g_z(z)\left[V(s,z,\overline{n})\right]^+$. Also, since V is bi-Lipschitz continuous with parameters $(\underline{\alpha}_V,\overline{\alpha}_V)$, for $\kappa\geq\overline{\alpha}_V$, the maximum is reached at $n_e=0$ and $\psi^{s,V}(\kappa)=\sum_z g_z(z)\left[V(s,z,0)\right]^+$. Let us show that $\psi^{s,V}$ is a decreasing function of κ. Take $\kappa_1<\kappa_2$ and the corresponding $n_{e,i}^{s,V}(z)$, $i=1,2$, that solve the maximization problem. Denote $\mathcal{Z}_i^{s,V}=\{z\in\mathcal{Z}|V^{s,V}(s,z,n_{e,i}^{s,V}(z))-\kappa_i n_{e,i}^{s,V}(z)\geq0\}$. Then we have

$$
\begin{aligned}
\psi^{s,V}(\kappa_1)-\psi^{s,V}(\kappa_2) &= \sum_z g_z(z)\left[V(s,z,n_{e,1}^{s,V}(z))-\kappa_1 n_{e,1}^{s,V}(z)\right]^+ \\
&\quad -\sum_z g_z(z)\left[V(s,z,n_{e,2}^{s,V}(z))-\kappa_2 n_{e,2}^{s,V}(z)\right]^+ \\
&\geq \sum_{z\in\mathcal{Z}_2^{s,V}} g_z(z)\left[V(s,z,n_{e,2}^{s,V}(z))-\kappa_1 n_{e,2}^{s,V}(z)\right] \\
&\quad -\sum_{z\in\mathcal{Z}_2^{s,V}} g_z(z)\left[V(s,z,n_{e,2}^{s,V}(z))-\kappa_2 n_{e,2}^{s,V}(z)\right] \\
&\geq (\kappa_2-\kappa_1)\sum_{z\in\mathcal{Z}_2^{s,V}} g_z(z)\, n_{e,2}^{s,V}(z).
\end{aligned}
$$

Symmetrically, we can establish that $\psi^{s,V}(\kappa_1)-\psi^{s,V}(\kappa_2)\leq(\kappa_2-\kappa_1)\sum_{z\in\mathcal{Z}_1^{s,V}} g_z(z)\, n_{e,1}^{s,V}(z)$. Thus ψ^V is decreasing. But this also tells us that if we denote $\overline{\kappa}$ the smallest κ such that $\psi^{s,V}(\kappa)=\sum_z g_z(z)\left[\varphi^{s,V}(z,0)\right]^+$ (i.e., for which $n_e=0$ is optimal for all z), then we have that ψ^V strictly decreases on $[0,\overline{\kappa}]$ from $\sum_z g_z(z)\left[V(s,z,\overline{n})\right]^+$ to $\sum_z g_z(z)\left[V(s,z,0)\right]^+$ and remains constant thereafter.

If $\sum_z g_z(z)\left[V(s,z,0)\right]^+<k_e<\sum_z g_z(z)\left[V(s,z,\overline{n})\right]^+$, the Intermediate Value Theorem tells us that there exists a unique $\kappa^V(s)$ such that $\psi^{s,V}\left(\kappa^V(s)\right)=k_e$. This establishes the existence of a solution to the free-entry problem. Part (1) of the proposition ensues:

$$\theta^V(s,x)>0\Leftrightarrow c/q(\theta(s,x))+x=\kappa^V(s).$$

Also, we have (2): there exists a $n_e^V(s,z)\geq 0$ chosen by entering firms so that

$$k_e=\sum_z g_z(z)\left[V\left(s,z,n_e^V(s,z)\right)-\kappa n_e^V(s,z)\right]^+$$

and a corresponding exit decision $d_e(s,z)$.

To conclude, we only need to check that

$$\sum_z g_z(z)\left[V(s,z,0)\right]^+<k_e<\sum_z g_z(z)\left[V(s,z,\overline{n})\right]^+.$$

The left-hand side is guaranteed by the fact that $V\in\mathcal{V}$. The right-hand side is guaranteed by assumption 4, as we have

$$\sum_z g_z(z)\left[V(s,z,\overline{n})\right]^+\geq\sum_z g_z(z)\,V(s,z,\overline{n})\geq\sum_z g_z(z)\left(V(s,z,0)+\underline{\alpha}_\varphi\overline{n}\right)\geq -k_f+\underline{\alpha}_V\overline{n}>k_e,$$

because of assumption 4.

(3) The complementary slackness condition (10) implies that either

$$\theta(s,x)=0\quad\text{ or }\quad c/q(\theta(s,x))+x=\kappa^V(s).$$

For $x>\kappa^V(s)-c$, the second expression admits no solution, as the probability q must remain below 1. So θ must be 0 in this region. For $x\leq\kappa^V(s)-c$, it admits the unique solution $q^{-1}\left(\frac{c}{\kappa^V(s)-x}\right)$. In this region: $c/q(0)+x<\kappa^V$, so

$\psi^{s,V}(c/q(0)+x) > k_e$. $\theta(s,x)$ cannot be 0 otherwise it would violate the free-entry condition (12). To summarize our results:

$$\theta^V(s,x) = \begin{cases} q^{-1}\left(\frac{c}{\kappa^V(s,x)-x}\right), & \text{for } \underline{x} \leq x \leq \kappa^V(s) - c, \\ 0, & \text{for } x \geq \kappa^V(s) - c. \end{cases}$$

□

We now prove the main proposition that establishes the existence of a quasi-equilibrium.

Proposition 3. *Under Assumptions 1-4, there exists a block-recursive solution to equations (1)-(12), i.e., the mapping $T : \mathcal{V} \longrightarrow \mathcal{V}$ such that*

$$\begin{aligned} TV(s,z,n) = \max_{\substack{d(s',z'),\, n_i(s',z'),\, x_i(s',z'),\\ \{\tau(s',z';j),\, x(s',z';j)\}_{j\in[0,n]}}} & e^{y(s)+z}F(n) - k_f + \beta\mathbb{E}\Big\{ nd\mathbf{U}^V(s') \\ & + (1-d)\left[\mathbf{U}^V(s')\int_0^n \tau \mathrm{dj} + \int_0^n (1-\tau)\lambda p\left(\theta^V(s',x)\right) x\mathrm{dj} \right. \\ & \left. -\kappa^V(s') n_i + V(s',z',n') \right]\Big\} \end{aligned}$$

with $n' = \int (1-\tau(j))(1-\lambda p\left(\theta^V(s,x(j))\right)x(j)\mathrm{d}j + n_i$, (θ^V, κ^V) solution to the free-entry problem (9)-(12) and $\mathbf{U}^V$ solution to (1) admits a fixed point.

Proof of proposition 3. To prove the existence, I will proceed in four steps: (1) establish existence, uniqueness and boundedness of $U^V(s)$ given some $V \in \mathcal{V}$, (2) show that T is a well-defined mapping from $\mathcal{V}$ to $\mathcal{V}$, (3) T is a continuous mapping, (4) $T(\mathcal{V})$ is an equicontinuous family. Since $\mathcal{V}$ is closed, bounded and convex, using Schauder's Fixed Point Theorem as stated in Stokey and Lucas, Theorem 17.4 p.520, this will establish the existence of a solution $\mathbf{V}$ in $\mathcal{V}$ to Bellman equation (7).

Step 1. For $V \in \mathcal{V}$, lemma 1 gives the existence and uniqueness of functions κ^V, n_e^V, d_e^V and θ^V. We are going to show that the following mapping M_V that defines U^V is a contraction from the space of functions $U : \mathcal{S} \longrightarrow \mathbb{R}$, bounded between some $\underline{U}$ and $\overline{U}$, to be defined later:

$$M^V U(s) = \max_{x_u(s')} b + \beta\mathbb{E}\left\{p\left(\theta^V(s',x_u)\right)x_u + \left(1 - p\left(\theta^V(s',x_u)\right)\right)U(s')\right\}.$$

Applying Blackwell's sufficient conditions for a contraction mapping, check *discounting*: for $a \geq 0$,

$$\begin{aligned} M^V(U+a) &= \max_{x_u(s')} b + \beta\mathbb{E}\left\{p\left(\theta^V(s',x_u)\right)x_u + \left(1 - p\left(\theta^V(s',x_u)\right)\right)(U(s')+a)\right\} \\ &\leq M^V U + \beta a. \end{aligned}$$

Check *monotonicity*: for $U_1 \leq U_2$, and corresponding optimal choices $x_u^{(i)}$, for $i = 1, 2$,

$$\begin{aligned} & M^V(U_2) - M^V(U_1) \\ & \geq \left(1 - p\left(\theta^V\left(s, x_u^{(2)}\right)\right)\right)\beta\mathbb{E}\left(U_2(s') - U_1(s')\right) \geq 0. \end{aligned}$$

It is easy to show now that if $\underline{U} \leq U \leq \overline{U}$, then

$$b + \beta\underline{U} \leq M^V U \leq b + \beta\left(\overline{x} + \overline{U}\right).$$

The unique fixed point of M^V is therefore bounded between $\underline{U} = (1-\beta)^{-1}b$ and $\overline{U} = (1-\beta)^{-1}(b+\beta\overline{x})$.

