Supplement to "Jump factor models in large cross-sections"

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This Online Supplementary Material contains all proofs for the results in the main text.

Appendix SA: Proofs

Throughout the proofs, we use K to denote a generic constant that may change from line to line. For a sub σ-field $\mathcal{G} \subseteq \mathcal{F}$ and a sequence X_n of random variables, we write $X_n \xrightarrow{\mathcal{L}|\mathcal{G}} X$ if the $\mathcal{G}$-conditional law of X_n converges in probability to that of X under a metric that is associated with the weak convergence of probability measures. By a standard localization procedure, we can strengthen Assumption 3 as the following without loss of generality:

Assumption SA1. *Suppose Assumption* 3 *holds with* $T_1 = \infty$. *Moreover, the processes* α_j, λ_j, λ_Z, b_f, σ_f, $\tilde{\sigma}_j^2$ *and* $\tilde{J}_{Y,j}$ *are bounded, uniformly in* j.

SA.1 *Preliminary results*

In this subsection, we introduce some notation and preliminary estimates that are used in the sequel. We consider a sequence Ω_n of random events defined by $\Omega_n \equiv \{$distinct jump times of the Poisson process $t \mapsto \mu([0, t], E)$ are at least $2k_n\Delta_n$ apart$\}$. Since $k_n\Delta_n \to 0$ and the jumps of Z is of finite activity, $\mathbb{P}(\Omega_n) \to 1$. Therefore, we can restrict our calculations to Ω_n without loss of generality. It is (notationally) convenient to extend the definition of the spot jump beta to all $t \in [0, T]$ such that, on each path, $\beta_{j,t} = \beta_{j,\tau}$ for $t \in [\tau - k_n\Delta_n, \tau + k_n\Delta_n]$. This extension is well behaved on Ω_n and our analysis only concerns the behavior of $\beta_{j,t}$ around shrinking neighborhoods around the jump times. (It should be noted that $\beta_{j,t}$ defined as such is not adapted to $\mathcal{F}_t$.)

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We also consider the following sequence of events:

$$\Omega'_n \equiv \left\{ \sum_{j=1}^{N_n} 1_{\{\Delta^n_{i(n,\tau)} \tilde{J}_{Y,j} \neq 0\}} \leq \lfloor N_n q_n^w \rfloor /2 \text{ for some } \tau \in \mathcal{T} \right\}. \tag{SA.1}$$

By Markov's inequality,

$$\mathbb{P}\left(\sum_{j=1}^{N_n} 1_{\{\Delta^n_{i(n,\tau)} \tilde{J}_{Y,j} \neq 0\}} > \lfloor N_n q_n^w \rfloor /2 \right) \leq \frac{2}{\lfloor N_n q_n^w \rfloor} \sum_{j=1}^{N_n} \mathbb{P}\big(\Delta^n_{i(n,\tau)} \tilde{J}_{Y,j} \neq 0\big) \leq K\Delta_n / q_n^w \to 0.$$

Since $\mathcal{T}$ is finite, we have $\mathbb{P}(\Omega'_n) \to 1$.

We denote the continuous part of Y_j and Z as, respectively,

$$Y'_{j,t} \equiv \int_0^t \alpha_{j,u}\, du + \int_0^t \lambda_{j,u}^\top \, df_u + \epsilon_{j,t}, \qquad Z'_t \equiv \int_0^t \lambda_{Z,u}^\top \, df_u. \tag{SA.2}$$

The diffusive residual process is then defined as

$$\widetilde{Y}'_{j,t} \equiv Y'_{j,t} - \beta_{j,t} Z'_t = \int_0^t \alpha_{j,u}\, du + \left(\int_0^t \lambda_{j,u}^\top \, df_u - \beta_{j,t} \int_0^t \lambda_{Z,u}^\top \, df_u \right) + \epsilon_{j,t}. \tag{SA.3}$$

We denote

$$\xi_{n,j,s} \equiv \frac{\Delta^n_{i(n,s)} \widetilde{Y}'_j}{\Delta^n_{i(n,s)} Z}, \tag{SA.4}$$

which can be decomposed as

$$\xi_{n,j,s} = \xi'_{n,j,s} + \xi''_{n,j,s}, \tag{SA.5}$$

where

$$\begin{cases} \xi'_{n,j,s} \equiv \dfrac{1}{\Delta Z_s} \big(\tilde{\lambda}_{j,s-}^\top (f_s - f_{(i(n,s)-1)\Delta_n}) + \tilde{\lambda}_{j,s}^\top (f_{i(n,s)\Delta_n} - f_s) \\ \qquad + \epsilon_{i(n,s)\Delta_n} - \epsilon_{(i(n,s)-1)\Delta_n} \big), \\ \xi''_{n,j,s} \equiv \dfrac{1}{\Delta Z_s} \displaystyle\int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n} \alpha_{j,u}\, du + \Delta^n_{i(n,s)} \widetilde{Y}'_j \left(\frac{1}{\Delta^n_{i(n,s)} Z} - \frac{1}{\Delta Z_s} \right) \\ \qquad + \dfrac{1}{\Delta Z_s} \displaystyle\int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{j,u} - \lambda_{j,s-})^\top \, df_u \\ \qquad - \dfrac{\beta_{j,s}}{\Delta Z_s} \displaystyle\int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{Z,u} - \lambda_{Z,s-})^\top \, df_u \\ \qquad + \dfrac{1}{\Delta Z_s} \displaystyle\int_{s}^{i(n,s)\Delta_n} (\lambda_{j,u} - \lambda_{j,s})^\top \, df_u - \dfrac{\beta_{j,s}}{\Delta Z_s} \int_{s}^{i(n,s)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,s})^\top \, df_u. \end{cases} \tag{SA.6}$$

We further rewrite $\xi'_{n,j,s}$ as

$$\xi'_{n,j,s} = \Delta_n^{1/2} \sum_{q \in \{s-, s+\}} w_{n,q} \big(\tilde{\lambda}_{j,q}^\top \Sigma_{f,q}^{1/2} \zeta_{n,q} + R_{n,j,q} \big), \tag{SA.7}$$

where we define

$$
\begin{cases}
\zeta_{n,s-} \equiv \Sigma_{f,s-}^{-1/2} \dfrac{f_s - f_{(i(n,s)-1)\Delta_n}}{\sqrt{s-\big(i(n,s)-1\big)\Delta_n}}, & \zeta_{n,s+} \equiv \Sigma_{f,s}^{-1/2} \dfrac{f_{i(n,s)\Delta_n} - f_s}{\sqrt{i(n,s)\Delta_n - s}}, \\
R_{n,j,s-} \equiv \dfrac{\epsilon_{j,s} - \epsilon_{j,(i(n,s)-1)\Delta_n}}{\sqrt{s-\big(i(n,s)-1\big)\Delta_n}}, & R_{n,j,s+} \equiv \dfrac{\epsilon_{j,i(n,s)\Delta_n} - \epsilon_{j,s}}{\sqrt{i(n,s)\Delta_n - s}}, \\
w_{n,s-} \equiv \dfrac{1}{\Delta Z_s}\sqrt{\dfrac{s-\big(i(n,s)-1\big)\Delta_n}{\Delta_n}}, & w_{n,s+} \equiv \dfrac{1}{\Delta Z_s}\sqrt{\dfrac{i(n,s)\Delta_n - s}{\Delta_n}}.
\end{cases}
\tag{SA.8}
$$

LEMMA SA1. *Under Assumptions* 3 *and* 4, *we have for* $p, q \in \{\tau-, \tau+, \eta-, \eta+\}$:

(a) $N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,p}R_{n,j,q} = O_p(N_n^{-1/2})$ *when* $p \neq q$;

(b) $N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,q}^2 \xrightarrow{\mathbb{P}} M_\epsilon(q)$;

(c) $N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,p}\tilde{\lambda}_{j,q} = O_p(N_n^{-1/2})$;

(d) $N_n^{-1}\Delta_n^{-1}\sum_{j=1}^{N_n}(\xi''_{n,j,\tau} - \xi''_{n,j,\eta})^2 = O_p(\Delta_n)$.

PROOF OF LEMMA SA1(A). We prove the case with $p = \tau-$ and $q = \tau+$ in detail, while noting that the other cases can be proved in exactly the same way. Note that the jump times of the Poisson measure μ are necessarily independent of the Brownian motions $\widetilde{W}_j$, $1 \le j \le N_n$. Let $\mathcal{G}_t$ be the smallest filtration such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ and the jump times of μ are $\mathcal{G}_t$-measurable. The processes $(\widetilde{W}_j)_{1\le j\le N_n}$ remain to be Brownian motions with respect to $(\mathcal{G}_t)_{t\ge 0}$. Consequently, ϵ_j is a $(\mathcal{G}_t)_{t\ge 0}$-martingale, and hence,

$$
\mathbb{E}[R_{n,j,\tau-}R_{n,j,\tau+}] = 0. \tag{SA.9}
$$

Moreover, for $j \neq m$,

$$
\mathbb{E}[R_{n,j,\tau-}R_{n,j,\tau+}R_{n,m,\tau-}R_{n,m,\tau+}] = \mathbb{E}\big[R_{n,j,\tau-}R_{n,m,\tau-}\mathbb{E}[R_{n,j,\tau+}R_{n,m,\tau+}|\mathcal{G}_\tau]\big] = 0, \tag{SA.10}
$$

where the first equality holds because $R_{n,j,\tau-}R_{n,m,\tau-}$ is $\mathcal{G}_\tau$-measurable and the second equality holds because $\widetilde{W}_j$ and $\widetilde{W}_m$ are orthogonal. Since the processes $\tilde{\sigma}_j^2$ are uniformly bounded, $\mathbb{E}[R_{n,j,\tau\pm}^4] \le K$ holds due to a standard estimate for continuous Itô processes. By the Cauchy–Schwarz inequality, this further implies that

$$
\mathbb{E}\big[R_{n,j,\tau-}^2 R_{n,j,\tau+}^2\big] \le K. \tag{SA.11}
$$

From (SA.9), (SA.10), and (SA.11), we deduce

$$
\mathbb{E}\left[\left(N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,p}R_{n,j,q}\right)^2\right] \le K N_n^{-1}.
$$

The assertion of part (a) then readily follows.

