Supplement to "Monetary policy switching and indeterminacy"

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Appendix A: Determinacy among Markovian equilibria

In this section, we revisit the results of Davig and Leeper (2007). We first define Markovian equilibria and then we prove a slightly extended proposition.

Definition 1. A solution, z_t, is said to be Markovian of order p, if it depends on the p past regimes, $\{s_t, s_{t-1}, \dots, s_{t-p}\}$, and all past shocks, ε^t, that is, there exists a measurable function, ϕ, mapping $\{1, \dots, N\}^{p+1} \times (\mathbb{R}^p)^\infty$ into $\mathbb{R}^n$ such that $z_t = \phi(\{s_t, s_{t-1}, \dots, s_{t-p}\}, \varepsilon^t)$.

This definition is a generalization of what Branch, Davig, and McGough (2007) call a "regime dependent equilibrium" and that we denote by $\mathcal{M}_0$ in Section 2.3. In our terminology, such an equilibrium is a 0-order Markovian solution.

We introduce a matrix, $\mathbf{M}$, as a combination between transition probabilities matrix, P, and a $(n.N \times n.N)$ real matrix, diagonal by blocks, $\text{diag}(\Gamma_1, \dots, \Gamma_N)$: $\mathbf{M} = (P \otimes \mathbb{1}_n) \times \text{diag}(\Gamma_1, \dots, \Gamma_N)$. The mathematical symbol $\otimes$ denotes the standard Kronecker product. Proposition 1 refines and complements the main proposition by Davig and Leeper (2007).

Proposition 1. 1. *Model* (13) *admits a unique Markovian bounded solution*, z_t^F, (*but also a unique bounded equilibrium in* $\mathcal{M}_0$) *if and only if the spectral radius of* $\mathbf{M}$, *that is, the largest eigenvalue in absolute value*, $\rho(\mathbf{M})$, *is strictly less than one.*

2. *Otherwise, all the bounded Markovian equilibria can be put into the following form*:

$$z_t = z_t^F + V_{s_t} w_t, \quad \text{where } w_t = J_w w_{t-1} + \xi_t,$$

with, the sunspot ξ_t *being any bounded zero mean process* ($\mathbb{E}_t \xi_{t+1} = 0$) *independent of current and past regimes. The sunspot can be either a sunspot shock as defined in Cass and Shell* (1983) *or a fundamental disturbance. Matrices* J_w *and* V_{s_t} *are defined in equations* (30) *and* (31).

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Proving the first point is done in two steps:

- If ϕ is a Markovian solution of equation (13), then $\phi \in \mathcal{M}_0$.
- Furthermore if $\phi \in \mathcal{M}$, then defining $\mathbf{\Phi}$ by

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix} \phi(1s^{t-1}, \varepsilon^t) \\ \vdots \\ \phi(Ns^{t-1}, \varepsilon^t) \end{bmatrix}.$$

$\mathbf{\Phi}$ is the solution of a linear rational expectations model with regime-independent parameters. We can thus apply the Blanchard and Kahn (1980) technique.

Assuming that there exists a p-order Markovian solution of (13), ϕ, we define $\mathcal{P}(q)$ as the statement that the solution only depends on the past q regimes:

$$\begin{aligned} &\mathcal{P}(q) : \phi(is_1 \cdots s_q w, \varepsilon^t) = \phi(is_1 \cdots s_q w', \varepsilon^t) \\ &\quad \forall (s_1, \ldots, s_q) \in \{1, \ldots, N\}^q, \forall w \in \{1, \ldots, N\}^\infty, \forall w' \in \{1, \ldots, N\}^\infty, \forall \varepsilon^t \in V^\infty. \end{aligned}$$

$\mathcal{P}(p)$ is satisfied by assumption. Let us assume that $\mathcal{P}(q)$ is met for $q \in \{1, \ldots, p\}$. Since ϕ is a solution of (13), for any w, we compute

$$\phi(s_t s_1 \cdots s_{q-1} w, \varepsilon^t) = \Gamma_{s_t}^{-1}(\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w, \varepsilon\varepsilon^t)\, d\varepsilon + \Gamma_{s_t}^{-1} \Omega_{s_t} \varepsilon_t.$$

Due to $\mathcal{P}(q)$, we know that

$$\Gamma_{s_t}^{-1}(\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w, \varepsilon\varepsilon^t)\, d\varepsilon = \Gamma_{s_t}^{-1}(\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w', \varepsilon\varepsilon^t)\, d\varepsilon$$

for any w', and hence, ϕ does not depend on w. $\mathcal{P}(q-1)$ is thus satisfied. By decreasing induction, we eventually show that ϕ is Markovian of order 0.

More generally, if the solution is Markovian, its order is the same as ψ_0. Here, ψ_0 is Markovian of order 0, thus ϕ is also Markovian of order 0.