Step 2. Let us now check that T is a well-defined mapping from $\mathcal{V}$ to $\mathcal{V}$. For what follows, it is useful to denote some policy $\gamma = \{\{\tau(s',z';j), x(s',z';j)\}_{j\in[0,n]}, d(s',z'), n_i(s',z'), x_i(s',z')\}$, and define

$$\Phi^V(s,z,n,\gamma) = \mathrm{e}^{y(s)+z}F(n) - k_f + \beta\mathbb{E}\Bigg\{nd\mathbf{U}^V(s') + (1-d)\Bigg[\mathbf{U}^V(s')\int_0^n \tau \mathrm{dj} + \int_0^n (1-\tau)\lambda p\left(\theta^V(s',x)\right)x\mathrm{dj} - \kappa^V(s')n_i + V(s',z',n')\Bigg]\Bigg\}.$$

Φ^V denotes the current joint surplus evaluated at some arbitrary policy γ.
(i) If $V \in \mathcal{V}$, then TV is strictly increasing in n. Take $n^{(1)} < n^{(2)}$ and the corresponding optimal policies $\gamma^{(1)}$ and $\gamma^{(2)}$.

$$\begin{aligned} TV(s,z,n^{(2)}) - TV(s,z,n^{(1)}) &= \Phi(s,z,n^{(2)},\gamma^{(2)}) - \Phi(s,z,n^{(1)},\gamma^{(1)}) \\ &\geq \Phi(s,z,n^{(2)},\tilde{\gamma}) - \Phi(s,z,n^{(1)},\gamma^{(1)}) \end{aligned}$$

with a suboptimal policy $\tilde{\gamma} = \left\{\{\tilde{\tau}(s',z';j), \tilde{x}(s',z';j)\}_{j\in[0,n^{(2)}]}, \tilde{d}, \tilde{n}_i, \tilde{x}_i\right\}$ such that $\tilde{x}(j) = x(j)^{(1)}$, $\tilde{d} = d^{(1)}$, $\tilde{n}_i = n_i^{(1)}$, $\tilde{x}_i = x_i^{(1)}$, and $\tilde{\tau}(j) = \tau(j)^{(1)}$ for $j \in [0,n^{(1)}]$ and 1 for $j \in [n^{(1)}, n^{(2)}]$. In that case, we have $\tilde{n} = n^{(1)}$, and many terms cancel to yield the desired result that TV is strictly increasing in n.

$$\begin{aligned} &TV(s,z,n^{(2)}) - TV(s,z,n^{(1)}) \geq \Phi(s,z,n^{(2)},\tilde{\gamma}) - \Phi(s,z,n^{(1)},\gamma_1) \\ &\geq e^{y(s)+z}\left(F\left(n^{(2)}\right) - F\left(n^{(1)}\right)\right) + \beta\mathbb{E}\left[\left(n^{(2)} - n^{(1)}\right)\mathbf{U}^V(s')\right] > 0. \end{aligned}$$

(ii) If $V \in \mathcal{V}$, then $\forall s \in \mathcal{S}, \sum_{z\in\mathcal{Z}} g_z(z)TV(s,z,0)^+ \leq \beta k_e$. Recall that

$$TV(s,z,0) = \max_{d,n_i,x_i} -k_f + \beta\mathbb{E}\Bigg\{(1-d)\left[-\kappa^V(s)n_i + \beta\mathbb{E}V(s',z',n_i)\right)\Big]\Bigg\}.$$

Since κ^V is the solution to the free-entry condition and because of Assumption 3, we have

$$TV(s,z,0) \leq -k_f + \beta k_e.$$

Since $TV(s,z,0) \geq -k_f$, we have

$$\begin{aligned} TV(s,z,0)^+ &= \max\{TV(s,z,0),0\} \leq \max\{TV(s,z,0)+k_f,0\} \\ &\leq TV(s,z,0)+k_f \leq \beta k_e, \end{aligned}$$

and therefore $\sum_{z\in\mathcal{Z}} g_z(z)TV(s,z,0)^+ \leq \beta k_e$.
(iii) If $V \in \mathcal{V}$, then TV is bounded in $[\underline{V},\overline{V}]$ with $\underline{V} = 0$ and $\overline{V} = (1-\beta)^{-1}[\mathrm{e}^{\overline{y}+\overline{z}}F(\overline{n}) - k_f + \beta\overline{n}\left(\lambda\overline{x} + (1-\beta)^{-1}(b+\beta\overline{x})\right)]$:

$$TV(s,z,n) \leq \mathrm{e}^{\overline{y}+\overline{z}}F(\overline{n}) - k_f + \beta\left(\overline{n}\overline{U} + \overline{n}\lambda\overline{x} + \overline{V}\right) \leq \overline{V}.$$

Now, for the lower bound:

$$
\begin{aligned}
TV(s,z,n) &\geq \Phi(s,z,n,\tilde{\gamma}) \\
&\geq e^{\underline{y}+\underline{z}} F(n) - k_f + \beta n \underline{U} \geq -k_f = \underline{V}
\end{aligned}
$$

with suboptimal policy $\tilde{\gamma}$ such that $\tilde{d} = 1$.

(iv) If $V \in \mathcal{V}$, then

$$
\forall (s,z), \forall n_2 \geq n_1, \quad \underline{\alpha}_V (n_2 - n_1) \leq TV(s,z,n_2) - TV(s,z,n_1) \leq \overline{\alpha}_V (n_2 - n_1).
$$

Take $n_2 \geq n_1$ and corresponding optimal policies γ_i, $i = 1,2$. Choose a suboptimal policy $\tilde{\gamma}$ such that $\tilde{d} = d_2$, $\tilde{x}(s',z';j) = x_2(s',z';j)$, $\tilde{n_i} = n_{i2}$, $\tilde{x}_i = x_{i2}$, $\tilde{\tau}(s',z';j) = \tau_2(s',z';j)$ for $j \in [0,n_1]$:

$$
\begin{aligned}
&TV(s,z,n_2) - TV(s,z,n_1) = \Phi(s,z,n_2,\gamma_2) - \Phi(s,z,n_1,\gamma_1) \\
&\leq \Phi(s,z,n_2,\gamma_2) - \Phi(s,z,n_1,\tilde{\gamma}) \\
&\leq \mathrm{e}^{y+z}(F(n_2) - F(n_1)) + \beta \mathbb{E} \Bigg\{ (n_2 - n_1) d_2 \mathbf{U}^V(s') + (1-d_2) \Bigg(\mathbf{U}^V(s') \int_{n_1}^{n_2} \tau_2 dj + \\
&+ \int_{n_1}^{n_2} (1-\tau_2) \lambda p^V(x_2) x_2 \mathrm{d}j + V(s,z,n_2') - V(s,z,\tilde{n}_1') \Bigg) \Bigg\} \\
&\leq \left[\mathrm{e}^{\overline{y}+\overline{z}} \overline{\alpha}_F + \beta \left(\overline{U} + + \lambda \overline{x} + \overline{\alpha}_V \right) \right] (n_2 - n_1) = \overline{\alpha}_V (n_2 - n_1).
\end{aligned}
$$

Proceed similarly for the other side and choose a policy $\tilde{\gamma}$ such that $\tilde{d} = d_1$, $\tilde{x}(s',z';j) = x_1(s',z';j)$, $\tilde{n_i} = n_{i1}$, $\tilde{x_i} = x_{i1}$, $\tilde{\tau}(s',z';j) = \tau_1(s',z';j)$ for $j \in [0,n_1]$ and 1 for $j \in [n_1,n_2]$:

$$
\begin{aligned}
&TV(s,z,n_2) - TV(s,z,n_1) = \Phi(s,z,n_2,\gamma_2) - \Phi(s,z,n_1,\gamma_1) \\
&\geq \Phi(s,z,n_2,\tilde{\gamma}) - \Phi(s,z,n_1,\gamma_1) \\
&\geq \mathrm{e}^{y+z}(F(n_2) - F(n_1)) + \beta \mathbb{E} \Bigg\{ (n_2 - n_1) d_1 \mathbf{U}^V(s') + (1-d_1)(n_2 - n_1) \mathbf{U}^V(s') \Bigg\} \\
&\geq \left[\mathrm{e}^{\underline{y}+\underline{z}} \underline{\alpha}_F + \beta \underline{U} \right] (n_2 - n_1) = \underline{\alpha}_V (n_2 - n_1).
\end{aligned}
$$

Therefore, TV is bi-Lipschitz continuous with the desired coefficients.