(b) We consider the case $q=\tau-$ in detail while noting that the other cases can be proved in the same way. By using Itô's formula, we can decompose

$$R^2_{n,j,\tau-} = U_{n,j} + U'_{n,j}, \quad \text{where}$$
$$U_{n,j} \equiv \frac{1}{\tau - \big(i(n,\tau)-1\big)\Delta_n} \int_{(i(n,\tau)-1)\Delta_n}^{\tau} \tilde{\sigma}^2_{j,u}\,du,$$
$$U'_{n,j} \equiv \frac{2}{\tau - \big(i(n,\tau)-1\big)\Delta_n} \int_{(i(n,\tau)-1)\Delta_n}^{\tau} (\epsilon_{j,u} - \epsilon_{j,(i(n,\tau)-1)\Delta_n})\,d\widetilde{W}_{j,u}.$$

We note that $\mathbb{E}[U'_{n,j}]=0$ for each j and $\mathbb{E}[U'_{n,j}U'_{n,m}]=0$ for $j\neq m$. In addition, $\mathbb{E}|U'_{n,j}|^2 \leq K$. From these estimates, it readily follows that

$$\frac{1}{N_n}\sum_{j=1}^{N_n} U'_{n,j} = O_p\big(N_n^{-1/2}\big). \tag{SA.12}$$

Next, note that by Assumption 3(v),

$$\mathbb{E}\big|U_{n,j} - \tilde{\sigma}^2_{j,\tau-}\big| \leq \mathbb{E}\Big[\sup_{s,t,|s-t|\leq\Delta_n} \big|\tilde{\sigma}^2_{j,s} - \tilde{\sigma}^2_{j,t}\big|\Big] \leq K\Delta_n^{1/2}.$$

From this estimate and Assumption 4, we deduce

$$\frac{1}{N_n}\sum_{j=1}^{N_n} U_{n,j} = \frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\sigma}^2_{j,\tau-} + o_p(1) = M_\epsilon(\tau-) + o_p(1). \tag{SA.13}$$

The assertion of part (b) then follows from (SA.12) and (SA.13).

(c) By Assumption 4, $\tilde{\lambda}_{j,q}$ is conditionally independent of $R_{n,j,p}$, and hence, $\mathbb{E}[R_{n,j,p}\tilde{\lambda}_{j,q}|\mathcal{G}_0] = \mathbb{E}[R_{n,j,p}|\mathcal{G}_0]\mathbb{E}[\tilde{\lambda}_{j,q}|\mathcal{G}_0]=0$. In addition, for $j\neq m$,

$$\mathbb{E}\big[R_{n,j,p}R_{n,m,p}\tilde{\lambda}_{j,q}\tilde{\lambda}^\top_{m,q}|\mathcal{G}_0\big] = \mathbb{E}[R_{n,j,p}R_{n,m,p}|\mathcal{G}_0]\mathbb{E}\big[\tilde{\lambda}_{j,q}\tilde{\lambda}^\top_{m,q}|\mathcal{G}_0\big]=0,$$

where the second equality follows from the orthogonality between $\widetilde{W}_j$ and $\widetilde{W}_m$. Since $\tilde{\lambda}_{j,q}$ is bounded, $R_{n,j,p}\tilde{\lambda}_{j,q}$ has bounded second moment. The assertion of part (c) readily follows from these facts.

(d) First, since the α_j's are uniformly bounded, it is easy to see that $(\Delta Z_s)^{-1}\times \int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n}\alpha_{j,u}\,du = O_p(\Delta_n)$ uniformly in j. Hence,

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\left(\frac{1}{\Delta Z_s}\int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n}\alpha_{j,u}\,du\right)^2 = O_p(\Delta_n). \tag{SA.14}$$

Further note that, uniformly in j, we have $\mathbb{E}|\Delta^n_{i(n,s)}\widetilde{Y}'_j|^2 \leq K\Delta_n$, and hence,

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\Delta^n_{i(n,s)}\widetilde{Y}'_j\big)^2 = O_p(1). \tag{SA.15}$$

It is easy to see that

$$\frac{1}{\Delta_{i(n,s)}^{n} Z} - \frac{1}{\Delta Z_s} = O_p\big(\Delta_n^{1/2}\big). \tag{SA.16}$$

From (SA.15) and (SA.16), we deduce

$$\frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} \bigg(\Delta_{i(n,s)}^{n} \widetilde{Y}_j' \bigg(\frac{1}{\Delta_{i(n,s)}^{n} Z} - \frac{1}{\Delta Z_s} \bigg) \bigg)^2 = O_p(\Delta_n). \tag{SA.17}$$

We then note that, since the processes λ_j's are $(1/2)$-Hölder continuous under L_2-norm uniformly in j (Assumption 3(v)), the following estimate also holds uniformly:

$$\mathbb{E}\bigg[\bigg(\int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{j,u} - \lambda_{j,s-})^\top df_u \bigg)^2 \bigg] \leq K \Delta_n^2.$$

Hence,

$$\frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} \bigg(\frac{1}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{j,u} - \lambda_{j,s-})^\top df_u \bigg)^2 = O_p(\Delta_n). \tag{SA.18}$$

Similarly,

$$\begin{cases} \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \bigg(\dfrac{\beta_{j,s}}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{Z,u} - \lambda_{Z,s-})^\top df_u \bigg)^2 = O_p(\Delta_n), \\ \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \bigg(\dfrac{1}{\Delta Z_s} \int_{s}^{i(n,s)\Delta_n} (\lambda_{j,u} - \lambda_{j,s})^\top df_u \bigg)^2 = O_p(\Delta_n), \\ \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \bigg(\dfrac{\beta_{j,s}}{\Delta Z_s} \int_{s}^{i(n,s)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,s})^\top df_u \bigg)^2 = O_p(\Delta_n). \end{cases} \tag{SA.19}$$

With an appeal to the Cauchy–Schwarz inequality, the assertion of part (d) then follows from (SA.14), (SA.17), (SA.18), and (SA.19). □

Next, we set

$$\begin{cases} A_n(s) \equiv \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \big(\xi_{n,j,s}'\big)^2, \quad s \in \{\eta, \tau\}, \\ B_n(\eta,\tau) \equiv \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \xi_{n,j,\eta}' \xi_{n,j,\tau}'. \end{cases} \tag{SA.20}$$

The following lemma collects some convergence results that we use for deriving limiting distributions.

LEMMA SA2. *Suppose that Assumptions* 3 *and* 4 *hold. Then*

$$\big(A_n(\eta), A_n(\tau), B_n(\eta,\tau)\big) \xrightarrow{\mathcal{L}\text{-}s} \big(\mathcal{A}(\eta), \mathcal{A}(\tau), \mathcal{B}(\eta,\tau)\big),$$

where $\xrightarrow{\mathcal{L}\text{-}s}$ *denotes* $\mathcal{F}$*-stable convergence in law.*

Proof of Lemma SA2. By Theorem 4.3.1 in Jacod and Protter (2012),

$$(w_{n,q}, \zeta_{n,q})_{q\in\{\eta-,\eta+,\tau-,\tau+\}} \xrightarrow{\mathcal{L}\text{-}s} (w_q, \zeta_q)_{q\in\{\eta-,\eta+,\tau-,\tau+\}}. \tag{SA.21}$$

Recall the definitions in (SA.7) and (SA.20). We have, for $s \in \{\eta, \tau\}$,

$$\begin{aligned} A_n(s) &= \frac{1}{N_n}\sum_{j=1}^{N_n}\Bigg(\sum_{q\in\{s-,s+\}} w_{n,q}\big(\tilde{\lambda}_{j,q}^\top \Sigma_{f,q}^{1/2}\zeta_{n,q} + R_{n,j,q}\big)\Bigg)^2 \\ &= \sum_{p,q\in\{s-,s+\}} w_{n,p}w_{n,q}\zeta_{n,p}^\top \Sigma_{f,p}^{1/2}\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\lambda}_{j,p}\tilde{\lambda}_{j,q}^\top\Bigg)\Sigma_{f,q}^{1/2}\zeta_{n,q} \\ &\quad + \sum_{q\in\{s-,s+\}} w_{n,q}^2\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n} R_{n,j,q}^2\Bigg) + O_p\big(N_n^{-1/2}\big), \end{aligned} \tag{SA.22}$$

where the rate for the $O_p(N_n^{-1})$ term in the last line is obtained using Lemma SA1(a), (c). Similarly,

$$\begin{aligned} &B_n(\eta, \tau) \\ &\quad = \frac{1}{N_n}\sum_{j=1}^{N_n}\Bigg(\sum_{p\in\{\tau-,\tau+\}} w_{n,p}\big(\tilde{\lambda}_{j,p}^\top \Sigma_{f,p}^{1/2}\zeta_{n,p} + R_{n,j,p}\big)\Bigg) \\ &\qquad \times \Bigg(\sum_{q\in\{\eta-,\eta+\}} w_{n,q}\big(\tilde{\lambda}_{j,q}^\top \Sigma_{f,q}^{1/2}\zeta_{n,q} + R_{n,j,q}\big)\Bigg) \\ &\quad = \sum_{p\in\{\tau-,\tau+\}}\sum_{q\in\{\eta-,\eta+\}} w_{n,p}w_{n,q}\zeta_{n,p}^\top \Sigma_{f,p}^{1/2}\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\lambda}_{j,p}\tilde{\lambda}_{j,q}^\top\Bigg)\Sigma_{f,q}^{1/2}\zeta_{n,q} \\ &\qquad + O_p\big(N_n^{-1/2}\big). \end{aligned} \tag{SA.23}$$

By Assumption 4 and Lemma SA1(b),

$$\Sigma_{f,p}^{1/2}\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\lambda}_{j,p}\tilde{\lambda}_{j,q}^\top\Bigg)\Sigma_{f,q}^{1/2} \xrightarrow{\mathbb{P}} M_C(p,q), \qquad \frac{1}{N_n}\sum_{j=1}^{N_n} R_{n,j,q}^2 \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.24}$$

We further note that the limiting variables $M_C(p,q)$ and $M_\epsilon(q)$ are $\mathcal{F}$-measurable. Hence, by the property of stable convergence in law, we can deduce the assertion of Lemma SA2 from (SA.21), (SA.22), (SA.23), and (SA.24). □

Finally, we show in Lemma SA3 some consistency results for the spot jump beta estimates.