If $\phi \in \mathcal{M}_0$ is a solution of (13), ϕ is a solution of

$$\forall i \in \{1, \ldots, N\} \phi(i, \varepsilon^t) - \Gamma_i^{-1}\left(p_{i1} \int \phi(1, \varepsilon\varepsilon^t)\, d\varepsilon + p_{i2} \int \phi(2, \varepsilon\varepsilon^t)\, d\varepsilon\right) = \Gamma_i^{-1} \Omega_i \varepsilon_t.$$

We consider

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix} \phi(1, \varepsilon^t) \\ \phi(2, \varepsilon^t) \\ \vdots \\ \phi(N, \varepsilon^t) \end{bmatrix}. \tag{27}$$

Thus, by introducing

$$\mathbf{D} = \begin{bmatrix} \Gamma_1^{-1}\Omega_1 \\ \vdots \\ \Gamma_N^{-1}\Omega_N \end{bmatrix},$$

the system is rewritten as

$$\mathbf{\Phi}(\varepsilon^t) - \mathbf{M}\int \mathbf{\Phi}(\varepsilon\varepsilon^t)\, d\varepsilon = D\varepsilon_t, \tag{28}$$

where $\mathbf{\Phi}$ is defined in equation (27). Model (28) is a standard linear rational expectations model with constant parameters. We hence easily prove Proposition 1 by applying the Blanchard and Kahn (1980) technique.

We denote by $\mathcal{B}_0$ the set of bounded functions on V^∞, and by $\mathcal{F}$ the bounded operator acting in $\mathcal{B}_0$:

$$\mathcal{F} : \mathbf{\Phi} \mapsto \left((\varepsilon^t) \mapsto \int \mathbf{\Phi}(\varepsilon\varepsilon^t)\, d\varepsilon \right).$$

We rewrite equation (28) as $[\mathbb{1} + \mathbf{M}\mathcal{F}]\mathbf{\Phi} = \boldsymbol{\Psi}_\mathbf{0}$, where $\boldsymbol{\Psi}_\mathbf{0}(\varepsilon_t) = \mathbf{D}\varepsilon_t$. The solution $\mathbf{\Phi}$ is then $\mathbf{\Phi} = \sum_{k=0}^{\infty}[-\mathbf{M}\mathcal{F}]^k \boldsymbol{\Psi}_\mathbf{0}$. Knowing that

$$\left(\mathcal{F}^k \boldsymbol{\Psi}_\mathbf{0}\right)(\varepsilon_t) = \mathbf{D}\mathbb{E}_t\varepsilon_{t+k}.$$

The solution is then given by

$$\mathbf{\Phi}(\varepsilon^t) = -\sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D}\mathbb{E}_t\varepsilon_{t+k}$$

thus,

$$\boldsymbol{\phi}(s_t, \varepsilon^t) = U_{s_t} \sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D}\mathbb{E}_t\varepsilon_{t+k},$$

where

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{bmatrix} = -\mathbb{1}.$$

Defining

$$z_t^F = U_{s_t} \sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D}\mathbb{E}_t\varepsilon_{t+k} \tag{29}$$

we notice that the solution z_t^F only depends on s_t and ε_t. In existing literature, z_t^F is called the fundamental or the minimum state variable solution.

In the case where $\rho(\mathbf{M}) > 1$, we apply the strategy of solving linear rational expectations models (see Blanchard and Kahn (1980) and Lubik and Schorfheide (2004)). There exists an invertible matrix Q such that

$$\mathbf{M} = \mathbf{Q} \times \begin{bmatrix} \Delta_u & R_u \\ 0 & \Delta_s \end{bmatrix} \mathbf{Q}^{-1}$$

with $\rho(\Delta_u) > 1$ and $\rho(\Delta_s) < 1$. Writing $Z_t = Q\left[\begin{smallmatrix} Z_t^u \\ Z_t^s \end{smallmatrix}\right]$, Z^s and Z^u are such that

$$Z_t^s = \sum_{k=0}^{\infty} (-\Delta_s)^k [0 \quad \mathbb{1}] Q \Omega \mathbb{E}_t \varepsilon_{t+k},$$

$$\Delta_u \mathbb{E}_t Z_{t+1}^u + R_u \mathbb{E}_t Z_{t+1}^s + Z_t^u = [\mathbb{1} \quad 0] Q \Omega \varepsilon_t.$$

A general solution of the previous equation is then

$$Z_t^u = Z_t^{u,0} + \sum_{k=0} (-\Delta_u)^{-k} \xi_{t-k},$$

where ξ_t is a zero mean sunspot and solution $Z_t^{u,0}$ is such that

$$Z_t^{u,0} = \sum_{k=0}^{\infty} (-\Delta_u)^{-k} [\mathbb{1} \quad 0] Q \Omega \varepsilon_{t-k-1} - \sum_{k=0}^{\infty} (-\Delta_u)^{-k} Z_{t-k}^s.$$

Then the solutions are given by

$$Z_t = z_t^F - Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} (-\Delta_u)^{-k} \xi_{t-k}.$$

And finally,

$$z_t = z_t^F + V_{s_t} w_t$$

with

$$V_1 = -[\mathbb{1} \quad 0] Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \quad \text{and} \quad V_2 = -[0 \quad \mathbb{1}] Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix}. \tag{30}$$

w satisfies

$$w_t = (-\Delta_u)^{-1} w_{t-1} + \xi_t.$$

Defining J_w by

$$J_w = (-\Delta_u)^{-1} \tag{31}$$

leads to the result.