Step 3. We are now going to show that $T : \mathcal{V} \longrightarrow \mathcal{V}$ is a continuous mapping. Denote by $||\cdot||$ the infinite norm, i.e., $||V|| = \sup_{(s,z,n) \in \mathcal{S} \times \mathcal{Z} \times [0,\overline{n}]} V(s,z,n)$. Take $V_1, V_2 \in \mathcal{V}$. For (s,z,n) fixed, denote by γ_k, $k = 1,2$, the corresponding optimal policies. Denote $\tilde{\gamma}$ the policy exactly equal to γ_1 except that $\tilde{x}(s',z';j)$ is chosen such that $p\left(\theta^{V_1}(s',x_1')\right) = p\left(\theta^{V_2}(s',\tilde{x}')\right)$.

This means in particular that $\tilde{x}\left(s', z'; j\right) = x_1\left(s', z'; j\right) + \kappa^{V_2}\left(s'\right) - \kappa^{V_1}\left(s'\right), \forall \left(s', z'\right)$.

$$\begin{aligned}
&TV_1(s,z,n) - TV_2(s,z,n) = \Phi^{V_1}(s,z,n,\gamma_1) - \Phi^{V_2}(s,z,n,\gamma_2)\\
&\leq \Phi^{V_1}(s,z,n,\gamma_1) - \Phi^{V_2}(s,z,n,\tilde{\gamma})\\
&\leq \beta\mathbb{E}\Bigg\{ d_1 n\left(\mathbf{U}^{V_1}(s') - \mathbf{U}^{V_2}(s')\right)\right) + (1-d_1)\Bigg(\left(\mathbf{U}^{V_1}(s') - \mathbf{U}^{V_2}(s')\right)\int \tau_1 \mathrm{d}j\\
&- (\kappa^{V_1}(s') - \kappa^{V_2}(s'))n_{i1} + \int (1-\tau_1)\lambda p\left(\theta^{V_1}\left(s', x_1\right)\right)\left(x_1 - \tilde{x}\right)\mathrm{d}j\\
&+ V_1(s', z', n_1') - V_2(s', z', n_1')\Bigg)\Bigg\}\\
&\leq \beta\Big[\overline{n}||\mathbf{U}^{V_1} - \mathbf{U}^{V_2}|| + \overline{n}||\kappa^{V_1} - \kappa^{V_2}|| + \overline{n}\lambda||\kappa^{V_1} - \kappa^{V_2}|| + ||V_1 - V_2||\Big].
\end{aligned}$$

According to lemma 2 below, we can control each term:

$$TV_1(y,s,z,n) - TV_2(y,s,z,n) \leq \beta\left[\overline{n}\alpha_U + \overline{n}\left(1+\lambda\right)\alpha_\kappa + 1\right]||V_1 - V_2||,$$

which can be made arbitrarily small as $||V_1 - V_2||$ gets smaller. Therefore, T is a continuous mapping.

Lemma 2. *If $V_1, V_2 \in \mathcal{V}$, then*
[(i)]

1. $||\kappa^{V_1} - \kappa^{V_2}|| \leq \alpha_\kappa ||V_1 - V_2||$, *with* $\alpha_\kappa = \frac{\beta}{n_{min}}$,
2. $||\theta^{V_1} - \theta^{V_2}|| \leq \alpha_\theta ||V_1 - V_2||$, *with* $\alpha_\theta = \frac{\beta}{c|q'(\theta_{max})|n_{min}}$,
3. $||\mathbf{U}^{V_1} - \mathbf{U}^{V_2}|| \leq \alpha_U ||V_1 - V_2||$, *with* $\alpha_U = (1-\beta)^{-1}\beta\alpha_\kappa$.

Proof. To prove the lemma, we first need to establish the following two results. Let us prove that there exists $\theta_{max} > 0$ such that

$$\forall V \in \mathcal{V}, \theta^V(\cdot) \leq \theta_{max},$$

and there exists $n_{min} > 0$ such that

$$\forall V \in \mathcal{V}, \sum_z g_z\left(z\right) n_e^V\left(s,z\right) \geq n_{min}.$$

The first result can be established by the fact that $\kappa_V \leq \overline{\alpha}_V$ as we showed in lemma 1. Then for some $x \in [\underline{x}, \overline{x}]$:

$$c/q(\theta^V(s,x)) + x \leq \overline{\alpha}_V \Rightarrow q(\theta^V(s,x)) \geq c(\overline{\alpha}_V - x)^{-1} \Rightarrow \theta^V(s,x) \leq q^{-1}[c(\overline{\alpha}_V - \underline{x})^{-1}].$$

Setting $\theta_{max} = q^{-1}[c(\overline{\alpha}_V - \underline{x})^{-1}]$ yields the desired result.
Now, for the second result, remember the free-entry condition:

$$k_e = \sum g_z\left(z\right)\left[V(s,z,n_e^{s,V}\left(z\right)) - \kappa^V\left(s\right) n_e^{s,V}\left(z\right)\right]^+.$$

Then, using the fact that V is bi-Lipschitz:

$$\begin{aligned}
k_e &\leq \sum g_z\left(z\right)\left[V(s,z,n_e^{s,V}\left(z\right))\right]^+ \leq \sum g_z\left(z\right)\left[V(s,z,0) + \overline{\alpha}_V n_e^{s,V}\left(z\right)\right]^+\\
&\leq \overline{\alpha}_V \sum g_z\left(z\right) n_e^{s,V}\left(z\right) + \sum g_z\left(z\right)\left[-k_f + \beta\mathbb{E}V(s',z',0)\right]^+.
\end{aligned}$$

Since $\sum g_z(z)\left[-k_f+\beta\mathbb{E}V(s',z',0)\right]^+ \leq \beta k_e$ as we argued before, we have

$$\mathrm{E}_{g_z} n_e^V \geq \overline{\alpha}_V^{-1}(1-\beta)k_e \equiv n_{min}.$$

(i) The free-entry condition gives us for $i=1,2$:

$$k_e = \sum_z g_z(z)\left[V\left(s,z,n_e^{s,V_i}(z)\right) - \kappa^{V_i}(s)\, n_e^{s,V_i}(z)\right]^+.$$

Denote $\mathcal{Z}_i = \{z \in \mathcal{Z} | V(s,z,n_e^{s,V_i}(z)) - \kappa^{V_i} n_e^{s,V_i}(z) \geq 0\}, i=1,2$. Substracting both:

$$\begin{aligned}
0 &= \sum_z g_z(z)\left[V\left(s,z,n_e^{s,V_1}(z)\right) - \kappa^{V_1}(s)\, n_e^{s,V_1}(z)\right]^+ \\
&- \sum_z g_z(z)\left[V\left(s,z,n_e^{s,V_2}(z)\right) - \kappa^{V_2}(s)\, n_e^{s,V_2}(z)\right]^+ \\
&\geq \sum_{z\in\mathcal{Z}_2} g_z(z)\left[\left(\kappa^{V_2}(s) - \kappa^{V_1}(s)\right) n_e^{s,V_2}(z)\right. \\
&\left. +\beta\mathbb{E}\left[V_1\left(s,z,n_e^{s,V_2}(z)\right) - V_2\left(s,z,n_e^{s,V_2}(z)\right)\right]\right]
\end{aligned}$$

which yields

$$\kappa^{V_2}(s) - \kappa^{V_1}(s) \leq \frac{\beta}{n_{min}}||V_1 - V_2||.$$

Symmetrically, establish that $\kappa^{V_1}(s) - \kappa^{V_2}(s) \leq \frac{\beta}{n_{min}}||V_1 - V_2||$, which establishes the desired result for $\alpha_\kappa = \beta/n_{min}$.
(ii) Pick an $s \in \mathcal{S}$ and $x \in [\underline{x}, \overline{x}]$, consider the case in which submarket x is open under value functions V_1 and V_2. In that case, we know that:

$$\kappa^{V_i}(s) = \frac{c}{q\left(\theta^{V_i}(s,x)\right)} + x,$$

therefore

$$\frac{c}{q\left(\theta^{V_1}(s,x)\right)} - \frac{c}{q\left(\theta^{V_2}(s,x)\right)} = \kappa^{V_1}(s) - \kappa^{V_2}(s),$$

so that

$$q\left(\theta^{V_2}(s,x)\right) - q\left(\theta^{V_1}(s,x)\right) = c^{-1} q\left(\theta^{V_1}(s,x)\right) q\left(\theta^{V_2}(s,x)\right)\left(\kappa^{V_1}(s) - \kappa^{V_2}(s)\right),$$

and we can easily conclude that

$$\left|\theta^{V_2}(s,x) - \theta^{V_1}(s,x)\right| \leq \frac{1}{c\left|q'\left(\theta_{max}\right)\right|}\left(\kappa^{V_1}(s) - \kappa^{V_2}(s)\right) \leq \frac{\beta}{c\left|q'\left(\theta_{max}\right)\right| n_{min}}||V_1 - V_2||.$$

