Supplement to "Quantile treatment effects and bootstrap inference under covariate-adaptive randomization"

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This paper gathers the supplementary material to the original paper. Sections A, B, C, and D contain the proofs for Theorems 3.1, 3.2, 4.1, and 5.1, respectively. Section E contains the proofs for the technical lemmas. A separate supplement (located in the replication file) contains the analysis of strata fixed effects quantile regression estimator as well as additional simulation results.

KEYWORDS. Bootstrap inference, quantile treatment effect.

JEL CLASSIFICATION. C12, C14.

APPENDIX A: PROOF OF THEOREM 3.1

Let $u = (u_0, u_1)' \in \Re^2$ and

$$L_n(u,\tau) = \sum_{i=1}^{n} \left[\rho_\tau\left(Y_i - \dot{A}_i'\beta(\tau) - \dot{A}_i'u/\sqrt{n}\right) - \rho_\tau\left(Y_i - \dot{A}_i'\beta(\tau)\right)\right].$$

Then, by the change of variable, we have that

$$\sqrt{n}\left(\hat{\beta}(\tau) - \beta(\tau)\right) = \arg\min_u L_n(u,\tau).$$

Notice that $L_n(u,\tau)$ is convex in u for each τ and bounded in τ for each u. In the following, we aim to show that there exists

$$g_n(u,\tau) = -u'W_n(\tau) + \frac{1}{2}u'Q(\tau)u$$

such that (1) for each u,

$$\sup_{\tau\in \Upsilon}\left|L_n(u,\tau) - g_n(u,\tau)\right| \xrightarrow{p} 0;$$

(2) the maximum eigenvalue of $Q(\tau)$ is bounded from above and the minimum eigenvalue of $Q(\tau)$ is bounded away from 0, uniformly over $\tau \in \Upsilon$; (3) $W_n(\tau) \rightsquigarrow \tilde{\mathcal{B}}(\tau)$ uniformly

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over $\tau \in \Upsilon$, in which $\tilde{\mathcal{B}}(\cdot)$ is some Gaussian process. Then by Kato (2009, Theorem 2), we have

$$\sqrt{n}\big(\hat{\beta}(\tau) - \beta(\tau)\big) = \big[Q(\tau)\big]^{-1} W_n(\tau) + r_n(\tau),$$

where $\sup_{\tau \in \Upsilon} \|r_n(\tau)\| = o_p(1)$. In addition, by (3), we have, uniformly over $\tau \in \Upsilon$,

$$\sqrt{n}\big(\hat{\beta}(\tau) - \beta(\tau)\big) \rightsquigarrow \big[Q(\tau)\big]^{-1} \tilde{\mathcal{B}}(\tau) \equiv \mathcal{B}(\tau).$$

The second element of $\mathcal{B}(\tau)$ is $\mathcal{B}_{\text{sqr}}(\tau)$ stated in Theorem 3.1. In the following, we prove requirements (1)–(3) in three steps.

Step 1. By Knight's identity (Knight (1998)), we have

$$\begin{aligned}
&L_n(u,\tau) \\
&= -u' \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \dot{A}_i\big(\tau - 1\{Y_i \le \dot{A}_i'\beta(\tau)\}\big) \\
&\quad + \sum_{i=1}^{n} \int_0^{\frac{\dot{A}_i u}{\sqrt{n}}} \big(1\{Y_i - \dot{A}_i'\beta(\tau) \le v\} - 1\{Y_i - \dot{A}_i'\beta(\tau) \le 0\}\big)\, dv \\
&\equiv -u' W_n(\tau) + Q_n(u,\tau),
\end{aligned}$$

where

$$W_n(\tau) = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \dot{A}_i\big(\tau - 1\{Y_i \le \dot{A}_i'\beta(\tau)\}\big)$$