Appendix B: Proof of Proposition 1

In this section, we prove Proposition 1. Assuming that Γ_i is invertible for any $i \in \{1, \dots, N\}$, we rewrite (13) as

$$z_t - \Gamma_{s_t}^{-1}\mathbb{E}_t z_{t+1} = -\Gamma_{s_t}^{-1}\Omega_{s_t}\varepsilon_t. \tag{32}$$

Then, considering $z_t = z(s^t, \varepsilon^t)$ as a function of all the past shocks $\{\varepsilon_t, \dots, \varepsilon_{-\infty}\}$ and regimes $\{s_t, \dots, s_{-\infty}\}$, introducing ψ_0 such that $\psi_0(s^t, \varepsilon^t) = \Gamma_{s_t}^{-1}\Omega_{s_t}\varepsilon_t$ and defining the operator $\mathcal{R}$ as

$$\mathcal{R} : z \mapsto \big((s^t, \varepsilon^t) \mapsto \Gamma_{s_t}^{-1}\mathbb{E}_t z\big(s^{t+1}, \varepsilon^{t+1}\big)\big). \tag{33}$$

Equation (32) is equivalent to the functional equation:

$$(\mathbb{1} - \mathcal{R})z = \psi_0. \tag{34}$$

This equation admits a unique solution if the operator $\mathbb{1} - \mathcal{R}$ is invertible, and thus, if $1 \notin \sigma(\mathcal{R})$. Consequently, conditions of existence and uniqueness of a solution of (13) rely on the spectrum of $\mathcal{R}$, this spectrum depending on the space of solutions we consider.

Before characterizing this spectrum, we first show that the sequence (u_k) in equation (17) is convergent.

B.1 *Behavior of the sequence u_k*

In this section, we prove the following result.

Lemma 1. *The sequence (u_k) in equation* (17) *has the following properties*:

- *The sequence $(u_k)^k$ is submultiplicative ($(u_{m+n})^{m+n} \le u_m^m u_n^n$), and thus, convergent.*
- *The limit, ν, does not depend on the chosen norm.*

We first show that (u_k^k) is submultiplicative. Using the submultiplicativity of a matricial norm, u_{m+n}^{m+n} satisfies

$$\begin{aligned}
&\sum_{(i_1,\dots,i_m,i_{m+1},\dots,i_{m+n})\in\{1,\dots,N\}^{m+n}} p_{i_1 i_2}\cdots p_{i_{m-1}i_m} \\
&\quad\times p_{i_m i_{m+1}}\cdots p_{i_{m+n-1}i_{m+n}}\big\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\big\| \\
&\le \sum_{(i_1,\dots,i_m,i_{m+1})\in\{1,\dots,N\}^{m+1}} p_{i_1 i_2}\cdots p_{i_{m-1}i_m} p_{i_m i_{m+1}}\big\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\big\| \\
&\quad\times\Bigg(\sum_{(i_{m+2},\dots,i_{m+n})\in\{1,\dots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}}\big\|\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\big\|\Bigg).
\end{aligned}$$

We find an upper bound for the second term by adding up i_{m+1} and as all the terms are positive:

$$\sum_{(i_{m+2},\dots,i_{m+n})\in\{1,\dots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}}\left\|\Gamma^{-1}_{i_{m+1}}\cdots\Gamma^{-1}_{i_{m+n}}\right\|$$
$$\leq \sum_{(i_{m+1},i_{m+2},\dots,i_{m+n})\in\{1,\dots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}}\left\|\Gamma^{-1}_{i_{m+1}}\cdots\Gamma^{-1}_{i_{m+n}}\right\| = (u_n)^n.$$

Thus,

$$(u_{n+m})^{n+m} \leq u_n^n \sum_{(i_1,\dots,i_m,i_{m+1})\in\{1,\dots,N\}^{m+1}} p_{i_1i_2}\cdots p_{i_{m-1}i_m}p_{i_mi_{m+1}}\left\|\Gamma^{-1}_{i_1}\cdots\Gamma^{-1}_{i_m}\right\| = u_n^n \times u_m^m$$

since $\sum_{i_{m+1}\in\{1,\dots,N\}} p_{i_mi_{m+1}} = 1$.

This shows that (u_k^k) is submultiplicative.

Besides, if a sequence of nonnegative real numbers (v_k) is submultiplicative, then $v_k^{1/k}$ is converging and $\lim_{k\to+\infty} v_k^{1/k} = \inf_k v_k^{1/k}$; see, for instance, Lemma 21, page 8 in Müller (2003). Thus, (u_k) is convergent.

Finally, because of the equivalence of the norms in $\mathcal{M}_n(\mathbb{R})$, it is immediate that ν does not depend on the chosen norm.

B.2 *Characterization of the spectral radius of $\mathcal{R}$*

We will prove the following lemma, describing the spectrum of $\mathcal{R}$ in $\mathcal{B}$.

LEMMA 2. *The operator $\mathcal{R}$ is bounded in $\mathcal{B}$ and its spectrum is given by*

$$\sigma(\mathcal{R}) = [-\nu, \nu].$$

First, $\mathcal{R}$ is bounded as the expectation operator is a bounded operator. The rest of the proof is based on two main arguments:

- The spectrum of $\mathcal{R}$ is symmetric convex.
- $$\lim_{k\to+\infty}\left\|\mathcal{R}^k\right\|^{1/k} = \nu.$$

The second point ensures that $\rho(\mathcal{R}) = \nu$ by applying the Gelfand characterization of the spectral radius for an operator (see, for instance, Theorem 22, page 8 in Müller (2003)), while the first point leads to the equality $\sigma(\mathcal{R}) = [-\nu, \nu]$.