Now consider the case in which submarket x is active under value V_2, but not under value V_1. We have:

$$\begin{aligned}
\kappa^{V_2}(s) &= \frac{c}{q\left(\theta^{V_2}(s,x)\right)} + x, \\
\kappa^{V_1}(s) &\leq \frac{c}{q\left(\theta^{V_1}(s,x)\right)} + x,
\end{aligned}$$

but also $\theta^{V_1}(s,x) = 0$ and $\theta^{V_2}(s,x) > 0$ from the complementary slackness condition. We can still derive the inequality:

$$\frac{c}{q\left(\theta^{V_2}(s,x)\right)} - \frac{c}{q\left(\theta^{V_1}(s,x)\right)} \leq \kappa^{V_2}(s) - \kappa^{V_1}(s),$$

so that:

$$0 \leq \theta^{V_2}(s,x) - \theta^{V_1}(s,x) \leq \frac{\beta}{c\,|q'(\theta_{max})|\,n_{min}}||V_1 - V_2||.$$

Finally, the case in which submarket x is closed for both value functions is trivial, $\theta^{V_1}(s,x) = \theta^{V_2}(s,x) = 0$.
(iii) Fix s. Denote by $x_{uk}, k = 1,2$ the corresponding optimal choices for unemployed workers. Pick the suboptimal policy $\tilde{x}_u(s')$ such that $p\left(\theta^{V_1}(s', x_{u1}(s'))\right) = p\left(\theta^{V_2}(s', \tilde{x}_u(s'))\right)$, i.e., $\tilde{x}_u(s') = x_{u1}(s') + \kappa^{V_2}(s') - \kappa^{V_1}(s'), \forall (s', z')$

$$\begin{aligned}
&U^{V_1}(s) - U^{V_2}(s) \\
&= \beta\mathbb{E}\left[p\left(\theta^{V_1}(s', x_{u1}(s))\right)x_{u1}(s') + \left(1 - p\left(\theta^{V_1}(s', x_{u1}(s))\right)\right)U^{V_1}(s')\right] \\
&\quad - \beta\mathbb{E}\left[p\left(\theta^{V_2}(s', x_{u2}(s))\right)x_{u2}(s') + \left(1 - p\left(\theta^{V_1}(s', x_{u2}(s))\right)\right)U^{V_2}(s')\right] \\
&\leq \beta\mathbb{E}\left[p\left(\theta^{V_1}(s', x_{u1}(s'))\right)(x_{u1}(s') - \tilde{x}_u(s'))\right. \\
&\qquad \left. + \left(1 - p\left(\theta^{V_1}(s', x_{u1}(s'))\right)\right)\left(U^{V_1}(s') - U^{V_2}(s')\right)\right] \\
&\leq \beta\mathbb{E}\left[p\left(\theta^{V_1}(s', x_{u1}(s'))\right)\left(\kappa^{V_1}(s') - \kappa^{V_2}(s')\right)\right. \\
&\qquad \left. + \left(1 - p\left(\theta^{V_1}(s', x_{u1}(s'))\right)\right)\left(U^{V_1}(s') - U^{V_2}(s')\right)\right] \\
&\leq \beta\alpha_\kappa||V_1 - V_2|| + \beta||U^{V_1} - U^{V_2}||
\end{aligned}$$

We can now conclude that

$$||U^{V_1} - U^{V_2}|| \leq (1-\beta)^{-1}\alpha_\kappa||V_1 - V_2||. \quad \square$$

Step 4. We can now proceed to the last step of the proof of proposition 3. We must show that the family $T(\mathcal{V})$ is equicontinuous, i.e., $\forall\varepsilon > 0$, there exists $\delta > 0$ such that for $\xi_k = (s_k, z_k, n_k)$,$k = 1,2$,

$$||\xi_1 - \xi_2|| < \delta \Rightarrow |TV(\xi_1) - TV(\xi_2)| < \varepsilon, \forall V \in \mathcal{V}.$$

Fix $\varepsilon > 0$ and denote

$$\begin{cases} \eta_s = \min\limits_{s_1 \neq s_2 \in \mathcal{S}}|s_1 - s_2| \\ \eta_z = \min\limits_{z_1 \neq z_2 \in \mathcal{Z}}|z_1 - z_2|. \end{cases}$$

Choose $\delta < \min(\eta_s, \eta_z, \varepsilon/\overline{\alpha}_V)$. Take (ξ_1, ξ_2) such that $||\xi_1 - \xi_2|| < \delta$. Therefore, $s_1 = s_2$ and $z_1 = z_2$. Take $V \in \mathcal{V}$. Using the fact that V is bi-Lipschitz:

$$|TV(\xi_1) - TV(\xi_2)| \leq \overline{\alpha}_V|n_1 - n_2| \leq \overline{\alpha}_V||\xi_1 - \xi_2|| < \varepsilon.$$

Conclusion: $T(\mathcal{V})$ is equicontinuous. Schauder's Fixed Point Theorem applies and tells us that there exists a fixed point $\mathbf{V}$ to the mapping T. All other equilibrium objects $\mathbf{U}$, $\mathbf{W}$, $\mathbf{J}$, θ, κ and optimal policy functions are then well defined. This achieves the proof of proposition 3 which corresponds to proposition 2.(i) in the text. $\square$

H.2.2 Proposition 2.(ii): Efficiency

Proof. To study efficiency, I now introduce the planning problem of this economy. I proceed in four steps. First, I define the planning problem. In step 2, I simplify one important constraint in the planner's problem and provide an equivalent formulation. In step 3, I show that the planner's problem is a well-defined pseudo-concave problem subject to quasiconcave constraints, so that the first order conditions of the Lagrangian problem are sufficient for optimality. Finally, I show in step 4 that a block-recursive allocation, when it exists, satisfies the first-order conditions of the planner's problem and is therefore efficient.

Step 1. Using the same convention as in part 2.8, I denote u_t and $g_t(z_t, n_t)$ the unemployment rate and distribution of firms at stage B of period t when production takes place. For notational simplicity, I also introduce distribution $g_t^A(z_t, n_{t-1})$ which is the distribution of firms at the beginning of the period in stage A. The two distributions are related in the following way:

$$\begin{aligned} g_t^A(z_t, n_{t-1}) &= \sum_{z_{t-1}} \pi_z(z_t \mid s_{t-1}, z_{t-1})\, g_{t-1}(z_{t-1}, n_{t-1}) \\ g_t(z_t, n_t) &= \sum_{n_{t-1}} \mathbb{I}\{n'(s_t, z_t; n_{t-1}) = n_t\}\, g_t^A(z_t, n_{t-1}) \\ &\quad + m_{e,t}\mathbb{I}\{n_e(s_t, z_t) = n_t\}\, g_z(z_t)\,. \end{aligned}$$