Lemma SA3. *Under Assumptions* 3 *and* 5, *the following holds for* $s \in \mathcal{T}$:

(a) $\sup_{1\le j\le N_n} |\hat{\beta}_{n,j,s} - \beta_{j,s}| 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}} = o_p(1)$;

(b) $N_n^{-1}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2=o_p(1)$.

PROOF OF LEMMA SA3(A). Note that

$$\hat{\beta}_{n,j,s}-\beta_{j,s}=\frac{\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z}{\Delta_{i(n,s)}^n Z}. \tag{SA.25}$$

By localization, we can assume that $\tilde{\sigma}_j^2$, Σ_f, λ_j and β_j are bounded. By a standard estimate for continuous Itô semimartingales (applied to the continuous parts of Y_j and Z), we have for any $p\geq 1$,

$$\mathbb{E}\left[\left|\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z\right|^p 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}\right]\leq K_p\Delta_n^{p/2},$$

for some constant K_p. By using a maximal inequality, we deduce that

$$\sup_{1\leq j\leq N_n}\left|\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z\right|1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}=O_p\left(\Delta_n^{1/2}N_n^{\iota}\right) \tag{SA.26}$$

for some arbitrarily small (but fixed) constant $\iota>0$. Then, by Assumption 5,

$$\sup_{1\leq j\leq N_n}\left|\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z\right|1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}=o_p(1).$$

Note that $1/\Delta_{i(n,s)}^n Z=O_p(1)$. The assertion of the lemma then readily follows from the above estimate and equation (SA.25).

(b) It is easy to see that $\hat{\beta}_{n,j,\tau}$, $1\leq j\leq N_n$, are uniformly bounded with probability approaching one. We then note that

$$\begin{aligned}
&\frac{1}{N_n}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2\\
&=\frac{1}{N_n}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}+\frac{1}{N_n}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}\neq 0\}}\\
&\leq\left(\sup_{1\leq j\leq N_n}|\hat{\beta}_{n,j,s}-\beta_{j,s}|1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}\right)^2+\frac{K}{N_n}\sum_{j=1}^{N_n}1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}\neq 0\}}\\
&=o_p(1),
\end{aligned}$$

as claimed in part (b). □

SA.2 *Proof of Proposition 1*

PROOF OF PROPOSITION 1. Recall that the spot jump betas $\beta_{j,s}$ are bounded by assumption. By Lemma SA3 and the boundedness of $\tilde{J}_{Y,j}$, we further deduce that the beta estimates $\hat{\beta}_{n,j,s}$ are uniformly (in j) bounded with probability approaching one. Since the loss function $L(\cdot)$ is Lipschitz on bounded sets (Assumption 1), we can now assume that $L(\cdot)$ is globally Lipschitz without loss of generality. Hence, by Lemma SA3,

$$\begin{aligned}
&\frac{1}{N_n}\sum_{j=1}^{N_n}\big|L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})-L(\chi_{j,\eta,\tau})\big| \\
&\quad\leq \frac{1}{N_n}\sum_{j=1}^{N_n}\big|L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})-L(\chi_{j,\eta,\tau})\big|1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&\quad\quad+\frac{K}{N_n}\sum_{j=1}^{N_n}1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}} \\
&\quad\leq K\max_{s\in\{\eta,\tau\},1\leq j\leq N_n}|\hat{\beta}_{n,j,s}-\beta_{j,s}|1_{\{\Delta^n_{i(n,s)}\tilde{J}_{Y,j}=0\}}+O_p(\Delta_n) \\
&\quad=o_p(1). \qquad \text{(SA.27)}
\end{aligned}$$

Next, we set

$$\xi_n=\frac{1}{N_n}\sum_{j=1}^{N_n}\big(L(\chi_{j,\eta,\tau})-\mathbb{E}\big[L(\chi_{j,\eta,\tau})|\mathcal{F}_{\eta-}\big]\big).$$

Under Assumption 6, ξ_n is the average of $\mathcal{F}_{\eta-}$-conditionally independent variables with zero conditional mean. Hence,

$$\begin{aligned}
\mathbb{E}\big[\xi_n^2|\mathcal{F}_{\eta-}\big]&=\frac{1}{N_n^2}\sum_{j=1}^{N_n}\mathbb{E}\big[\big(L(\chi_{j,\eta,\tau})-\mathbb{E}\big[L(\chi_{j,\eta,\tau})|\mathcal{F}_{\eta-}\big]\big)^2|\mathcal{F}_{\eta-}\big] \\
&\leq\frac{1}{N_n^2}\sum_{j=1}^{N_n}\mathbb{E}\big[L(\chi_{j,\eta,\tau})^2|\mathcal{F}_{\eta-}\big]=O_p\big(N_n^{-1}\big)=o_p(1).
\end{aligned}$$

In particular, this implies that $\mathbb{E}[|\xi_n|\wedge 1|\mathcal{F}_{\eta-}]=o_p(1)$. By the bounded convergence theorem, we further deduce $\mathbb{E}[|\xi_n|\wedge 1]\to 0$. But this is equivalent to $\xi_n=o_p(1)$. This, together with (SA.27), implies that

$$\frac{1}{N_n}\sum_{j=1}^{N_n}L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})=\frac{1}{N_n}\sum_{j=1}^{N_n}\mathbb{E}\big[L(\chi_{j,\eta,\tau})|\mathcal{F}_{\eta-}\big]+o_p(1). \qquad \text{(SA.28)}$$

Since $q_n^w\to 0$, the winsorized estimator $\hat{V}_n$ differs from $N_n^{-1}\sum_{j=1}^{N_n}L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})$ by an $o_p(1)$ term. The assertion of the proposition then follows from (SA.28). □

SA.3 *Proof of Theorem 1*

Proof of Theorem 1. Step 1. The proof proceeds in two steps. Recall Ω'_n from (SA.1). Since $\mathbb{P}(\Omega'_n)\to 1$, we can restrict our calculations to Ω'_n without loss of generality. In this step, we show that

$$\Delta_n^{-1}\hat{V}_n = \frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) + o_p(1). \tag{SA.29}$$

From (SA.26), we see that

$$\sup_{1\le j\le N_n} |\hat{\beta}_{n,j,s} - \beta_{j,s}| 1_{\{\Delta^n_{i(n,s)}\tilde{J}_{Y,j}=0\}} = O_p\big(\Delta_n^{1/2}N_n^{\iota}\big) \tag{SA.30}$$

for some fixed but arbitrarily small constant $\iota > 0$. In restriction to Ω'_n and the null hypothesis, $\bar{B}_{n,\eta,\tau}$ is bounded by two times of the left-hand of the above display. Hence,

$$\bar{B}_{n,\eta,\tau} = O_p\big(\Delta_n^{1/2}N_n^{\iota}\big). \tag{SA.31}$$

We note that

$$\begin{aligned}
&\left|\frac{1}{\Delta_n N_n}\sum_{j=1}^{N_n}\right| L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau}\big) \\
&\quad - L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}|\big)| 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}}| \\
&\le \frac{\lceil q_n^w N_n\rceil}{\Delta_n N_n} \sup_{1\le j\le N_n} L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}|\big) 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&= O_p\big(q_n^w N_n^{2\iota}\big) = o_p(1),
\end{aligned} \tag{SA.32}$$

where the inequality follows from the fact that the winsorization is active for at most $\lceil q_n^w N_n\rceil$ terms ($\lceil\cdot\rceil$ denotes the ceiling function); the first equality follows from (SA.30); the second equality follows from Assumptions 2 and 5 with ι chosen sufficiently small. Note that in restriction to $\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}| + |\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}| = 0\}$ and the null hypothesis, $\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta} = \xi_{n,j,\tau} - \xi_{n,j,\eta}$. Hence,

$$\begin{aligned}
&\frac{1}{\Delta_n N_n}\sum_{j=1}^{N_n} L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau}\big) 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&\quad = \frac{1}{\Delta_n N_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&\qquad + o_p(1).
\end{aligned} \tag{SA.33}$$

Next, we note that

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau}\big) 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}}$$

$$\leq \frac{L(\bar{B}_{n,\eta,\tau})}{N_n\Delta_n}\sum_{j=1}^{N_n} 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}}$$

$$= O_p\big(\Delta_n N_n^{2\iota}\big) = o_p(1), \tag{SA.34}$$

where the inequality follows from the monotonicity of $L(\cdot)$ and the last line follows from (SA.31) and the fact that $\mathbb{P}(\Delta^n_{i(n,s)}\tilde{J}_{Y,j} \neq 0) \leq K\Delta_n$. Similarly, we can show that

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau}-\xi_{n,j,\eta}) 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}} = o_p(1). \tag{SA.35}$$

From (SA.33), (SA.34), and (SA.35), we deduce (SA.29) as wanted.