First, we introduce operators $\mathcal{F}_i$, for $i \in \{1,\dots N\}$, $\mathcal{F}$ and $\mathcal{L}$ on $\mathcal{B}$ defined by

$$\mathcal{F}_i : \phi \mapsto (\left(s^t, \varepsilon^t\right) \mapsto \int_V \phi\left(is^t, \varepsilon\varepsilon^t\right)d\varepsilon,$$
$$\mathcal{L} : \phi \mapsto (\left(s^t, \varepsilon^t\right) \mapsto \phi\left(s^{t-1}, \varepsilon^{t-1}\right),$$
$$\mathcal{F}(\phi)\left(s^t, \varepsilon^t\right) = (p_{s_t1}\mathcal{F}_1 + p_{s_t2}\mathcal{F}_2)(\phi)\left(s^t, \varepsilon^t\right).$$

Operators $\mathcal{F}_i$ and $\mathcal{L}$ have the following straightforward properties:

1. $\mathcal{F}_i\mathcal{L} = \mathbb{1}$, and $\mathcal{F}\mathcal{L} = \mathbb{1}$,
2. $|||\mathcal{F}_i||| = 1$ and $|||\mathcal{L}||| = 1$,

where $|||\cdot|||$ is the triple norm associated with the infinite norm $\|\cdot\|_\infty$ on $\mathcal{B}$. Then $\mathcal{R}$ can be rewritten as

$$\mathcal{R}(\phi)(s^t, \varepsilon^t) = \Gamma_{s_t}^{-1}(p_{s_t1}\mathcal{F}_1 + p_{s_t2}\mathcal{F}_2)(\phi)(s^t, \varepsilon^t).$$

We define $\tilde{\mathcal{R}}$ by

$$\tilde{\mathcal{R}} : \phi \mapsto \Gamma_{s_{t-1}}\mathcal{L}(\phi)(s^t, \varepsilon^t).$$

We have that

$$\tilde{\mathcal{R}}\mathcal{R} = \mathcal{L}\mathcal{F}, \qquad \mathcal{R}\tilde{\mathcal{R}} = \mathbb{1}.$$

We copy the techniques used to study the spectrum of isometries in Banach spaces such as that of Conway (1990). We refer to this publication and to Müller (2003) for the different types of spectrum. We know that the spectrum of $\mathcal{R}$ is a closed subset of $[-\|\mathcal{R}\|, \|\mathcal{R}\|]$, and that the boundary $\partial\sigma(\mathcal{R})$ of $\sigma(\mathcal{R})$ is included in the point approximate spectrum, that is, the set of values λ such that $\mathcal{R} - \lambda\mathbb{1}$ is neither injective nor bounded below. We assume that $\sigma(\mathcal{R})$ is not convex, and that there exists $\lambda_0 \in (0, \nu)$ such that $\lambda \in \partial\sigma(\mathcal{R})$. We then prove that λ_0 is an eigenvalue. Actually, $\mathcal{R} - \lambda\mathbb{1}$ is bounded below for any $\lambda < \|\mathcal{R}\|$. $\mathcal{R}$ is the composition of an invertible operator and an isometry, and is thus bounded below. Moreover, we notice that

$$\|\mathcal{R}\| = \sup_{v\in \operatorname{Im}(\tilde{\mathcal{R}})} \frac{\|\mathcal{R}v\|}{\|v\|} = \left(\inf_u \frac{\|\tilde{\mathcal{R}}u\|}{\|u\|}\right)^{-1}$$

which implies that

$$\|u - \lambda\tilde{\mathcal{R}}u\| \leq \left(1 - \frac{\lambda}{\|\mathcal{R}\|}\right)\|u\|.$$

We show now that for any α such that $|\alpha| < 1$, then $\lambda\alpha$ belongs to $\sigma(\mathcal{R})$. We know that λ is an eigenvalue of $\mathcal{R}$, let $\phi_0 \in \mathcal{B}$ an eigenvector of $\mathcal{R}$ associated with λ,

$$\mathcal{R}\phi_0 = \lambda\phi_0.$$

We define f by

$$f = \phi_0 - \lambda\tilde{\mathcal{R}}\phi_0.$$

We notice that $\mathcal{R}(f) = o$, and that $\|(\lambda\tilde{\mathcal{R}})^k(f)\| \leq \|\phi_0\|$. Fix α such that $|\alpha| < 1$. We define $\tilde{\phi}_0$ by

$$\tilde{\phi}_0 = \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k (f).$$

We compute

$$\mathcal{R}(\tilde{\phi}_0) = \sum_{k=0}^{\infty} \alpha^k \mathcal{R}(\lambda\tilde{\mathcal{R}})^k(f),$$

$$\mathcal{R}(\tilde{\phi}_0) = \alpha\lambda \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k(f) = \alpha\lambda\tilde{\phi}_0.$$

Thus, $\alpha\lambda$ is an eigenvalue of $\mathcal{R}$, which contradicts $\lambda \in \partial\sigma(\mathcal{R})$, and $\partial\sigma(\mathcal{R}) = \nu$.

As regards the second point, we first prove that $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \le \nu$. We then construct, for any k, a function ϕ_k, such that

$$\left\|\mathcal{R}^k(\phi_k)\right\|^{1/k} \ge \rho(S_k)^{1/k}.$$

This construction is a generalization to the multivariate cases of Farmer, Waggoner, and Zha (2009) and Farmer, Waggoner, and Zha (2010).