Since the planner can freely allocate workers between firms without respect to any promised utility, the only relevant information concerning each labor market segment is its tightness. Let us therefore label each submarket by its tightness, θ, instead of its corresponding contract, x. Denote by $(\theta_x, \theta_i, \theta_u)$ the markets chosen respectively by firms for on-the-job search, for hirings, and the one chosen by unemployed workers to search. Furthermore, all workers are identical in the eyes of the planner. Given the strict concavity of the problem in $\theta_{x,t}$, I focus directly on allocations in which $\theta_{x,t}$ is the same across workers within a same firm. Similarly, as proposition 1 will make it clear, only the total number of layoffs at the firm level is determined in equilibrium, whereas the exact distribution of layoffs across workers in the same firm is not. I thus focus directly on allocations in which τ_t is the same across workers, so that the total number of layoffs is $n_{t-1}\tau_t$. It should be understood that transformations of τ_t across workers that leave the total number of layoffs unchanged are also solutions of the planning problem. All decisions at time t depend implicitly on the entire history of past aggregate shocks $s^t = \{s_t, s_{t-1}, ...\}$. The planner's objective is to maximize the total welfare in the economy,

$$\begin{aligned} &\max_{\substack{u_t, g_{t+1}^A, \theta_{u,t}, d_t, n_t, \tau_t, \\ n_{i,t}, \theta_{i,t}, d_{e,t}, n_{e,t}, \theta_{e,t}}} \\ &\mathbb{E}\sum_t \beta^t \Bigg\{ u_t b + \sum_{z_t, n_{t-1}} g_t^A(z_t, n_{t-1})\,(1 - d_t(z_t, n_{t-1})) \times ... \\ &\qquad ... \times \left(e^{y(s_t)+z_t} F(n_t(z_t, n_{t-1})) - k_f - \frac{c}{q(\theta_{i,t}(z_t, n_{t-1}))} n_{i,t}(z_t, n_{t-1}) \right) \\ &+ m_{e,t} \Bigg[-k_e + \sum_{z_t} g_z(z_t)\,(1 - d_{e,t}(z_t)) \times ... \\ &\qquad ... \times \left(e^{y(s_t)+z_t} F(n_{e,t}(z_t)) - k_f - \frac{c}{q(\theta_{e,t}(z_t))} n_{e,t}(z_t) \right) \Bigg] \Bigg\}, \end{aligned} \tag{25}$$

which is the discounted sum of production net of operating cost k_f and vacancy posting cost c over all existing firms, minus total entry costs for new firms $m_{e,t}$ every period, plus home production b of unemployed agents. The planner is subject to the laws of motion of the unemployment rate,

$$\begin{aligned} u_t &= \Big(1 - p(\theta_{u,t})\Big) u_{t-1} + ... \\ &\quad + \sum_{z_t, n_{t-1}} n_{t-1}\left[d_t(z_t, n_{t-1}) + (1 - d_t(z_t, n_{t-1}))\,\tau_t(z_t, n_{t-1})\right] g_t^A(z_t, n_{t-1}), \end{aligned} \tag{26}$$

the level employment for every firm, $\forall(z_t, n_{t-1})$

$$n_t(z_t, n_{t-1}) = n_{t-1}(1 - \tau_t(z_t, n_{t-1}))(1 - \lambda p(\theta_{x,t})) + n_{i,t}(z_t, n_{t-1})\,, \tag{27}$$

and the distribution of firms, $\forall(z_t, n_{t-1})$

$$
\begin{aligned}
g^A_{t+1}(z_{t+1}, n_t) = & \sum_{\substack{(z_t, n_{t-1}) \\ |n_t(z_t, n_{t-1}) = n_t}} \left(1 - d_t\left(z_t, n_{t-1}\right)\right) \pi_z(z_{t+1} \mid s_t, z_t) g^A_t(z_t, n_{t-1}) \\
& ... + m_{e,t} \sum_{z_t | n_{e,t}(z_t) = n_t} \left(1 - d_{e,t}\left(z_t\right)\right) \pi_z(z_{t+1} \mid s_t, z_t) g_z(z_t),
\end{aligned} \tag{28}
$$

In addition, the planner is subject to two additional types of constraints: a non-negativity constraint for entry, and a constraint verifying that each labor market segment is in equilibrium, i.e., that the number of workers finding a job is equal to the number of successful job openings on a given submarket. More precisely, in every period, the planner is subject to:

$$
m_{e,t} \geq 0, \tag{29}
$$

$$
JF^w_t\left(\theta\right) + JF^u_t\left(\theta\right) = JC^f_t\left(\theta\right) + JC^e_t\left(\theta\right), \quad \forall \theta, \tag{30}
$$

where $JF^W_t\left(\theta\right)$ is the total number of jobs found by incumbent workers, equal to the number of successful job-to-job transitions,

$$
\begin{aligned}
JF^W_t\left(\theta\right) = & \sum_{\substack{(z_t, n_{t-1})| \\ \theta_{x,t}\left(z_t, n_{t-1}\right) = \theta}} g^A_t(z_t, n_{t-1}) \left(1 - d_t\left(z_t, n_{t-1}\right)\right) \\
& ... \times n_{t-1} \left(1 - \tau_t\left(z_t, n_{t-1}\right)\right) \lambda p\left(\theta_{x,t}\left(z_t, n_{t-1}\right)\right),
\end{aligned}
$$

$JF^U_t\left(\theta\right)$ is the number of jobs found for unemployed, equal to the number of successful unemployed candidates,

$$
JF^u_t\left(\theta\right) = \mathbb{1}(\theta_{u,t} = \theta) p(\theta_{u,t}) u_{t-1},
$$

$JC^f_t\left(\theta\right)$ is the number of jobs created by incumbent firms on market θ,

$$
JC^f_t\left(\theta\right) = \sum_{(z_t, n_{t-1}) | \theta_{i,t}(z_t, n_{t-1}) = \theta} g^A_t(z_t, n_{t-1})(1 - d_t\left(z_t, n_{t-1}\right)) n_{i,t}\left(z_t, n_{t-1}\right),
$$

and $JC^e_t\left(\theta\right)$ that of entering firms,

$$
JC^e_t\left(\theta\right) = m_{e,t} \sum_{z | \theta_{e,t}(z) = \theta} g_z(z)(1 - d_{e,t}\left(z\right)) n_{e,t}\left(z\right).
$$

As a summary, the planner's problem is to maximize (25) subject to constraints (26)-(30).

Step 2. The constraints defined in (30) are difficult to handle in practice. We now provide an easier equivalent formulation of the problem. Notice first that under constraint (30), we have the following equality:

$$
\begin{aligned}
& \sum_{z_t, n_{t-1}} g^A_t(z_t, n_{t-1}) \left(1 - d_t\right) \frac{c}{q\left(\theta_{i,t}\right)} n_{i,t} + m_{e,t} \sum_{z_t} g_z\left(z_t\right) \left(1 - d_{e,t}\right) \frac{c}{q\left(\theta_{e,t}\right)} n_{e,t} \\
= \quad & c \sum_{z_t, n_{t-1}} g^A_t\left(z_t, n_{t-1}\right) \left(1 - d_t\right) \left(1 - \tau_t\right) \lambda n_{t-1} \theta_{x,t} + \theta_{u,t} u_{t-1},
\end{aligned} \tag{31}
$$

where I have used the identity $p\left(\theta\right) = \theta q\left(\theta\right)$. Substituting (31) into the objective function (25), we notice that the markets for hiring $\theta_{i,t}$ and $\theta_{e,t}$ do not affect the objective function:

$$\max_{\substack{u_t,\, g^A_{t+1},\, \theta_{u,t},\, d_t,\, n_t,\\ \tau_t,\, n_{i,t},\, d_{e,t},\, n_{e,t},}} \mathbb{E}\sum_t \beta^t \Big\{ u_t b + \sum_{z_t, n_{t-1}} g^A_t(z_t, n_{t-1})\,(1-d_t)\left(e^{y(s_t)+z_t}F\left(n_t\right)-k_f\right)$$
$$+\, m_{e,t}\left[-k_e + \sum_{z_t} g_z\left(z_t\right)\left(1-d_{e,t}\right)\left(e^{y(s_t)+z_t}F\left(n_{e,t}\right)-k_f\right)\right]$$
$$-\, c\left(\theta_{u,t}u_{t-1} + \sum_{z_t,n_{t-1}} g_t\left(z_t,n_{t-1}\right)n_{t-1}\lambda\left(1-d_t\right)\left(1-\tau_t\right)\theta_{x,t}\right)\Big\}. \tag{32}$$

This means that, as long as constraint (30) is satisfied, variables $(\theta_{i,t}, \theta_{e,t})$ leave aggregate welfare unchanged. This result echoes our finding in the competitive equilibrium that firms are indifferent between markets. What this means is that we can replace constraint (30) by an easier one. Summing over all the submarkets, constraint (30) gives an expression for the measure of entrants:

$$m_{e,t} = \left(\sum_{z_t} g_z\left(z_t\right)\left(1-d_{e,t}\right)n_{e,t}\right)^{-1}$$
$$\times \left[\sum_{z_t,n_{t-1}} g_t\left(z_t,n_{t-1}\right)\left(1-d_t\right)\left(\lambda n_{t-1}p\left(\theta_{x,t}\right)-n_{i,t}\right) + p\left(\theta_{u,t}\right)u_{t-1}\right]. \tag{33}$$

Because $\theta_{i,t}$ and $\theta_{e,t}$ do not affect welfare under constraint (30), it is equivalent to maximize (25) under constraint (30) as to maximize (32) under constraint (33). Indeed, as long as (33) is satisfied, we can always arbitrarily distribute incumbent and entering firms across markets so that (30) is satisfied for all active submarket.