Step 2. It remains to derive the convergence of $(N_n\Delta_n)^{-1}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau}-\xi_{n,j,\eta})$. Recall the definition of $\xi'_{n,j,s}$ from (SA.6). Let L_n be defined as

$$L_n \equiv \frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\xi'_{n,j,\tau}-\xi'_{n,j,\eta}\big)^2. \tag{SA.36}$$

Recalling the definitions in (SA.20), we can rewrite L_n as

$$L_n = A_n(\eta) + A_n(\tau) - 2B_n(\eta,\tau). \tag{SA.37}$$

Then, by Lemma SA2,

$$L_n \xrightarrow{\mathcal{L}\text{-}s} \mathcal{L}(\eta,\tau) \equiv \mathcal{A}(\eta) + \mathcal{A}(\tau) - 2\mathcal{B}(\eta,\tau). \tag{SA.38}$$

From (SA.5), we further see that

$$\begin{aligned}
&\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau}-\xi_{n,j,\eta}) \\
&= L_n + \frac{2}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\xi'_{n,j,\tau}-\xi'_{n,j,\eta}\big)\big(\xi''_{n,j,\tau}-\xi''_{n,j,\eta}\big) \\
&\quad + \frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\xi''_{n,j,\tau}-\xi''_{n,j,\eta}\big)^2.
\end{aligned} \tag{SA.39}$$

By Lemma SA1(d), the last term in (SA.39) is $O_p(\Delta_n)$. By the Cauchy–Schwarz inequality, this estimate and (SA.38) further imply that the second term on the right-hand side of (SA.39) is $o_p(1)$. Therefore,

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau}-\xi_{n,j,\eta}) = L_n + o_p(1).$$

The assertion of the theorem then follows from (SA.29) and (SA.38). □

SA.4 *Proof of Theorem 2*

We start with the proof of part (a) and part (b). We provide details for the case with $q = \tau-$, while noting that the case with $q = \tau+$ only requires a change of notation. Hence, we suppress (in most cases) the dependence on q in our notations for simplicity. More specifically, we write $\hat{X}_n$, $\hat{F}_n$, $\hat{\Lambda}_n$, Λ_n^*, $\mathcal{E}_n$, H, Σ_f, M_Λ^* and M_C^* in place of $\hat{X}_n(q)$, $\hat{F}_n(q)$, $\hat{\Lambda}_n(q)$, $\Lambda_n^*(q)$, $\mathcal{E}_n(q)$, H_q, $\Sigma_{f,q}$, $M_\Lambda^*(q,q)$, and $M_C^*(q,q)$, respectively. We denote the jth column of a generic matrix A by $A_{\cdot j}$. Recall the sequence Ω_n of events defined as in Section SA.1. Since $\mathbb{P}(\Omega_n) \to 1$, we can restrict our calculations below in Ω_n without loss of generality.

Below, we denote $\Gamma_n \equiv \{\gamma \in \mathbb{R}^{k_n} : \gamma^\top \gamma = k_n\}$. Note that each column of $\hat{F}_n$ is an element of Γ_n. We collect some useful estimates in Lemma SA4, where we denote

$$\tilde{\Lambda}_n^* \equiv (\lambda_{1,\tau-} - \tilde{\beta}_{n,1,\tau}\lambda_{Z,\tau-}, \ldots, \lambda_{N_n,\tau-} - \tilde{\beta}_{n,N_n,\tau}\lambda_{Z,\tau-})^\top. \tag{SA.40}$$

We also consider an $N_n \times k_n$ matrix $\mathcal{E}_n' = [e_{j,l}']_{1\le j\le N_n, 1\le l\le k_n}$ defined as

$$\begin{aligned} e_{j,l}' \equiv\ & \Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \alpha_{j,s}\,ds + \Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{j,u} - \lambda_{j,\tau-})^\top df_u \\ & - \tilde{\beta}_{n,j,\tau}\Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,\tau-})^\top df_u. \end{aligned} \tag{SA.41}$$

LEMMA SA4. *Under the conditions of Theorem* 2, *the following statements hold*:

(a) $\sup_{\gamma\in\Gamma_n} k_n^{-2}N_n^{-1}\gamma^\top\mathcal{E}_n^\top\mathcal{E}_n\gamma = o_p(1)$;

(b) $\sup_{\gamma\in\Gamma_n} k_n^{-2}N_n^{-1}\gamma^\top\mathcal{E}_n'^\top\mathcal{E}_n'\gamma = o_p(1)$;

(c) $\sup_{\gamma\in\Gamma_n} k_n^{-1}N_n^{-1}|\gamma^\top\mathcal{E}_n^\top\Lambda_n^*| = o_p(1)$;

(d) $\sup_{\gamma\in\Gamma_n} k_n^{-1}N_n^{-1}|\gamma^\top\mathcal{E}_n'^\top\Lambda_n^*| = o_p(1)$;

(e) $N_n^{-1}\tilde{\Lambda}_n^{*\top}\Lambda_n^* = M_\Lambda^* + o_p(1)$ *and* $N_n^{-1}\tilde{\Lambda}_n^{*\top}\tilde{\Lambda}_n^* = M_\Lambda^* + o_p(1)$.

PROOF OF LEMMA SA4. (a) Recall that the (j,l) element of $\mathcal{E}_n$ is given by $e_{j,l} \equiv \Delta_{i(n,\tau-)+l}^n\epsilon_j/\Delta_n^{1/2}$. We observe

$$\begin{aligned} &\frac{1}{k_n^2N_n}\gamma^\top\mathcal{E}_n^\top\mathcal{E}_n\gamma \\ &= \frac{1}{k_n^2N_n}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\gamma_l\gamma_m\sum_{j=1}^{N_n}e_{j,l}e_{j,m} \\ &\le \left(\frac{1}{k_n^2}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\gamma_l^2\gamma_m^2\right)^{1/2}\left(\frac{1}{k_n^2}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}e_{j,m}\right)^2\right)^{1/2} \\ &= \left(\frac{1}{k_n^2}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}e_{j,m}\right)^2\right)^{1/2}, \end{aligned} \tag{SA.42}$$

where the first equality is by definition, the inequality is by the Cauchy–Schwarz inequality, and the last line follows from $\gamma^\top \gamma = k_n$.

We decompose the majorant side of (SA.42) as

$$\frac{1}{k_n^2}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2$$
$$= \frac{1}{k_n^2}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}^2\right)^2 + \frac{1}{k_n^2}\sum_{l,m,l\neq m}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2. \tag{SA.43}$$

By a standard estimate for continuous Itô semimartingales, $\mathbb{E}[e_{j,l}^4] \leq K$; this holds uniformly in $j \in \{1,\dots,N_n\}$ because the idiosyncratic variances $\tilde{\sigma}_j^2$ are uniformly (locally) bounded under Assumption 3(iii). Hence, by Jensen's inequality,

$$\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}^2\right)^2\right] \leq \mathbb{E}\left[\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}^4\right] \leq K.$$

From here, it follows that the first term on the right-hand side of (SA.43) is $o_p(1)$. In view of (SA.42) and (SA.43), it remains to show that the second term on the right-hand side of (SA.43) is also $o_p(1)$.

To this end, we observe the following for $l \neq m$: (i) $\mathbb{E}[e_{j,l}e_{j,m}] = 0$ because the process ϵ_j is a martingale; (ii) $\mathbb{E}[e_{j,l}^2 e_{j,m}^2] \leq K$; and (iii) the variables $(e_{j,l}e_{j,m})_{1\leq j\leq N_n}$ are uncorrelated, which can be shown by using repeated conditioning and the orthogonality among the Brownian motions $(\widetilde{W}_j)_{j\geq 1}$. Hence,

$$\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2\right] \leq K N_n^{-1} \to 0,$$

which implies, as wanted,

$$\frac{1}{k_n^2}\sum_{l,m,l\neq m}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2 = O_p\left(N_n^{-1}\right) = o_p(1).$$

This finishes the proof of part (a).

(b) Similar to (SA.42), we can derive

$$\sup_{\gamma\in\Gamma_n} \frac{1}{k_n^2 N_n}\gamma^\top \mathcal{E}_n'^\top \mathcal{E}_n' \gamma \leq \left(\frac{1}{k_n^2}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}'e_{j,m}'\right)^2\right)^{1/2}. \tag{SA.44}$$

In addition, we observe

$$\mathbb{E}\left|\Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{j,u} - \lambda_{j,\tau-})^\top df_u\right|^4$$

$$\leq K\Delta_n^{-2}\mathbb{E}\left[\left(\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \|\lambda_{j,u}-\lambda_{j,\tau-}\|^2\,du\right)^2\right]$$
$$\leq K\Delta_n^{-1}\mathbb{E}\left[\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \|\lambda_{j,u}-\lambda_{j,\tau-}\|^4\,du\right]$$
$$\leq K\Delta_n^{-1}\mathbb{E}\left[\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \|\lambda_{j,u}-\lambda_{j,\tau-}\|^2\,du\right]\leq K\Delta_n, \tag{SA.45}$$

where the first inequality is by the Burkholder–Davis–Gundy inequality, the second inequality is by Jensen's inequality, and the last line holds because $\lambda_{j,u}$ is bounded and $(1/2)$-Hölder continuous under L_2-norm uniformly in j. Similarly,

$$\mathbb{E}\left|\Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n}(\lambda_{Z,u}-\lambda_{Z,\tau-})^\top df_u\right|^4\leq K\Delta_n. \tag{SA.46}$$

Under Assumption 7, $(\tilde{\beta}_{j,n,\tau})_{1\leq j\leq N_n}$ are uniformly bounded with probability approaching one, so we can assume that these variables are bounded without loss of generality. Hence, from (SA.45) and (SA.46), we deduce that

$$\mathbb{E}\left|e'_{j,l}\right|^4\leq K\Delta_n. \tag{SA.47}$$

Hence, by the Cauchy–Schwarz inequality, we further have

$$\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e'_{j,l}e'_{j,m}\right)^2\right]\leq K\Delta_n. \tag{SA.48}$$

The assertion of part (b) then follows from (SA.44) and (SA.48).

(c) We denote the (j,k) element of Λ_n^* by $\lambda_{j,k}^*$. We note that for each $k\in\{1,\ldots,r\}$ (recalling that $\Lambda_{n,\cdot k}^*$ denotes the kth column of Λ_n^*),

$$\frac{1}{k_nN_n}\left|\gamma^\top\mathcal{E}_n^\top\Lambda_{n,\cdot k}^*\right|=\left|\frac{1}{k_n}\sum_{l=1}^{k_n}\gamma_l\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}\lambda_{j,k}^*\right)\right|$$
$$\leq\left(\frac{1}{k_n}\sum_{l=1}^{k_n}\gamma_l^2\right)^{1/2}\left(\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}\lambda_{j,k}^*\right)^2\right)^{1/2}$$
$$=\left(\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}\lambda_{j,k}^*\right)^2\right)^{1/2}, \tag{SA.49}$$

where the first line is by definition, the second line is by the Cauchy–Schwarz inequality and the last line follows from $\gamma\in\Gamma_n$. Under Assumption 8, $e_{j,l}$ is independent of $\lambda_{j,k}^*$; hence, the variables $(e_{j,l}\lambda_{j,k}^*)_{1\leq j\leq N_n}$ are uncorrelated and have zero mean and bounded second moment. It is then easy to see that

$$\mathbb{E}\left[\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}\lambda^*_{j,k}\right)^2\right] \le K/N_n.$$

Therefore,

$$\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}\lambda^*_{j,k}\right)^2 = O_p\big(N_n^{-1}\big) = o_p(1). \tag{SA.50}$$

The assertion of part (c) then follows from (SA.49) and (SA.50).