We compute

$$\mathcal{R}^k(\phi)\big(s^t\big) = \sum_{i_1,\dots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} A_{i_1} \cdots \Gamma_{i_{k-1}}^{-1} \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}(\phi)\big(s^t\big).$$

We will find an upper bound and a lower bound for $\|\mathcal{R}^k\|$, in terms of a sequence (u_k) associated with well-chosen norms on $\mathcal{M}_n(\mathbb{R})$. First, we consider the triple norm associated with the infinite norm on $\mathcal{M}_n(\mathbb{R})$ and the associated sequence u_k. For any ϕ such that $\|\phi\|_\infty = 1$, we obtain by subadditivity of the norm,

$$\left\|\mathcal{R}^k(\phi)\right\|_\infty \le \sum_{i_1,\dots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \left|\!\left|\!\left| \Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1} \right|\!\right|\!\right| = u_k^k$$

which leads to $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \le \nu$.

Reciprocally, we consider on $\mathcal{M}_{r,s}(\mathbb{R})$ the norm $|\cdot|$ defined by

$$|M| = \sum_{i,j} |m_{i,j}|, \quad \text{where } M = [m_{i,j}]_{(i,j)\in\{1,\dots,r\}\times\{1,\dots,s\}}.$$

This norm satisfies:

- $|M| \le r |\!|\!| M |\!|\!|_\infty$
- If we write $M = [M_1, M_2, \dots, M_l]$ by blocks, we notice the following useful property:

$$|M| = \sum_{i=1}^{l} |M_i|.$$

Fix $s_t \in \{1,\dots,N\}$ and let us denote by $\{w_{i_1\cdots i_{k+1}}, \forall(i_1\cdots i_{k+1}\in\{1,\dots,N\})\}$ a family of $n\times 1$ vectors and rewrite the following sum as a product of matrices by blocks:

$$\sum_{(i_1,\dots,i_k)\in\{1,\dots,N\}^k} p_{s_t i_1} p_{i_1 i_2}\cdots p_{i_{k-1}i_k} A_{s_t} A_{i_1}\cdots A_{i_{k-1}} w_{s_t i_1\cdots i_{k-1}}$$

$$= \Big[p_{s_t 1}\cdots p_{11}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}\big] \quad \cdots \quad |p_{s_t N}\cdots p_{NN}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}\big]\Big]\times\begin{bmatrix} w_{s_t 1\cdots 1}\\ \vdots \\ w_{s_t N\cdots N}\end{bmatrix}.$$

Thus,

$$\sup_{\|w_{i_1\cdots i_p}\|_\infty\le 1}\left\|\sum_{i_1,\dots,i_k} p_{s_t i_1} p_{i_1 i_2}\cdots p_{i_{k-1}i_k} A_{s_t} A_{i_1}\cdots A_{i_{k-1}} w_{i_1\cdots i_p}\right\|_\infty$$

$$= \sup_{\|w_{i_1\cdots i_p}\|_\infty\le 1}\left\|\Big[p_{s_t 1}\cdots p_{11}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}\big] \quad \cdots \quad |p_{s_t N}\cdots p_{NN}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}\big]\Big]\right.$$

$$\left.\times\begin{bmatrix} w_{s_t 1\cdots 1}\\ \vdots \\ w_{s_t N\cdots N}\end{bmatrix}\right\|_\infty$$

$$= \left|\left|\left|\Big[p_{s_t 1}\cdots p_{11}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}\big] \quad \cdots \quad |p_{s_t N}\cdots p_{NN}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}\big]\Big]\right|\right|\right|_\infty$$

$$\ge \frac{1}{Nn}\left|\Big[p_{s_t 1}\cdots p_{11}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_1^{-1}\big] \quad \cdots \quad |p_{s_t N}\cdots p_{NN}\big[\Gamma_{s_t}^{-1}\cdots\Gamma_N^{-1}\big]\Big]\right|$$

$$\ge \frac{1}{Nn}\sum_{i_1,\dots,i_k} p_{s_t i_1} p_{i_1 i_2}\cdots p_{i_{k-1}i_k} |A_{s_t} A_{i_1}\cdots A_{i_{k-1}}|.$$

Furthermore, as the considered space is a bounded subset of finite-dimensional vectorial space, the supremum is reached and there exist N^k vectors $(w_{s_t i_1\cdots i_{k-1}})$ for $(i_1,\dots,i_{k-1})\in\{1,\dots,N\}^{k-1}$ such that

$$\left\|\sum_{i_1,\dots,i_k} p_{s_t i_1} p_{i_1 i_2}\cdots p_{i_{k-1}i_k}\Gamma_{s_t}^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1} w_{i_1\cdots i_p}\right\|$$

$$\ge \frac{1}{Nn}\sum_{i_1,\dots,i_k} p_{s_t i_1} p_{i_1 i_2}\cdots p_{i_{k-1}i_k}\big|\Gamma_{s_t}^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\big|.$$

We define the function ϕ_0 by $\phi_0(s^t) = w_{s_t s_{t-1} s_{t-2}\cdots s_{t-k}}$. This function is bounded and of norm 1. Moreover, ϕ_0 satisfies

$$\sum_{s_t}\big\|\mathcal{R}^k(\phi_0)(s^t)\big\| \ge \frac{1}{Nn}\sum_{s_t,i_1,\dots,i_k} p_{s_t i_1} p_{i_1 i_2}\cdots p_{i_{k-1}i_k}\big|\Gamma_{s_t}^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\big| = \frac{1}{Nn}(\tilde{u}_k)^k$$

which leads to

$$\left\|\mathcal{R}^k(\phi_0)\right\|_\infty \geq \frac{1}{N^2 n}(\tilde{u}_k)^k.$$

Finally, this implies that

$$\left\|\mathcal{R}^k\right\|^{1/k} \geq \left(N^2 n\right)^{-1/k} \tilde{u}_k.$$

Taking the limit, we obtain $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \geq \nu$. This completes the proof of Lemma 2.