Step 3. We now show that the planner's problem is a well-behaved pseudo-concave problem. To show this, I rewrite the maximization of (32) under constraints (26), (27), (28) and (33) in such a way that the objective function is pseudoconcave and all constraints are quasiconcave. In that purpose, it is useful to write the summation over distribution $g^A_t\left(z_t,n_{t-1}\right)$ as a summation over firms' indices, so that we can ignore the law of motion of g^A_t. Every firm is indexed by the period it was born, t_0, and an firm-specific index, j, among that cohort. Let me also introduce the variables $\xi_{u,t} = p\left(\theta_{u,t}\right)$ and $\xi^{(t_0,j)}_{x,t} = p\left(\theta^{(t_0,j)}_{u,t}\right)$ which are useful to turn the problem concave along some dimensions. The planning problem may be equivalently written:

$$\max_{\substack{u_t,\, \theta_{u,t},\, \xi_{u,t} h_t,\, v_t,\\ \left\{d^{(t_0,j)}_t,\, n^{(t_0,j)}_t,\, \tau^{(t_0,j)}_t,\, n^{(t_0,j)}_{i,t},\, \theta^{(t_0,j)}_{x,t},\, \xi^{(t_0,j)}_{x,t}\right\}_{(t,t_0,j)}}} \tag{34}$$
$$\mathbb{E}\sum_t \beta^t \left[\sum_{t_0=-\infty}^{t} \int \prod_{l=t_0}^{t}\left(1-d^{(t_0,j)}_l\right)\left(e^{y(s_t)+z^{(t_0,j)}_t}F\left(n^{(t_0,j)}_t\right)-k_f\right)dj\right.$$
$$\left. \ldots + u_t b - c v_t - m_{e,t}k_e\right]$$

subject to

$$n^{(t_0,j)}_{t-1}\left(1-\tau^{(t_0,j)}_t\right)\left(1-\lambda\xi^{(t_0,j)}_{x,t}\right) + n^{(t_0,j)}_{i,t} - n^{(t_0,j)}_t = 0, \tag{35}$$

$$\sum_{t_0=-\infty}^{t} \int \left[\prod_{l=t_0}^{t-1} \left(1 - d_l^{(t_0,j)}\right)\right] \left(1 - d_t^{(t_0,j)}\right) \left(1 - \tau_t^{(t_0,j)}\right) \lambda n_{t-1}^{(t_0,j)} \theta_{x,t}^{(t_0,j)} dj$$
$$... + \theta_{u,t} u_{t-1} - v_t = 0, \tag{36}$$

$$\left(1 - \xi_{u,t}\right) u_{t-1} + \sum_{t_0=-\infty}^{t} \int \left[\prod_{l=t_0}^{t-1} \left(1 - d_l^{(t_0,j)}\right)\right] n_{t-1}^{(t_0,j)} \left(d_t^{(t_0 j)} + \left(1 - d_t^{(t_0,j)}\right) \tau_t^{(t_0,j)}\right) - u_t = 0, \tag{37}$$

$$\sum_{t_0=-\infty}^{t} \int \left[\prod_{l=t_0}^{t-1} \left(1 - d_l^{(t_0,j)}\right)\right] \left\{\left(1 - d_t^{(t_0,j)}\right) \left[n_{t-1}^{(j)} \left(1 - \tau_t^{(t_0,j)}\right) \lambda \xi_{x,t}^{(t_0,j)} - n_{i,t}^{(t_0,j)}\right] dj\right.$$
$$... + \xi_{u,t} u_{t-1} = 0, \tag{38}$$

$$\int \left(1 - d_t^{(t,j)}\right) dj - m_{e,t} = 0, \tag{39}$$

$$p\left(\theta_{u,t}\right) - \xi_{u,t} = 0 \text{ and } p\left(\theta_{x,t}^{(t_0,j)}\right) - \xi_{x,t}^{(t_0,j)} = 0. \tag{40}$$

The objective function is concave and non-stationary. It is therefore pseudoconcave. The constraints are all sums of linear and positive cross-product terms and are therefore quasiconcave. We may then conclude that the first-order conditions of the Lagrangian problem are sufficient to guarantee optimality.

Step 4. I will now show that a block-recursive equilibrium solves the planner's first order conditions. For that purpose, let us write the Lagrangian of version (25) of the planner's problem, summing over firms' indices. Write μ_t the Lagrange multiplier on constraint (26) and $\eta_t(\theta)$ the one for each submarket equilibrium (30).

$$\mathcal{L} = \mathbb{E}\sum_t \beta^t \Bigg\{ \sum_{t_0=-\infty}^{t} \int \left[\prod_{l=t_0}^{t-1} \left(1 - d_l^{(t_0,j)}\right)\right] \left[\left(1 - d_t^{(t_0,j)}\right) \left(e^{y(s_t)+z_t(t_0,j)} F\left(n_t^{(t_0,j)}\right) - k_f\right.\right. \tag{41}$$
$$... - \frac{c}{q\left(\theta_{i,t}^{(t_0,j)}\right)} n_{i,t}^{(t_0,j)} - \eta_t\left(\theta_{i,t}^{(t_0,j)}\right) n_{i,t}^{(t_0,j)} + \eta_t\left(\theta_{x,t}^{(t_0,j)}\right) n_{t-1}^{(t_0,j)} \left(1 - \tau_t^{(t_0,j)}\right) \lambda p\left(\theta_{x,t}^{(t_0,j)}\right))$$
$$... + \mu_t n_{t-1}^{(t_0,j)} \left(d_t^{(t_0,j)} + \left(1 - d_t^{(t_0,j)}\right) \tau_t^{(t_0,j)}\right)\Big]$$
$$... - m_{e,t} k_e + u_t b - \mu_t \left(u_t - u_{t-1}\left(1 - p(\theta_{u,t})\right)\right) + \eta_t\left(\theta_{u,t}\right) u_{t-1} p\left(\theta_{u,t}\right)\Bigg\},$$

where constraint (27) is implicitly substituted. To complete the proof, I am now going to show that a block-recursive competitive equilibrium (with non-negative entry) satisfies the first-order conditions of the planner. Pick a block-recursive equilibrium by $\{\mathbf{V}, \mathbf{U}, \kappa^*(s), \theta^*(s,x)\}$. Guess the following Lagrange multipliers:

$$\mu_t(s^t) = \mathbf{U}(s_t)$$
$$\eta_t(s^t, \theta) = x(s_t, \theta) \text{ s.t } x(s_t, \theta) = \theta^{*-1}(s_t, \theta).$$

In particular, notice that the Lagrange multipliers only depend on the current aggregate state of the economy, s_t, and not on its entire history. One may worry here about the invertibility of the equilibrium function θ^*, but we know thanks to lemma 1 that there always exists a corresponding promised utility x for all values of θ in $[0, \infty)$ given by $x = \kappa(s) - c/q(\theta)$.[4] Given this guess, we can now recognize that the planner's objective is to sum the joint-surplus $\mathbf{V}$ of incumbent and entering firms and the utility of unemployed workers $\mathbf{U}$. Each of these problems can be solved independently and we know that the policies obtained in the competitive equilibrium maximize each of them. To see this, let us have a look at the parts of the Lagrangian

[4] The bounds $[\underline{x}, \overline{x}]$ are chosen so that the optimal x lies in the interior, so that we are not constraining the equilibrium.