(d) Like (SA.49), we can derive

$$\frac{1}{k_n N_n}\left|\gamma^\top \mathcal{E}_n'^\top \Lambda^*_{n,\cdot k}\right| \le \left(\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e'_{j,l}\lambda^*_{j,k}\right)^2\right)^{1/2}. \tag{SA.51}$$

We further note that

$$\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e'_{j,l}\lambda^*_{j,k}\right)^2\right] \le \mathbb{E}\left[\frac{1}{N_n}\sum_{j=1}^{N_n}\big(e'_{j,l}\lambda^*_{j,k}\big)^2\right] \le \frac{K}{N_n}\sum_{j=1}^{N_n}\mathbb{E}\big[\big(e'_{j,l}\big)^2\big] \le K\Delta_n,$$

where the first inequality is by Jensen's inequality, the second inequality holds because $\lambda^*_{j,k}$ is bounded and the last inequality can be derived similarly as (SA.47). In view of (SA.51), the assertion of part (d) readily follows.

(e) From the definitions of Λ^*_n and $\tilde{\Lambda}^*_n$ respectively from (3.9) and (SA.40), we see that (recall $q = \tau-$)

$$\tilde{\Lambda}^*_n - \Lambda^*_n = \big(\big(\beta^*_{1,\tau} - \tilde{\beta}_{n,1,\tau}\big)\lambda_{Z,\tau-}, \dots, \big(\beta^*_{n,N_n,\tau} - \tilde{\beta}_{n,N_n,\tau}\big)\lambda_{Z,\tau-}\big)^\top.$$

Therefore, by Assumption 7,

$$\frac{1}{N_n}\big(\tilde{\Lambda}^*_n - \Lambda^*_n\big)^\top\big(\tilde{\Lambda}^*_n - \Lambda^*_n\big) = o_p(1). \tag{SA.52}$$

That is, $N_n^{-1}\|\tilde{\Lambda}^*_n - \Lambda^*_n\|^2 = o_p(1)$. Since $N_n^{-1}\Lambda_n^{*\top}\Lambda^*_n \xrightarrow{\mathbb{P}} M^*_\Lambda$ by Assumption 8, the estimate above readily implies the assertions in part (e). □

We are now ready to prove part (a) and part (b) of Theorem 2. We remind the reader that we fix $q = \tau-$ for proving these parts.

Proof of Theorem 2(a). Step 1. We prove part (a) of Theorem 2 in several steps. In this step, we show that

$$\sup_{\gamma\in\Gamma_n}\left|\varXi_n(\gamma) - \varXi^*_n(\gamma)\right| = o_p(1), \tag{SA.53}$$

where $\Xi_n(\cdot)$ and $\Xi_n^*(\cdot)$ are defined as

$$\Xi_n(\gamma) \equiv \frac{1}{k_n^2 N_n}\gamma^\top \hat{X}_n^\top \hat{X}_n \gamma, \qquad \Xi_n^*(\gamma) \equiv \frac{1}{k_n^2 N_n}\gamma^\top F_n \Lambda_n^{*\top}\Lambda_n^* F_n^\top \gamma. \tag{SA.54}$$

Below, we denote the (j, l) element of $\hat{X}_n$ by

$$\xi_{n,j,l} \equiv \frac{\Delta_{i(n,\tau-)+l}^n Y_j \wedge u_n \vee (-u_n) - \tilde{\beta}_{n,j,\tau}\Delta_{i(n,\tau-)+l}^n Z}{\sqrt{\Delta_n}}.$$

We set

$$\begin{aligned}\xi'_{n,j,l} \equiv{}& \Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \alpha_{j,s}\, ds \\ &+ \Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{j,s} - \tilde{\beta}_{n,j,\tau}\lambda_{Z,s})^\top\, df_s + \Delta_n^{-1/2}\Delta_{i(n,\tau-)+l}^n \epsilon_j.\end{aligned}$$

Note that

$$\mathbb{E}\big|\xi_{n,j,l} - \xi'_{n,j,l}\big|^2 \le K\Delta_n. \tag{SA.55}$$

We now define $\hat{X}'_n$ as a $N_n \times k_n$ matrix whose (j, l) element is given by $\xi'_{n,j,l}$ and let

$$\Xi'_n(\gamma) = \frac{1}{k_n^2 N_n}\gamma^\top \hat{X}_n'^\top \hat{X}'_n \gamma.$$

By (SA.55),

$$\frac{1}{k_n N_n}\big\|\hat{X}_n - \hat{X}'_n\big\|^2 = \frac{1}{k_n N_n}\sum_{j=1}^{N_n}\sum_{l=1}^{k_n}\big|\xi_{n,j,l} - \xi'_{n,j,l}\big|^2 = o_p(1). \tag{SA.56}$$

By the Cauchy–Schwarz inequality and the triangle inequality,

$$\begin{aligned}\sup_{\gamma\in\Gamma_n}\big|\Xi_n(\gamma) - \Xi'_n(\gamma)\big| &= \frac{1}{k_n^2 N_n}\sup_{\gamma\in\Gamma_n}\big|\gamma^\top\big(\hat{X}_n^\top \hat{X}_n - \hat{X}_n'^\top \hat{X}'_n\big)\gamma\big| \\ &\le \frac{1}{k_n^2 N_n}\sup_{\gamma\in\Gamma_n}\|\gamma\|^2\big\|\hat{X}_n^\top \hat{X}_n - \hat{X}_n'^\top \hat{X}'_n\big\| \\ &= \frac{1}{k_n N_n}\big\|\hat{X}_n^\top \hat{X}_n - \hat{X}_n'^\top \hat{X}'_n\big\| \\ &\le \frac{2}{k_n N_n}\big\|\hat{X}'_n\big\|\big\|\hat{X}_n - \hat{X}'_n\big\| + \frac{1}{k_n N_n}\big\|\hat{X}_n - \hat{X}'_n\big\|^2.\end{aligned}$$

It is easy to see that $\|\hat{X}'_n\| = O_p(\sqrt{k_n N_n})$. Hence, by (SA.56),

$$\sup_{\gamma\in\Gamma_n}\big|\Xi_n(\gamma) - \Xi'_n(\gamma)\big| = o_p(1). \tag{SA.57}$$

To show (SA.53), it remains to show that $\sup_{\gamma\in\Gamma_n}|\Xi'_n(\gamma)-\Xi^*_n(\gamma)|=o_p(1)$. We note that, by a standard result for spot covariance estimation

$$F_n^\top F_n/k_n \xrightarrow{\mathbb{P}} \Sigma_f. \tag{SA.58}$$

In particular, $\|F_n\|=O_p(k_n^{1/2})$. Hence,

$$\sup_{\gamma\in\Gamma_n}\|\gamma^\top F_n/k_n\| \le \sup_{\gamma\in\Gamma_n}\|\gamma\|\|F_n\|/k_n = O_p(1). \tag{SA.59}$$

Under Assumption 8, $\Lambda_n^{*\top}\Lambda_n^*=O_p(N_n)$. It then follows that

$$\sup_{\gamma\in\Gamma_n}\Xi_n^*(\gamma)=O_p(1). \tag{SA.60}$$

Recall the definitions in (3.9), (SA.40), and (SA.41). We can decompose $\hat{X}'_n$ as

$$\hat{X}'_n=\tilde{\Lambda}_n^* F_n^\top+\mathcal{E}_n+\mathcal{E}'_n. \tag{SA.61}$$

Hence,

$$\hat{X}'_n-\Lambda_n^*F_n^\top=(\tilde{\Lambda}_n^*-\Lambda_n^*)F_n^\top+\mathcal{E}_n+\mathcal{E}'_n. \tag{SA.62}$$

We can then decompose

$$\begin{aligned}\Xi'_n(\gamma)-\Xi^*_n(\gamma)={}&\frac{2}{k_n^2N_n}\gamma^\top F_n\Lambda_n^{*\top}\big[(\tilde{\Lambda}_n^*-\Lambda_n^*)F_n^\top+\mathcal{E}_n+\mathcal{E}'_n\big]\gamma\\&+\frac{1}{k_n^2N_n}\gamma^\top\big((\tilde{\Lambda}_n^*-\Lambda_n^*)F_n^\top+\mathcal{E}_n+\mathcal{E}'_n\big)^\top\big((\tilde{\Lambda}_n^*-\Lambda_n^*)F_n^\top+\mathcal{E}_n+\mathcal{E}'_n\big)\gamma.\end{aligned} \tag{SA.63}$$

By Lemma SA4(a), (b),

$$\frac{1}{k_n^2N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top\mathcal{E}_n^\top\mathcal{E}_n\gamma=o_p(1),\qquad \frac{1}{k_n^2N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top\mathcal{E}_n'^\top\mathcal{E}'_n\gamma=o_p(1). \tag{SA.64}$$

Further using the Cauchy–Schwarz inequality, we can deduce that $\sup_{\gamma\in\Gamma_n}|\gamma^\top\mathcal{E}_n^\top\mathcal{E}'_n\gamma|=o_p(1)$; hence,

$$\frac{1}{k_n^2N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top(\mathcal{E}_n+\mathcal{E}'_n)^\top(\mathcal{E}_n+\mathcal{E}'_n)\gamma=o_p(1). \tag{SA.65}$$

In addition, by Lemma SA4(e) and (SA.59),

$$\sup_{\gamma\in\Gamma_n}\frac{1}{k_n^2N_n}\gamma^\top F_n^\top(\tilde{\Lambda}_n^*-\Lambda_n^*)^\top(\tilde{\Lambda}_n^*-\Lambda_n^*)F_n^\top\gamma=o_p(1). \tag{SA.66}$$

By (SA.65) and (SA.66), as well as the Cauchy–Schwarz inequality, we deduce

$$\frac{1}{k_n^2N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top\big((\tilde{\Lambda}_n^*-\Lambda_n^*)F_n^\top+\mathcal{E}_n+\mathcal{E}'_n\big)^\top\big((\tilde{\Lambda}_n^*-\Lambda_n^*)F_n^\top+\mathcal{E}_n+\mathcal{E}'_n\big)\gamma=o_p(1). \tag{SA.67}$$

By (SA.60) and the Cauchy–Schwarz inequality, (SA.67) further implies that

$$\frac{2}{k_n^2 N_n} \sup_{\gamma\in\Gamma_n} \left|\gamma^\top F_n \Lambda_n^{*\top}\left[\left(\tilde{\Lambda}_n^* - \Lambda_n^*\right)F_n^\top + \mathcal{E}_n + \mathcal{E}_n'\right]\gamma\right| = o_p(1). \tag{SA.68}$$

By (SA.63), (SA.67), and (SA.68), we deduce $\sup_{\gamma\in\Gamma_n} |\Xi_n'(\gamma) - \Xi_n^*(\gamma)| = o_p(1)$, and hence, (SA.53) as wanted.