B.3 *Proof of Proposition 1*

A consequence of Lemma 2 is that $1 \in \sigma(\mathcal{R})$ if and only if $\nu \geq 1$, and thus, $(\mathbb{1} - \mathcal{R})$ is invertible if and only if $\nu < 1$, which proves Proposition 1.

Appendix C: Asymptotic behavior of u_k

For computational reasons, it is often quicker to compute eigenvalues rather than sums with an increasing number of terms.

Let us start with the univariate case. In this case, the sequence u_k^k is the sum of the terms of a matrix S^k where S is defined as follows:

$$S = \left(p_{ij}\left\|\Gamma_i^{-1}\right\|\right)_{ij}.$$

Actually, matrices Γ_i^{-1} reduce to scalars in this case, and hence, are commutative. Thus, for univariate model, u_k^k behaves as $\|S^k\|$ and limit ν is equal to the spectral radius of S, $\rho(S)$. Farmer, Waggoner, and Zha (2009) find a comparable result in the specific context of the Fisherian model of inflation determination.

In the general case, we introduce matrix S_k.

$$S_k = \left(\sum_{(i_1,\ldots,i_{k-1})\in\{1,\ldots,N\}^{k-1}} p_{ii_1}\cdots p_{i_{k-1}j}\left\|\Gamma_i^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\right\|\right)_{ij}.$$

For any k, an (i, j) element of matrix S_k corresponds to an upper bound of the expected impact (expressed as a norm) of the endogenous variables along trajectories from regime i to regime j in k steps weighted by the probability of each trajectory.

The following result links the behavior of u_k to that of the spectral radius $\rho(S_k)$.

Lemma 3. *Sequence $(\rho(S_k)^{1/k})$ is equivalent to (u_k) when k tends to ∞.*

We now consider norm $|\cdot|_\infty$ on $\mathcal{M}_2(\mathbb{R})$ defined by $|M|_\infty = \sum_{i,j}|m_{ij}|$. One may observe that

$$|S_k|_\infty = \sum_{i,i_1,\ldots,i_{k-1},j} p_{ii_1}\cdots p_{i_{k-1}j}\left\|\Gamma_i^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\right\| = u_{k-1}^{k-1}. \tag{35}$$

As the spectral radius is the infimum of matricial norms, Equation (35) leads to

$$\rho(S_k) \leq u_{k-1}^{k-1}. \tag{36}$$

Furthermore,

$$(S_k^q)_{ij} = \sum_{i_1,\dots,i_{k-1},i_k,i_{k+1},\dots,i_{2k},\dots,i_{k(q-1)+1},\dots,i_{kq-1}} p_{ii_1}\cdots p_{i_{k-1}i_k} p_{i_{(q-1)k}i_{(q-1)k+1}}\cdots p_{i_{qk-1}j}$$
$$\times \left\|\Gamma_i^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\right\|\cdots\left\|\Gamma_{i_{(q-1)k}}^{-1}\cdots\Gamma_{i_{qk-1}}^{-1}\right\|.$$

And using the submultiplicativity of matricial norms:

$$(S_k^q)_{ij} \geq \sum_{i_1,\dots,i_{k-1},i_k,i_{k+1},\dots,i_{2k},\dots,i_{k(q-1)+1},\dots,i_{kq-1}} p_{ii_1}\cdots p_{i_{k-1}i_k} p_{i_{(q-1)k}i_{(q-1)k+1}}\cdots p_{i_{qk-1}j}$$
$$\times \left\|\Gamma_i^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\cdots\Gamma_{i_{(q-1)k}}^{-1}\cdots\Gamma_{i_{qk-1}}^{-1}\right\|$$

and hence,

$$\left|S_k^q\right|_\infty \geq u_{kq-1}^{kq-1}. \tag{37}$$

Equation (36) can be rewritten as follows:

$$\left|S_k^q\right|_\infty^{(1/q)} \geq (u_{kq-1})^{k-1/q}.$$

For any norm, Gelfand's theorem shows that $\lim_{q\to\infty}\|X^q\|_\infty^{(1/q)} = \rho(X)$. Thus, when q tends to infinity, (36) leads to

$$\lim_{p\to\infty} u_p^k \leq \rho(S_k).$$

Thus, as $k > 1$,

$$\lim_{p\to\infty} u_p \leq \rho(S_k)^{1/k}. \tag{38}$$

Combining equations (36) and (38), we find the following upper and lower bounds:

$$\lim_{p\to\infty} u_p \leq \rho(S_k)^{1/k} \leq u_{k-1}^{1-1/k}$$

and thus, $(\rho(S_k)^{1/k})$ is convergent and has the same limit as (u_k).

Appendix D: Proof of Proposition 2

Proposition 2 follows directly from equation (38).