corresponding to a single existing firm given our choice of Lagrange multipliers:

$$
\begin{aligned}
&\max_{\{\tau_t,\theta_{x,t},d_t,n_{i,t},\theta_{it}\}_t} \\
&\mathbb{E}\sum_t \beta^t \left[\prod_{l=-\infty}^{t-1}(1-d_l)\right]\Big[(1-d_t)\Big(e^{y(s_t)+z_t}F(n_t)-k_f-\left(\frac{c}{q\left(\theta_{i,t}\right)}+x\left(s_t,\theta_{i,t}\right)\right)n_{i,t} \\
&\ldots+n_{t-1}\left(1-\tau_t\right)\lambda p\left(\theta_{x,t}\right)x\left(s_t,\theta_{x,t}\right)\Big)+n_{t-1}(d_t+(1-d_t)\tau_t)\mathbf{U}(s_t)\Big],
\end{aligned}
$$

which is the sequential formulation of the surplus maximization problem in the competitive equilibrium. Turning to firms entering at date t:

$$
\begin{aligned}
&\max_{\{\tau_{t'},\theta_{x,t'},d_{t'},n_{i,t'},\theta_{i,t'}\}_{t'\geq t}} m_{e,t}\Big\{-k_e+\mathbb{E}\sum g_z\left(z_t\right)\times\ldots \\
&\sum_{t'=t}^{\infty}\beta^{t'-t}\left[\prod_{l=t}^{t'-1}(1-d_l)\right]\Big[(1-d_{t'})\Big(e^{y\left(s_{t'}\right)+z_{t'}}F(n_{t'})-k_f-\left(\frac{c}{q\left(\theta_{i,t'}\right)}+x\left(s_{t'},\theta_{i,t'}\right)\right)n_{i,t} \\
&+n_{t'-1}(1-\tau_{t'})\lambda p(\theta_{x,t'})x(s_{t'},\theta_{x,t'})\Big)+n_{t'-1}(d_{t'}+(1-d_{t'})\tau_{t'})\mathbf{U}(s_{t'})\Big]\Big\}.
\end{aligned}
$$

This is the sequential formulation of the free-entry problem solved in the competitive equilibrium. The planner increases the number of entrants $m_{e,t}$ as long as the expected surplus from entering is equal to the entry cost k_e. Now, let us examine the part of the Lagrangian related to unemployed workers:

$$
\max_{\{\theta_{u,t},u_t\}_t}\sum_t \beta^t\Big[u_t b-\mathbf{U}(s_t)\big(u_t-u_{t-1}(1-p(\theta_{ut}))\big)+u_{t-1}p(\theta_{u,t})x(s_t,\theta_{u,t})\Big]
$$

The first-order conditions with respect to u_{t+1} and θ_{ut} are equal to

$$
\begin{aligned}
&[u_t] \quad b-\mathbf{U}(s_t)+\beta\mathbb{E}\left[\left(1-p\left(\theta_{ut+1}\right)\right)\mathbf{U}\left(s_{t+1}\right)+p\left(\theta_{ut+1}\right)x\left(s_{t+1},\theta_{ut+1}\right)\right]=0 \\
&[\theta_{u,t}] \quad -u_{t-1}p'\left(\theta_{ut}\right)\mathbf{U}(s_t)+u_{t-1}p'\left(\theta_{ut}\right)x\left(s_t,\theta_{ut}\right)+u_{t-1}p\left(\theta_{u,t}\right)x_\theta(s_t,\theta_{u,t})=0.
\end{aligned}
$$

We recognize in the first equation the Bellman equation faced by unemployed workers and, in the second equation, the first-order condition corresponding to their problem. Therefore, the policies obtained from the competitive equilibrium maximize the planner's problem given our choice of Lagrange multipliers. The first-order conditions are thus satisfied. Block-recursive equilibria are thus efficient. □

H.3 Proofs of part F

(ii) First, recall that

$$
p(\theta\left(s,x\right))=p\circ q^{-1}\left(\frac{c}{\kappa\left(s\right)-x}\right).
$$

Under assumption 2, $p\left(\theta(s,x)\right)$ is a strictly decreasing, strictly concave function of $x\in[\underline{x},\kappa(s)-c]$.

Proof of proposition 1. (i) Pick a contract $\omega=\{w,\tau,x,W',d\}$ that implement the firm's optimal policy. Consider now the modified contract $\tilde{\omega}=\{w+a\Delta,\tau,x,W'-\Delta,d\}$ where $a=\beta\mathbb{E}\left[(1-d)(1-\tau)\left(1-\lambda p(\theta\left(s',x\right))\right)\right]$. The worker's utility under this new contract is

$$
\begin{aligned}
\mathbf{W}(s,z,\tilde{\omega}) &= w+a\Delta+\beta\mathbb{E}\Big[\left(d+\left(1-d\right)\tau\right)\mathbf{U}\left(s'\right)+\left(1-d\right)\left(1-\tau\right)\lambda p\left(\theta\left(s',x\right)\right)x \\
&\qquad \ldots+\left(1-d\right)\left(1-\tau\right)\Big(1-\lambda p\left(\theta\left(s',x\right)\right)\Big)\left(W'-\Delta\right)\Big] \\
&= \mathbf{W}(s,z,\omega)
\end{aligned}
$$

The worker's utility is unchanged. His promise-keeping constraint is thus still satisfied. Turning to the firm's profits:

$$\begin{aligned}
&\mathbf{J}\left(s, z, n, \{W(j)\}_{j\in[0,n]}\right) = e^{y(s)+z}F(n) - k_f - \int_0^n w(j)dj \\
&\quad + \beta\mathbb{E}\left[(1-d')\left(-n_i'\frac{c}{q(\theta(s', x_i'))} + \mathbf{J}\left(s', z', n', \left\{\hat{W}'\right\}\right)\right)\right] \\
&= e^{y(s)+z}F(n) - k_f - \int_0^n w(j)dj + \beta\mathbb{E}\Big[(1-d')\Big(\mathbf{V}(s', z', n') \\
&\quad - \int_0^n (1-\tau)(1-\lambda p(\theta(s', x)))W'dj - n_i'(c/q(\theta(s', x_i)) + x_i)\Big)\Big].
\end{aligned}$$

Under the new contract $\tilde{\omega}$, we have

$$\begin{aligned}
& -\int_0^n \tilde{w}(j)dj + \beta\mathbb{E}\int_0^n (1-d)(1-\tau)(1-\lambda p(\theta(s', x)))\tilde{W}'(s', z'; j)\,dj \\
= & -\int_0^n w(j)dj + \beta\mathbb{E}\int_0^n (1-d)(1-\tau)(1-\lambda p(\theta(s', x)))W'(s', z'; j)\,dj,
\end{aligned}$$

so the firm's profit is unchanged. The new contract leaves the firm and workers indifferent and implements the firm's optimal policy as well.

(ii) It is useful to rewrite the surplus maximization problem as a two-step problem

$$\begin{aligned}
\mathbf{V}(s, z, n) \quad = \quad & \max_{d,n} \mathrm{e}^{y(s)+z}F(n) - k_f + \beta\mathbb{E}\Big\{dn\mathbf{U}(s') + (1-d)\Big[v(s', z', n, n') \\
& + \mathbf{V}(s', z', n')\Big]\Big\}
\end{aligned}$$

with

$$\begin{aligned}
v(s, z, n, n') \quad = \quad & \max_{n_i, x_i, \{\tau(j), x(j)\}} \mathbf{U}(s)\int_0^n \tau(j)\mathrm{d}j + \int_0^n (1-\tau(j))\,\lambda p(\theta(s, x(j)))\,x(j)\mathrm{d}j \\
& -\left(\frac{c}{q(\theta(s, x_i))} + x_i\right)n_i \\
\text{subject to} \quad & n' = \int_0^n (1-\tau)(1-\lambda p(\theta(s, x)))\,dj + n_i.
\end{aligned}$$

First, it is easy to show that if $n' \geq n$, then it is optimal to set $n_i = n' - n$, $\tau = 0$ and $x = \kappa(s) - c$ so that $p(\theta(s, x)) = 0$. Indeed, since it is costly to hire workers, it is never optimal to layoff or let any worker leave for another firm if it wants to expand. Let us now focus on the case in which $n' < n$. Again, it is easy to show in this case that $n_i = 0$. However, the firm must solve a trade-off between layoffs and job-to-job transitions which we can write as

$$\begin{aligned}
\max_{\{\tau(j), x(j)\}} \quad & \mathbf{U}(s)\int_0^n \tau(j)\mathrm{d}j + \int_0^n (1-\tau(j))\,\lambda p(\theta(s, x(j)))\,x(j)\mathrm{d}j \\
\text{subject to} \quad & n' = \int_0^n (1-\tau)(1-\lambda p(\theta(s, x)))\,dj.
\end{aligned}$$

Proceeding with the change of variables $\theta(j) = q^{-1}(c/(\kappa(s) - x(j)))$, the problem becomes strictly concave in $\theta(j)$. Taking the first order conditions with respect to $x(j)$,

$$(\kappa(s) + \mu)\,p'(\theta(j)) = c,$$

where μ is the Lagrange multiplier on the constraint. We thus conclude that $\theta(j)$ and thus $x(j)$ are identical across workers within a given firm.
(iii) Imposing that $x(j) = x, \quad \forall j \in [0,n]$, it is trivial to see that any permutation of the τ's between workers or any transformation that leave the total mass of layoff unchanged does not affect the objective function. The total number of layoff though is uniquely determined using the constraint: $\int_0^n \tau(s',z';j)dj = n - (1-\lambda p(\theta))^{-1} n'$. $\square$
Proof of proposition 2. I will prove the result in two steps. I will first show that if the firm can choose any continuing utility $W'(s',z')$, it is possible to find a schedule $W'(s',z';x')$ that makes the worker choose x exactly. We will then show that this continuing utility must satisfy the participation constraint, i.e., $\lambda p(\theta(s',x))x + (1-\lambda p(\theta(s',x)))W'(s',z';x') \geq \mathbf{U}(s')$.