Step 2. In this step, we show that

$$S_n^*\left(\hat{F}_n^{*\top} F_n/k_n\right)\Sigma_f^{-1/2} H \xrightarrow{\mathbb{P}} I_r, \tag{SA.69}$$

where we recall that $S_n^* = \operatorname{diag}(\operatorname{sign}(\hat{F}_n^{*\top} F_n (F_n^\top F_n/k_n)^{-1/2} H))$ and H is the ordered eigenvector matrix of M_C^*. Below, we denote by D_j the jth largest eigenvalue of M_C^* and write $D = \operatorname{diag}(D_1, \ldots, D_r)$.

We first show that

$$\sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma) \xrightarrow{\mathbb{P}} D_1. \tag{SA.70}$$

To see this, we note that we can represent $\gamma \in \Gamma_n$ as

$$\gamma = F_n\left(F_n^\top F_n/k_n\right)^{-1/2} H\delta + \tilde{\gamma}, \tag{SA.71}$$

where $\tilde{\gamma}$ is the projection error of γ onto the column space of F_n such that $F_n^\top \tilde{\gamma} = 0$. We can then rewrite

$$\sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma) = \sup_{\|\delta\|\le 1} \delta^\top H^\top M_{C,n}^* H\delta, \quad \text{where}$$

$$M_{C,n}^* \equiv \left(\frac{F_n^\top F_n}{k_n}\right)^{1/2}\left(\frac{\Lambda_n^{*\top}\Lambda_n^*}{N_n}\right)\left(\frac{F_n^\top F_n}{k_n}\right)^{1/2}.$$

Hence, $\sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma)$ is the largest eigenvalue of $M_{C,n}^*$. By (SA.58) and Assumption 8,

$$M_{C,n}^* \xrightarrow{\mathbb{P}} \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} \equiv M_C^*.$$

Since the mapping for calculating the unique largest eigenvalue is continuous, we deduce (SA.70) by using the continuous mapping theorem.

By the construction of $\hat{F}_n$, its first column $\hat{F}_{n,\cdot 1}$ satisfies

$$\Xi_n(\hat{F}_{n,\cdot 1}) = \sup_{\gamma\in\Gamma_n} \Xi_n(\gamma).$$

By (SA.53), $\sup_{\gamma\in\Gamma_n} \Xi_n(\gamma) = \sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma) + o_p(1)$, which implies $\Xi_n(\hat{F}_{n,\cdot 1}) \xrightarrow{\mathbb{P}} D_1$ because of (SA.70). Using the uniform convergence result in (SA.53), we further deduce

$$\Xi_n^*(\hat{F}_{n,\cdot 1}) \xrightarrow{\mathbb{P}} D_1. \tag{SA.72}$$

We now represent $\hat{F}_{n,\cdot 1}$ in the format of (SA.71), that is,

$$\hat{F}_{n,\cdot 1} = F_n\left(F_n^\top F_n/k_n\right)^{-1/2} H\hat{\delta}_1 + \tilde{\gamma}_1, \tag{SA.73}$$

such that $F_n^\top \tilde{\gamma}_1 = 0$. From (SA.72) and (SA.73), we see

$$\begin{aligned} o_p(1) &= \Xi_n^*(\hat{F}_{n,\cdot 1}) - D_1 \\ &= \hat{\delta}_1^\top H^\top M_{C,n}^* H \hat{\delta}_1 - D_1 \\ &= \hat{\delta}_1^\top H^\top (M_{C,n}^* - M_C^*) H \hat{\delta}_1 + \hat{\delta}_1^\top H^\top M_C^* H \hat{\delta}_1 - D_1 \\ &= \hat{\delta}_1^\top H^\top (M_{C,n}^* - M_C^*) H \hat{\delta}_1 + \hat{\delta}_1^\top D \hat{\delta}_1 - D_1, \end{aligned}$$

where the last line follows from the eigenvalue decomposition $M_C^* = HDH^\top$. Since $\|\hat{\delta}_1\| \leq 1$ and $M_{C,n}^* - M_C^* = o_p(1)$, the above display implies that

$$\hat{\delta}_1^\top D \hat{\delta}_1 - D_1 = o_p(1).$$

Since D_1 is the unique largest eigenvalue, this further implies that $\hat{\delta}_{11}^2 \xrightarrow{\mathbb{P}} 1$ and $\hat{\delta}_{1j}^2 \xrightarrow{\mathbb{P}} 0$ for $j \geq 2$. In particular, $\|\hat{\delta}_1\| \xrightarrow{\mathbb{P}} 1$ which implies that $\tilde{\gamma}_1^\top \tilde{\gamma}_1 / k_n \xrightarrow{\mathbb{P}} 0$.

Let $S_{n,j}^*$ denote the jth diagonal element of S_n^*. Note that by (SA.73),

$$\hat{F}_{n,\cdot 1}^\top F_n / k_n = \hat{\delta}_1^\top H^\top \big(F_n^\top F_n / k_n\big)^{1/2}.$$

Hence,

$$\hat{\delta}_1^\top = \big(\hat{F}_{n,\cdot 1}^\top F_n / k_n\big)\big(F_n^\top F_n / k_n\big)^{-1/2} H.$$

By the definition of $S_{n,1}^*$, the first element of $S_{n,1}^*(\hat{F}_{n,\cdot 1}^\top F_n / k_n)(F_n^\top F_n / k_n)^{-1/2} H$ is nonnegative. Hence,

$$\begin{aligned} & S_{n,1}^*\big(\hat{F}_{n,\cdot 1}^\top F_n / k_n\big)\big(F_n^\top F_n / k_n\big)^{-1/2} H \\ & \quad = \big(|\hat{\delta}_{11}|, S_{n,1}^* \hat{\delta}_{12}, \ldots, S_{n,1}^* \hat{\delta}_{1r}\big) \xrightarrow{\mathbb{P}} (1, 0, \ldots, 0). \end{aligned}$$

By (SA.58), we further deduce that

$$S_{n,1}^*\big(\hat{F}_{n,\cdot 1}^\top F_n / k_n\big) \Sigma_f^{-1/2} H \xrightarrow{\mathbb{P}} (1, 0, \ldots, 0),$$

which shows the convergence in (SA.69) for the first row.

By repeating the same argument (by setting Γ_n as the subspace orthogonal to previous eigenvectors), we can prove the convergence in (SA.69) for the jth row, $2 \leq j \leq r$.

Step 3. In this step, we finish the proof for part (a) of Theorem 2. We denote

$$\tilde{D}_n = N_n^{-1}\big(\hat{\Lambda}_n^* - \Lambda_n^* \Sigma_f^{1/2} H S_n^*\big)^\top \big(\hat{\Lambda}_n^* - \Lambda_n^* \Sigma_f^{1/2} H S_n^*\big).$$

The assertion of part (a) can be rewritten as $\mathrm{Trace}[\tilde{D}_n] = o_p(1)$.

We decompose

$$\tilde{D}_n = \tilde{D}_{n,1} - \tilde{D}_{n,2} - \tilde{D}_{n,2}^\top + \tilde{D}_{n,3},$$

where

$$\begin{cases} \tilde{D}_{n,1} \equiv N_n^{-1} \hat{\Lambda}_n^{*\top} \hat{\Lambda}_n^*, & \tilde{D}_{n,2} \equiv N_n^{-1} \hat{\Lambda}_n^{*\top} \Lambda_n^* \Sigma_f^{1/2} H S_n^*, \\ \tilde{D}_{n,3} \equiv N_n^{-1} S_n^* H^\top \Sigma_f^{1/2} \Lambda_n^{*\top} \Lambda_n^* \Sigma_f^{1/2} H S_n^*. \end{cases}$$

To prove $\mathrm{Trace}[\tilde{D}_n] = o_p(1)$, it suffices to show that

$$\tilde{D}_{n,k} \xrightarrow{\mathbb{P}} D, \quad k = 1, 2, 3, \tag{SA.74}$$

where we recall that D is the diagonal matrix that collects the ordered eigenvalues of M_C^*. Below, we prove (SA.74) for each case.