To prove Proposition 2, we notice that

$$u_k^k = \sum_{(i_1,\dots,i_k,i_{k+1})\in\{1,\dots,N\}^k} p_{i_1i_2}\cdots p_{i_{k-1}i_k} p_{i_ki_{k+1}}\left\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_k}^{-1}\right\|.$$

Then by considering the multiples of p ($k = np$) and by only keeping the diverging trajectory (the hypothesis of the lemma), we can rewrite the above equation as follows:

$$u_{np}^{np} \geq \left[p_{i_0i_1}p_{i_1i_2}\cdots p_{i_{p_0}i_0}\left\|\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1}\right\|\right]^n$$

and hence,

$$u_{np} \geq \left[p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{p_0} i_0} \left\| \Gamma_{i_0}^{-1} \cdots \Gamma_{i_{p_0}}^{-1} \right\|\right]^{1/p}.$$

Besides,

$$\left[p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{p_0} i_0} \left\| \Gamma_{i_0}^{-1} \cdots \Gamma_{i_{p_0}}^{-1} \right\|\right]^{1/p} \geq \rho\left(p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{p_0} i_0} \Gamma_{i_0}^{-1} \cdots \Gamma_{i_{p_0}}^{-1}\right).$$

Thus,

$$\lim_{n \to \infty} u_{np} \geq \rho\left(p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{p_0} i_0} \Gamma_{i_0}^{-1} \cdots \Gamma_{i_{p_0}}^{-1}\right).$$

The right-hand side of the inequality is larger than one by hypothesis which implies that $\nu > 1$.

Appendix E: Proof of Proposition 3

The existence of a set of matrix $\{R_1^h, \ldots, R_N^h\}$ satisfying equation (18) whose joint spectral radius $\rho(\{R^h\})$ is less than one ensures that there exists at least one stable equilibrium defined by equation (21). Then Proposition 1 proves that this equilibrium is unique if and only if the limit $\nu(\{R^h\})$ is less than one. In some cases, however, it is not possible to find such a set of matrices. If instead we find a set of matrices satisfying equation (18) associated with a greater than one joint spectral radius $\rho(\{R^h\})$, it may be because there is no stable equilibria. It is the case when the associated limit $\nu(\{R^h\})$ is lower than one. In such a case, we know that if z_t is bounded then w_t as defined by equation (19) is also bounded. However, there exists only one stable solution to equation (20), $w_t = -(R_{s_t}^h)^{-1} D_{s_t} \varepsilon_t$, because $\nu(\{R^h\})$ is less than one. Thus, any bounded equilibrium can be written recursively as $z_t = (R_{s_t}^h)^{-1} C_{s_t} z_{t-1} - (R_{s_t}^h)^{-1} D_{s_t} \varepsilon_t$. This later solution is not bounded because the joint spectral radius $\rho(\{R^h\}) > 1$, there is thus a contradiction, and hence no bounded equilibrium.

Appendix F: Proof of Proposition 4

This section is devoted to the proof of Proposition 4.

- First point is a consequence of Theorem 2.35 in Bini, Iannazzo, and Meini (2011). In Bini, Iannazzo, and Meini (2011), this is shown that if:

(i) There exists a stable solution P $(\rho(P) < 1)$ of

$$AP^2 + BP + C = [0]. \tag{39}$$

(ii) There exists a stable solution Q $(\rho(Q) < 1)$ of

$$A + QB + Q^2 C = [0]. \tag{40}$$

Then the quadratic matrix polynomial $Az^2 + Bz + C$ admits a strong (n, n) splitting of eigenvalues

$$|\lambda_1| \leq \cdots \leq |\lambda_n| < 1 < |\lambda_{n+1}| \leq \cdots \leq |\lambda_{2n}|.$$

These eigenvalues correspond to the generalized eigenvalues of the pencil $\langle F, G\rangle$. This means that the conditions of Blanchard–Kahn are satisfied.

- To prove second point, we assume that there exists R^h solution of (23), and such that $\rho((R^h)^{-1}C) < 1$, and such that $\rho(A(R^h)^{-1}) > 1$ then we consider two bounded solutions w_t^1 and w_t^2 of

$$w_t = \left(R^h\right)^{-1} A\mathbb{E}_t w_{t+1} - \left(R^h\right)^{-1} D\varepsilon_t.$$

They induce two solutions of model (22). Precisely, introducing λ and w_0 such that

$$\left(R^h\right)^{-1} Aw_0 = \lambda w_0, \quad \lambda > 1.$$

Then, for any t_0, we can build several bounded solutions

$$z_t^1 = \sum_{k=0}^{\infty}\left(\left(R^h\right)^{-1}C\right)^k \left(\left(R^h\right)^{-1}C\right)\varepsilon_{t-k}, \qquad z_t^2 = z_t^1 + \left(\mathbb{1} - \frac{\left(R^h\right)^{-1}C}{\lambda}\right)^{-1} \frac{w_0}{\lambda^{t-t_0}} \mathbb{1}_{t\geq t_0}.$$

- To prove the third point, let us assume now that all the solutions R^h satisfy $\rho((R^h)^{-1}C) > 1$, and that there exists at least one R^h such that $\rho(A(R^h)^{-1}) < 1$. In this case, there are strictly less than n roots of the matrix polynomial

$$Az^2 + Bz + C = [0]$$

which are inside the unit disk. According to Blanchard–Kahn, there is no bounded solution.

- It remains to show that the case where the solutions R^h satisfying (22) cannot all satisfy $\rho(A(R^h)^{-1}) > 1$ and $\rho((R^h)^{-1}C) > 1$. $(-A(R^h)^{-1})$ is solution of

$$A + QB + Q^2C = [0]$$

so the matrix polynomial

$$A + Bz + Cz^2 = z^2\left(A(1/z)^2 + B(1/z) + C\right)$$

admits at least $(n+1)$ eigenvalues outside the unit disk. Similarly, the matrix $(-(R^h)^{-1}C)$ is solution of

$$AP^2 + BP + C = [0]$$

so the matrix polynomial

$$Az^2 + Bz + C$$

admits at least $n+1$ roots outside the unit disk. Noticing that both polynomials are dual and with degree $2n$, we get a contradiction.