Step 1. Fix (s',z'). Recall that workers solve the problem[5]

$$x = \underset{\tilde{x}\in[\underline{x},\kappa(s')-c]}{\operatorname{argmax}} \; p\left(\theta(s',\tilde{x})\right)\left(\tilde{x} - W'(s',z')\right).$$

Define

$$\tilde{D}(x,W') = p(\theta(s',x))(x-W') \text{ and } \begin{cases} D(s',W') = \underset{x\in[\underline{x},\kappa(s)-c]}{\max} \tilde{D}(x,W') \\ C(s',W') = \underset{\tilde{x}\in[\underline{x},\kappa(s')-c]}{\operatorname{argmax}} \tilde{D}(x,W') \end{cases}$$

$\tilde{D}$ is a continuous function of x and W'. It reaches a non-negative maximum in x on $[W',\kappa(s')-c]$. Assumption 2 guarantees that $\tilde{D}$ is strictly concave in x on $[W',\kappa(s')-c]$. The Theorem of the Maximum tells us therefore that $D(W')$ and $C(W')$ are continuous functions of W'. Thus, p being strictly positive over $[\underline{x},\kappa(s')-c)$, D is strictly decreasing on $[-\infty,\kappa(s')-c]$. Therefore, C is strictly increasing on $[-\infty,\kappa(s')-c]$, as can be seen from the following: take $W_1 < W_2 \leq \kappa(s')-c$. Denote $x_k = C(W_k), k=1,2$. Then the following is true:

$$p\left(\theta\left(s',x_1\right)\right)\left(x_1 - W'_1\right) - p\left(\theta\left(s',x_2\right)\right)\left(x_2 - W'_2\right) < p\left(\theta\left(s',x_1\right)\right)\left(W'_2 - W'_1\right),$$

and

$$p\left(\theta\left(s',x_1\right)\right)\left(x_1 - W'_1\right) - p\left(\theta\left(s',x_2\right)\right)\left(x_2 - W'_2\right) > p\left(\theta\left(s',x_2\right)\right)\left(W'_2 - W'_1\right).$$

Therefore, $\theta(s',x_1) > \theta(s',x_2)$, and since in equilibrium $\theta(s',x) = q^{-1}\left(c/\left(\kappa(s')-x\right)\right)$ is decreasing in x, we have:$x_2 > x_1$ and C is strictly increasing.
Now, let us show that C reaches $\underline{x}$ and $\kappa(s)-c$. For $W' = \kappa(s)-c$, function $\tilde{D}$ trivially reaches its maximum at $x = W' = \kappa(s')-c$. Does it reach $\underline{x}$? Rewrite the maximization problem of the worker over θ:

$$\begin{aligned} \tilde{D} &= \max_{\theta\in[0,\theta(s,\underline{x})]} p(\theta)\left(\tilde{x}(\theta) - W'\right) \\ &= \max_{\theta\in[0,\theta(s,\underline{x})]} p(\theta)\left(\kappa(s') - W'\right) - c\theta, \end{aligned}$$

where I have used the equilibrium relationship: $\kappa(s) = x + c/q\left(\theta(s,x)\right)$. This is a well defined strictly concave maximization problem and its derivative with respect to θ is

$$p'(\theta)(\kappa(s') - W') - c,$$

so that $\theta = (p')^{-1}\left(c/\left(\kappa(s')-W'\right)\right)$. Therefore, setting W' to equal $\kappa(s') - c/p'\left(\theta(s',\underline{x})\right)$, the optimum is reached at $\theta(s',\underline{x})$ and the worker chooses to search in submarket $\underline{x}$. $C(W')$ is thus a continuous strictly increasing function that reaches $\underline{x}$ and $\kappa(s')-c$. By the Intermediate Value Theorem, for any $x \in [\underline{x},\kappa(s')-c]$, there exists a unique $W'_{IC}(x')$ such that

[5] Remember that $x = \kappa(s) - c$ is the highest active submarket in equilibrium. It satisfies $\theta(s,x) = 0$.

$\max_{\tilde{x}} \tilde{D}(\tilde{x}, W'_{IC}(x))$ is reached at x exactly. In other words, there exists a unique continuation utility $W'_{IC} \in [-\infty, \kappa(s') - c]$ that makes the worker choose exactly x. To finish this first step, we must choose the rest of the contract. Set $\tau_{IC} = \tau$ and $d_{IC} = d$. Now, in an optimal allocation, w_{IC} must be chosen so that the promise-keeping constraint is binding. The worker's expected utility is

$$\begin{aligned}\mathbf{W}(s, z, \{w, \tau, x, d, W'\}) =& w + \beta\mathbb{E}\Big[(1-d)(1-\tau)\lambda p(\theta(s', x))\, x \\ &+ (d + (1-d)\tau)\mathbf{U}(s') + (1-d)(1-\tau)(1 - \lambda p(\theta(s', x)))W'\Big].\end{aligned}$$

Given $\{\tau_{IC}, x_{IC}, d_{IC}, W'_{IC}\}$, there exists a unique wage w_{IC} that matches exactly the promised utility. This does not affect the joint surplus, which is maximized by assumption. From proposition 1, the firm's profit is maximized when the level of promised utility is exactly achieved. We have thus found a contract that implements the optimal allocation.

Step 2. I will now proceed to the second step of the proof and show that the participation constraint is satisfied by $W'(s', z'; x)$. Let us first have a look at the problem faced by the worker choosing whether or not to leave the firm at the time of separation. The participation constraint is satisfied if

$$\max_x \lambda p(\theta(s', x))x + (1 - \lambda p(\theta(s', x)))W'(s', z'; x) \geq \mathbf{U}(s).$$

Abusing notation slightly, denote $p(s', x) \equiv p(\theta(s', x))$, we can derive the first-order condition for the worker:

$$\lambda p'(s', x)(x - W') + \lambda p(s', x) = 0.$$

Turning back to the joint surplus maximization, the terms related to x and τ are

$$\begin{aligned}&\mathbf{U}(s')\int_0^n \tau \mathrm{d}j + \lambda p(s', x)\, x \int_0^n (1-\tau)\mathrm{d}j \\ &+ \mathbf{V}(s', z', \int_0^n (1 - \lambda p(s', x))(1-\tau)\mathrm{d}j + n_i).\end{aligned}$$

To simplify the notatinon, write $nT = \int_0^n \tau dj$, T being the total fraction of layoffs. We can rewrite the above term as

$$nT\mathbf{U}(s') + n(1-T)\lambda p(s', x)\, x + \mathbf{V}(s', z', n(1-T)(1 - \lambda p(s', x)) + n_i).$$

The first-order condition with respect to x is

$$n(1-T)\lambda p'(s', x)\Big(x - \mathbf{V}_n\big(s', z', n(1-T)(1-\lambda p(s', x)) + n_i\big)\Big) + n(1-T)\lambda p(s', x) = 0.$$

Notice that it is possible to identify W' from the two first-order conditions. The incentive compatible contract must be such that

$$W'(s', z') = \mathbf{V}_n\Big(s', z', n(1-T)(1 - \lambda p(s', x)) + n_i\Big).$$

To verify whether the participation constraint is satisfied, it is informative to look at the first-order condition with respect to T (ignoring the irrelevant case where $T = 1$):

$$n\mathbf{U}(s') - n\Big(\lambda p(s', x)x + (1 - \lambda p(s', x))\mathbf{V}_n\Big) \leq 0,$$

which is exactly equivalent to the participation constraint

$$\lambda p(s', x)x + (1 - \lambda p(s', x))W'(s', z'; x) \geq \mathbf{U}(s').$$

The incentive-compatible contract therefore satisfies the participation constraint. $\square$

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