Case $k=1$: Recall that we partition $\hat{F}_n = [\hat{F}_n^* \vdots \hat{F}_n^0]$, where $\hat{F}_n^*$ collects the first r columns of $\hat{F}_n$. We set

$$\hat{\Lambda}_n'^* = \frac{1}{k_n} \hat{X}_n' \hat{F}_n^* = \frac{1}{k_n} \big(\tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}_n' \big) \hat{F}_n^*. \tag{SA.75}$$

Note that

$$\begin{aligned} \big\| \tilde{D}_{n,1} - N_n^{-1} \hat{\Lambda}_n'^{*\top} \hat{\Lambda}_n'^* \big\| &= N_n^{-1} \big\| \hat{\Lambda}_n^{*\top} \hat{\Lambda}_n^* - \hat{\Lambda}_n'^{*\top} \hat{\Lambda}_n'^* \big\| \\ &= k_n^{-2} N_n^{-1} \big\| \hat{F}_n^{*\top} \hat{X}_n^\top \hat{X}_n \hat{F}_n^* - \hat{F}_n^{*\top} \hat{X}_n'^\top \hat{X}_n' \hat{F}_n^* \big\| = o_p(1), \end{aligned} \tag{SA.76}$$

where the first two equalities are by definition and the last one is by (SA.57). Subsequently, by (SA.75), we can decompose $\tilde{D}_{n,1}$ as

$$\begin{aligned} \tilde{D}_{n,1} &= \frac{1}{k_n^2 N_n} \hat{F}_n^{*\top} \big(\tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}_n' \big)^\top \big(\tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}_n' \big) \hat{F}_n^* + o_p(1) \\ &= \tilde{D}_{n,1,1} + \tilde{D}_{n,1,2} + \tilde{D}_{n,1,2}^\top + \tilde{D}_{n,1,3} + o_p(1), \end{aligned}$$

where

$$\begin{cases} \tilde{D}_{n,1,1} \equiv \big(\hat{F}_n^{*\top} F_n / k_n \big) \big(\tilde{\Lambda}_n^{*\top} \tilde{\Lambda}_n^* / N_n \big) \big(F_n^\top \hat{F}_n^* / k_n \big), \\ \tilde{D}_{n,1,2} \equiv \big(k_n^{-1} N_n^{-1} \hat{F}_n^{*\top} \big(\mathcal{E}_n + \mathcal{E}_n' \big)^\top \tilde{\Lambda}_n^* \big) \big(F_n^\top \hat{F}_n^* / k_n \big), \\ \tilde{D}_{n,1,3} \equiv k_n^{-2} N_n^{-1} \hat{F}_n^{*\top} \big(\mathcal{E}_n + \mathcal{E}_n' \big)^\top \big(\mathcal{E}_n + \mathcal{E}_n' \big) \hat{F}_n^*. \end{cases}$$

From (SA.69),

$$\frac{1}{k_n} \hat{F}_n^{*\top} F_n - S_n^* H^\top \Sigma_f^{1/2} = o_p(1), \qquad \frac{1}{k_n} \hat{F}_n^{*\top} F_n = O_p(1). \tag{SA.77}$$

Hence, recalling that H is the eigenvector matrix of $M_C^* = \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2}$ and S_n^* is a diagonal matrix with ± 1 on its diagonal, we deduce

$$\tilde{D}_{n,1,1} = S_n^* H^\top \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} H S_n^* + o_p(1) = D + o_p(1).$$

By Lemma SA4, we see that $\tilde{D}_{n,1,2}$ and $\tilde{D}_{n,1,3}$ are both $o_p(1)$. From these estimates, (SA.74) for the case $k=1$ readily follows.

Case $k=2$: By (SA.76) and the Cauchy–Schwarz inequality,

$$\tilde{D}_{n,2} \equiv N_n^{-1} \hat{\Lambda}_n^{\prime * \top} \Lambda_n^* \Sigma_f^{1/2} H S_n^* + o_p(1).$$

By (SA.75), we can thus decompose $\tilde{D}_{n,2}$ as $\tilde{D}_{n,2} = \tilde{D}_{n,2,1} + \tilde{D}_{n,2,2} + o_p(1)$ where

$$\begin{cases} \tilde{D}_{n,2,1} \equiv (\hat{F}_n^{*\top} F_n / k_n)(\tilde{\Lambda}_n^{*\top} \Lambda_n^* / N_n) \Sigma_f^{1/2} H S_n^*, \\ \tilde{D}_{n,2,2} \equiv (k_n^{-1} N_n^{-1} \hat{F}_n^{*\top} (\mathcal{E}_n + \mathcal{E}_n')^\top \Lambda_n^*) \Sigma_f^{1/2} H S_n^*. \end{cases}$$

By (SA.77) and Lemma SA4(e), we deduce

$$\tilde{D}_{n,2,1} = S_n^* H^\top \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} H S_n^* + o_p(1) = D + o_p(1).$$

By Lemma SA4(c), (d), $\tilde{D}_{n,2,2} = o_p(1)$. This proves (SA.74) for the case $k=2$.

Case $k=3$: By Assumption 8, it is obvious that

$$\tilde{D}_{n,3} = S_n^* H^\top \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} H S_n^* + o_p(1) = D + o_p(1).$$

This finishes the proof of (SA.74), and hence, part (a) of Theorem 2. □

Proof of Theorem 2(b). We fix $j \in \{r+1, \dots, \bar{r}\}$. Recall that $\hat{\Lambda}_{n,\cdot j}$ denote the jth column of $\hat{\Lambda}_n$. By the definitions of $\hat{\Lambda}_n$ and $\hat{F}_n$,

$$\frac{1}{N_n} \hat{\Lambda}_{n,\cdot j}^\top \hat{\Lambda}_{n,\cdot j} = \Xi_n(\hat{F}_{n,\cdot j}). \tag{SA.78}$$

Like in (SA.73), for each $k \in \{1, \dots, r\}$, we can represent

$$\hat{F}_{n,\cdot k} = F_n (F_n^\top F_n / k_n)^{-1/2} H \hat{\delta}_k + \tilde{\gamma}_k, \tag{SA.79}$$

where $F_n^\top \tilde{\gamma}_k = 0$. Following a similar argument as in Step 2 of the proof of Theorem 2(a), we can show that, for each $k, k' \in \{1, \dots, r\}$ with $k \neq k'$,

$$\hat{\delta}_{kk}^2 \xrightarrow{\mathbb{P}} 1, \qquad \hat{\delta}_{kk'} \xrightarrow{\mathbb{P}} 0, \qquad \tilde{\gamma}_k^\top \tilde{\gamma}_k / k_n \xrightarrow{\mathbb{P}} 0. \tag{SA.80}$$

We also represent

$$\hat{F}_{n,\cdot j} = F_n (F_n^\top F_n / k_n)^{-1/2} H \hat{\delta}_j + \tilde{\gamma}_j, \tag{SA.81}$$

where $F_n^\top \tilde{\gamma}_j = 0$. Since $\hat{F}_{n,\cdot j}^\top \hat{F}_{n,\cdot k} / k_n = 0$ for $1 \le k \le r$ (because $\hat{F}_n$ collects the eigenvectors of $\hat{X}_n^\top \hat{X}_n$), we have

$$\hat{\delta}_j^\top \hat{\delta}_k + \tilde{\gamma}_j^\top \tilde{\gamma}_k / k_n = 0. \tag{SA.82}$$

Since $\tilde{\gamma}_k^\top \tilde{\gamma}_k / k_n \xrightarrow{\mathbb{P}} 0$ and $\tilde{\gamma}_j^\top \tilde{\gamma}_j / k_n \le 1$, we have $\tilde{\gamma}_j^\top \tilde{\gamma}_k / k_n = o_p(1)$ by the Cauchy–Schwarz inequality. Therefore, $\hat{\delta}_j^\top \hat{\delta}_k = o_p(1)$ for $1 \le k \le r$. By (SA.80) above, this implies $\hat{\delta}_j = o_p(1)$. Hence,

$$\Xi_n^*(\hat{F}_{n,\cdot j}) = \hat{\delta}_j^\top H M_{C,n}^* H^\top \hat{\delta}_j = o_p(1). \tag{SA.83}$$

By (SA.53), $\varXi_n(\hat{F}_{n,\cdot j}) = o_p(1)$. The assertion of part (b) readily follows from (SA.78). □

Proof of Theorem 2(c). By Assumption 8,

$$H_p^\top \Sigma_{f,p}^{1/2} \frac{\Lambda_n^*(p)^\top \Lambda_n^*(q)}{N_n} \Sigma_{f,q}^{1/2} H_q \xrightarrow{\mathbb{P}} H_p^\top M_C^*(p,q) H_q. \tag{SA.84}$$

We observe

$$\begin{aligned}
&\frac{1}{N_n}\left\| \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) - S_n^*(p) H_p^\top \Sigma_{f,p}^{1/2} \Lambda_n^*(p)^\top \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q S_n^*(q) \right\| \\
&\quad \le \frac{1}{N_n}\left\| \big(\hat{\Lambda}_n^*(p) - \Lambda_n^*(p)\Sigma_{f,p}^{1/2} H_p S_n^*(p)\big)^\top \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q \right\| \\
&\qquad + \frac{1}{N_n}\left\| H_p^\top \Sigma_{f,p}^{1/2} \Lambda_n^*(p)^\top \big(\hat{\Lambda}_n^*(q) - \Lambda_n^*(q)\Sigma_{f,q}^{1/2} H_q S_n^*(q)\big) \right\| \\
&\qquad + \frac{1}{N_n}\left\| \big(\hat{\Lambda}_n^*(p) - \Lambda_n^*(p)\Sigma_{f,p}^{1/2} H_p S_n^*(p)\big)^\top \big(\hat{\Lambda}_n^*(q) - \Lambda_n^*(q)\Sigma_{f,q}^{1/2} H_q S_n^*(q)\big) \right\|.
\end{aligned}$$

By the Cauchy–Schwarz inequality and Theorem 2(a), we deduce that the terms on the majorant side of the above display are all $o_p(1)$. Hence, by (SA.84),

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) - S_n^*(p) H_p^\top M_C^*(p,q) H_q S_n^*(q) = o_p(1). \tag{SA.85}$$

In particular,

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) = O_p(1). \tag{SA.86}$$

By Theorem 2(b),

$$\frac{1}{N_n} \hat{\Lambda}_n^0(p)^\top \hat{\Lambda}_n^0(q) = o_p(1). \tag{SA.87}$$

By the Cauchy–Schwarz inequality, (SA.86), and (SA.87), we deduce

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^0(q) = o_p(1). \tag{SA.88}$$

The assertion of part (c) then follows from (SA.85), (SA.87), and (SA.88). □

Proof of Theorem 2(d). By part (c) of Theorem 2,

$$\begin{aligned}
\operatorname{Trace}\big[\widehat{M}_{C,n}(q,q)\big] &= \operatorname{Trace}\big[S_n^*(q) H_q^\top M_C^*(q,q) H_q S_n^*(q)\big] + o_p(1) \\
&= \operatorname{Trace}\big[M_C^*(q,q)\big] + o_p(1) \\
&= \operatorname{Trace}\big[M_\Lambda^*(q,q)\Sigma_{f,q}\big] + o_p(1),
\end{aligned}$$

where the second inequality follows from the orthogonality of $H_q S_n^*(q)$ and the last line holds because $M_C^*(q,q) = \Sigma_{f,q}^{1/2} M_\Lambda^*(q,q) \Sigma_{f,q}^{1/2}$. We also note from (SA.56) that

$$\frac{1}{k_n N_n}\left\| \hat{X}_n(q) \right\|^2 = \frac{1}{k_n N_n}\left\| \hat{X}_n'(q) \right\|^2 + o_p(1).$$