Appendix G: Sunspot equilibria of order q

In this section, we prove Proposition 6.

For $\alpha = [\alpha(i_0, i_1, \ldots, i_{q-1}, i_q)]_{(i_0,i_1,\ldots,i_{q-1},i_q)\in\{1,\ldots,N\}^{q+1}}$, we denote by $K(\alpha)$ the matrix introduced in Proposition 6 as follows:

$$K(\alpha) = \left[\sum_{(i_1,\ldots,i_{q-1})\in\{1,N\}^{q-1}} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{q-1}j} \Gamma_i^{-1} \cdots \Gamma_{i_{q-1}}^{-1} \alpha(i, i_1, \ldots, i_{q-1}, j)\right]_{(i,j)}. \tag{41}$$

The assumptions of Proposition 6 imply that, for a certain integer q, there exist N^{q+1} real numbers in the (interior of the) unit disk, $\alpha(i_0, \ldots, i_q)$, and a $(nN \times 1)$ column vector, U, satisfying:

$$K(\alpha)U = U. \tag{42}$$

Then we introduce, for $k \in \{0, \ldots, q\}$ the $(k+1)^N$ vectors $V(s_{t-k}, \ldots, s_t)$. For $k = q$, $V(s_{t-q}, \ldots, s_t)$ satisfies

$$V(s_{t-q}, \ldots, s_t) = \alpha(s_{t-q}, \ldots, s_t) U_{s_t} \tag{43}$$

and, for k from $q-1$ to 0, $V(s_{t-k}, \ldots, s_t)$ is defined by backward induction by

$$\Gamma_{s_t} V(s_{t-k}, \ldots, s_t) = p_{s_t 1} V(s_{t-k}, \ldots, s_t, 1) + p_{s_t 2} V(s_{t-k}, \ldots, s_t, 2). \tag{44}$$

By construction, we see that

$$V(s_t) = U_{s_t}.$$

Now, we construct some specific solutions.

Lemma 4. *Under the assumptions of Proposition* 6, *a unique Markovian stable equilibrium co-exists with multiple stable cyclical equilibria. An example of such bounded equilibria is, for any given* t_0,

$$\begin{cases} z_t = z_t^F, \quad \textit{for } t < t_0, \\ z_t = z_t^F + w_t, \quad \textit{for } t \geq t_0, \\ w_t = \dfrac{V'(s_{t-1-(t-1-t_0)[q]}, \ldots, s_{t-1}) w_{t-1}}{V'(s_{t-1-(t-1-t_0)[q]}, \ldots, s_{t-1}) V(s_{t-1-(t-1-t_0)[q]}, \ldots, s_{t-1})} V(s_{t-(t-t_0)[q]}, \ldots, s_t) \\ \qquad + V(s_{t-(t-t_0)[q]}, \ldots, s_t)\xi_t, \end{cases}$$

where $(t - t_0)[q]$ *represents the rest of the division of* $(t - t_0)$ *by* q. *Vectors* V *are defined in equations* (43), ξ_t *is any bounded real-valued zero mean sunspots independent of* s^t.

Lemma 4 gives the explicit form of sunspots, and thus implies Proposition 6. Moreover, we notice that for any $\lambda \in]1, \frac{1}{\max(\alpha)}[$, matrix $K(\lambda\alpha)$ has an eigenvalue larger than 1, thus according to (43), there exists a continuum of solutions w.

To prove Lemma 4, we have to check that w is a solution.

We first notice that by construction, w_t is collinear with $V(s_{t-(t-t_0)[q]}, \dots, s_t)$, for any $t \geq t_0$. Moreover, we notice that $(t-t_0)[q]$ belongs to $\{0, \dots, q-1\}$, thus:

$$\mathbb{E}_t(w_{t+1}) = \left(\frac{V'(s_{t-(t-t_0)[q]}, \dots, s_t) w_t}{V'(s_{t-(t-t_0)[q]}, \dots, s_t) V(s_{t-(t-t_0)[q]}, \dots, s_t)} + \mathbb{E}_t \xi_{t+1} \right) \\ \times \mathbb{E}_t V(s_{t+1-(t+1-t_0)[q]}, \dots, s_{t+1}).$$

We know that $\mathbb{E}_t(\xi_{t+1}) = 0$, and according to equation (44), that

$$\mathbb{E}_t V(s_{t+1-(t+1-t_0)[q]}, \dots, s_{t+1}) = \Gamma_{s_t} V(s_{t+1-(t+1-t_0)[q]}, \dots, s_t).$$

And finally,

$$\begin{aligned} \mathbb{E}_t(w_{t+1}) &= \frac{V'(s_{t-(t-t_0)[q]}, \dots, s_t) w_t}{V'(s_{t-(t-t_0)[q]}, \dots, s_t) V(s_{t-(t-t_0)[q]}, \dots, s_t)} \Gamma_{s_t} V(s_{t+1-(t+1-t_0)[q]}, \dots, s_t) \\ &= \Gamma_{s_t} w_t. \end{aligned}$$

We represent in Figure 3, the different determinacy regions, for $q = 1$ and $q = 6$ for the calibration chosen in Davig and Leeper (2007).

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