Hence, it remains to show that

$$\frac{1}{k_n N_n}\left\|\hat{X}'_n(q)\right\|^2 \xrightarrow{\mathbb{P}} \mathrm{Trace}\left[M^*_\Lambda(q,q)\Sigma_{f,q}\right] + M_\epsilon(q). \tag{SA.89}$$

To show (SA.89), we consider the following decomposition:

$$\begin{aligned}\left\|\hat{X}'_n(q)\right\|^2 &= \mathrm{Trace}\left[\hat{X}'_n(q)^\top \hat{X}'_n(q)\right] \\ &= \mathrm{Trace}\left[\tilde{\Lambda}^*_n(q)^\top \tilde{\Lambda}^*_n(q) F_n(q)^\top F_n(q)\right] \\ &\quad + \mathrm{Trace}\left[\left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)^\top \left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)\right] \\ &\quad + 2\,\mathrm{Trace}\left[F_n(q)\tilde{\Lambda}^*_n(q)^\top \left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)\right].\end{aligned} \tag{SA.90}$$

By Lemma SA4(e) and (SA.58),

$$\frac{1}{k_n N_n}\mathrm{Trace}\left[\tilde{\Lambda}^*_n(q)^\top \tilde{\Lambda}^*_n(q) F_n(q)^\top F_n(q)\right] \xrightarrow{\mathbb{P}} \mathrm{Trace}\left[M^*_\Lambda(q,q)\Sigma_{f,q}\right]. \tag{SA.91}$$

In the proof of Lemma SA4(c), (d), we have shown that

$$\frac{1}{k_n}\left\|\frac{1}{N_n}\left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)^\top \Lambda^*_n(q)\right\|^2 = o_p(1).$$

In addition, by (SA.52),

$$\begin{aligned}&\frac{1}{k_n}\left\|\frac{1}{N_n}\left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)^\top \left(\tilde{\Lambda}^*_n(q)-\Lambda^*_n(q)\right)\right\|^2 \\ &\quad \le \frac{\left\|\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right\|^2}{k_n N_n}\cdot\frac{\left\|\tilde{\Lambda}^*_n(q)-\Lambda^*_n(q)\right\|^2}{N_n} = o_p(1).\end{aligned}$$

Hence, $\|(\mathcal{E}_n(q)+\mathcal{E}'_n(q))^\top \tilde{\Lambda}^*_n(q)\| = o_p(N_n k_n^{1/2})$. Also note that $\|F_n(q)\| = O_p(k_n^{1/2})$. Therefore, by the Cauchy–Schwarz inequality,

$$\left\|\frac{1}{k_n N_n}\left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)^\top \tilde{\Lambda}^*_n(q) F_n(q)^\top\right\| \le \frac{1}{k_n N_n}\left\|F_n(q)\right\|\left\|\left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)^\top \tilde{\Lambda}^*_n(q)\right\| = o_p(1).$$

Consequently,

$$\frac{1}{k_n N_n}\mathrm{Trace}\left[F_n(q)\tilde{\Lambda}^*_n(q)^\top \left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)\right] = o_p(1). \tag{SA.92}$$

In view of (SA.90), (SA.91), and (SA.92), (SA.89) will be implied by

$$\frac{1}{k_n N_n}\mathrm{Trace}\left[\left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)^\top \left(\mathcal{E}_n(q)+\mathcal{E}'_n(q)\right)\right] \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.93}$$

Finally, we show (SA.93). For each j, we denote

$$\xi_{n,j} \equiv \frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{\Delta^n_{i(n,q)+l}\epsilon_j}{\sqrt{\Delta_n}}\right)^2,$$

$$\xi'_{n,j} \equiv \frac{1}{k_n \Delta_n} \int_{i(n,q)\Delta_n}^{i(n,q)\Delta_n + k_n \Delta_n} \tilde{\sigma}^2_{j,u}\, du, \qquad \xi''_{n,j} \equiv \xi_{n,j} - \xi'_{n,j}.$$

Then we can decompose

$$\begin{aligned}\frac{1}{k_n N_n} \operatorname{Trace}\big[\mathcal{E}_n(q)^\top \mathcal{E}_n(q)\big] &= \frac{1}{k_n N_n} \sum_{j=1}^{N_n} \sum_{l=1}^{k_n} \left(\frac{\Delta^n_{i(n,q)+l} \epsilon_j}{\sqrt{\Delta_n}} \right)^2 \\ &= \frac{1}{N_n} \sum_{j=1}^{N_n} \xi'_{n,j} + \frac{1}{N_n} \sum_{j=1}^{N_n} \xi''_{n,j}.\end{aligned}$$

We note that conditional on $\mathcal{F}_{i(n,q)\Delta_n}$, the variables $(\xi''_{n,j})_{1 \le j \le N_n}$ are uncorrelated with zero mean and bounded variances. Hence,

$$\frac{1}{N_n} \sum_{j=1}^{N_n} \xi''_{n,j} = o_p(1). \tag{SA.94}$$

In addition, we note that

$$\begin{aligned}\frac{1}{N_n} \sum_{j=1}^{N_n} \xi'_{n,j} - \frac{1}{N_n} \sum_{j=1}^{N_n} \tilde{\sigma}^2_{j,q} &= \frac{1}{N_n} \sum_{j=1}^{N_n} \frac{1}{k_n \Delta_n} \int_{i(n,q)\Delta_n}^{i(n,q)\Delta_n + k_n \Delta_n} \big(\tilde{\sigma}^2_{j,u} - \tilde{\sigma}^2_{j,q}\big)\, du \\ &= O_p\big(k_n^{1/2} \Delta_n^{1/2}\big) = o_p(1).\end{aligned}$$

It readily follows that

$$\frac{1}{N_n} \sum_{j=1}^{N_n} \xi'_{n,j} \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.95}$$

By (SA.94) and (SA.95),

$$\frac{1}{k_n N_n} \operatorname{Trace}\big[\mathcal{E}_n(q)^\top \mathcal{E}_n(q)\big] \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.96}$$

We further note that

$$\frac{1}{k_n N_n} \operatorname{Trace}\big[\mathcal{E}'_n(q)^\top \mathcal{E}'_n(q)\big] = \frac{1}{k_n N_n} \sum_{j=1}^{N_n} \sum_{l=1}^{k_n} \big(e'_{j,l}\big)^2 = O_p(\Delta_n). \tag{SA.97}$$

With an appeal to the Cauchy–Schwarz inequality, we deduce (SA.93) from (SA.96) and (SA.97). This finishes the proof of part (d) of Theorem 2. □

SA.5 *Proof of Theorem 3*

(a) First, by Theorem 2(c), (d), it is obvious that $\tilde{L}_n(\eta, \tau) = O_p(1)$. Hence, the quantile $cv_{n,\alpha} = O_p(1)$. Next, we consider the case under the null hypothesis, so $M^*_C(p,q)$ coincides with $M_C(p,q)$.

We partition $\tilde{\zeta}_q^\top = (\tilde{\zeta}_q^{*\top}, \tilde{\zeta}_q^{0\top})$, where $\tilde{\zeta}_q^*$ is r-dimensional. By Theorem 2(c), (d), we have, for $s \in \{\eta, \tau\}$,

$$\begin{cases} \tilde{A}_n(s) = \sum_{p,q\in\{s-,s+\}} \tilde{w}_{n,p}\tilde{w}_{n,q}\tilde{\zeta}_p^{*\top} S_n^*(p) H_p^\top M_C(p,q) H_q S_n^*(q)\tilde{\zeta}_q^* \\ \qquad + \sum_{q\in\{s-,s+\}} \tilde{w}_{n,q}^2 M_\epsilon(q) + o_p(1), \\ \tilde{B}_n(\eta,\tau) = \sum_{p\in\{\tau-,\tau+\}} \sum_{q\in\{\eta-,\eta+\}} \tilde{w}_{n,p}\tilde{w}_{n,q}\tilde{\zeta}_p^{*\top} S_n^*(p) H_p^\top M_C(p,q) H_q S_n^*(q)\tilde{\zeta}_q^* + o_p(1). \end{cases}$$

We note that the r-dimensional vectors $H_q S_n^*(q)\tilde{\zeta}_q^*$ are, conditionally on $\mathcal{F}$, standard normal and mutually independent across $q \in \{\tau-, \tau+, \eta-, \eta+\}$. We also observe that for $s \in \{\eta, \tau\}$, $\Delta_{i(n,s)}^n Z \xrightarrow{\mathbb{P}} \Delta Z_s$. Hence,

$$\big(H_q S_n^*(q)\tilde{\zeta}_q^*, \tilde{w}_{n,q}\big)_{q\in\{\tau-,\tau+,\eta-,\eta+\}} \xrightarrow{\mathcal{L}|\mathcal{F}} (\zeta_q, w_q)_{q\in\{\tau-,\tau+,\eta-,\eta+\}}, \tag{SA.98}$$

where $\xrightarrow{\mathcal{L}|\mathcal{F}}$ denotes the convergence of conditional law in probability. It follows that

$$\big(\tilde{A}_n(\eta), \tilde{A}_n(\tau), \tilde{B}_n(\eta,\tau)\big) \xrightarrow{\mathcal{L}|\mathcal{F}} \big(\mathcal{A}(\eta), \mathcal{A}(\tau), \mathcal{B}(\eta,\tau)\big).$$

Consequently, $\tilde{L}_n(\eta,\tau) \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{L}(\eta,\tau)$. We further note that the $\mathcal{F}$-conditional distribution function of $\mathcal{L}(\eta,\tau)$ is continuous and strictly increasing. Hence, $cv_{n,\alpha} \xrightarrow{\mathbb{P}} cv_\alpha$.

(b) The assertion on the asymptotic level follows from part (a) and Theorem 1. Under the alternative, $\Delta_n^{-1}\hat{V}_n$ diverges to $+\infty$ in probability by Proposition 1. The power property then follows from $cv_{n,\alpha} = O_p(1)$.

References

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