and

$$\begin{aligned}
Q_n(u,\tau) &= \sum_{i=1}^{n} \int_0^{\frac{\dot{A}_i' u}{\sqrt{n}}} \big(1\{Y_i - \dot{A}_i'\beta(\tau) \le v\} - 1\{Y_i - \dot{A}_i'\beta(\tau) \le 0\}\big)\, dv \\
&= \sum_{i=1}^{n} A_i \int_0^{\frac{u_0+u_1}{\sqrt{n}}} \big(1\{Y_i(1) - q_1(\tau) \le v\} - 1\{Y_i(1) - q_1(\tau) \le 0\}\big)\, dv \\
&\quad + \sum_{i=1}^{n} (1-A_i) \int_0^{\frac{u_0}{\sqrt{n}}} \big(1\{Y_i(0) - q_0(\tau) \le v\} - 1\{Y_i(0) - q_0(\tau) \le 0\}\big)\, dv \\
&\equiv Q_{n,1}(u,\tau) + Q_{n,0}(u,\tau).
\end{aligned}$$

We first consider $Q_{n,1}(u,\tau)$. Following Bugni, Canay, and Shaikh (2018), we define $\{(Y_i^s(1), Y_i^s(0)) : 1 \le i \le n\}$ as a sequence of i.i.d. random variables with marginal distributions equal to the distribution of $(Y_i(1), Y_i(0))|S_i = s$. The distribution of $Q_{n,1}(u,\tau)$ is the same as the counterpart with units ordered by strata and then ordered by $A_i = 1$

first and $A_i = 0$ second within each stratum, that is,

$$Q_{n,1}(u,\tau) \stackrel{d}{=} \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \int_0^{\frac{u_0+u_1}{\sqrt{n}}} \left(1\left\{Y_i^s(1) - q_1(\tau) \le v\right\} - 1\left\{Y_i^s(1) - q_1(\tau) \le 0\right\}\right) dv$$
$$= \sum_{s\in\mathcal{S}} \left[\Gamma_n^s\left(N(s) + n_1(s), \tau\right) - \Gamma_n^s\left(N(s), \tau\right)\right], \tag{A.1}$$

where $N(s) = \sum_{i=1}^n 1\{S_i < s\}$, $n_1(s) = \sum_{i=1}^n 1\{S_i = s\}A_i$, and

$$\Gamma_n^s(k,\tau) = \sum_{i=1}^k \int_0^{\frac{u_0+u_1}{\sqrt{n}}} \left(1\left\{Y_i^s(1) - q_1(\tau) \le v\right\} - 1\left\{Y_i^s(1) - q_1(\tau) \le 0\right\}\right) dv.$$

In addition, note that

$$\mathbb{P}\Big(\sup_{t\in(0,1),\tau\in\Upsilon} \left|\Gamma_n^s\left(\lfloor nt\rfloor, \tau\right) - \mathbb{E}\Gamma_n^s\left(\lfloor nt\rfloor, \tau\right)\right| > \varepsilon\Big)$$
$$= \mathbb{P}\Big(\max_{1\le k\le n} \sup_{\tau\in\Upsilon} \left|\Gamma_n^s(k,\tau) - \mathbb{E}\Gamma_n^s(k,\tau)\right| > \varepsilon\Big)$$
$$\le 3 \max_{1\le k\le n} \mathbb{P}\Big(\sup_{\tau\in\Upsilon} \left|\Gamma_n^s(k,\tau) - \mathbb{E}\Gamma_n^s(k,\tau)\right| > \varepsilon/3\Big)$$
$$\le 9\mathbb{P}\Big(\sup_{\tau\in\Upsilon} \left|\Gamma_n^s(n,\tau) - \mathbb{E}\Gamma_n^s(n,\tau)\right| > \varepsilon/30\Big)$$
$$\le \frac{270\mathbb{E}\sup_{\tau\in\Upsilon} \left|\Gamma_n^s(n,\tau) - \mathbb{E}\Gamma_n^s(n,\tau)\right|}{\varepsilon} = o(1). \tag{A.2}$$

The first inequality holds due to Lemma E.1 with $S_k = \Gamma_n^s(k,\tau) - \mathbb{E}\Gamma_n^s(k,\tau)$ and $\|S_k\| = \sup_{\tau\in\Upsilon} |\Gamma_n^s(k,\tau) - \mathbb{E}\Gamma_n^s(k,\tau)|$. The second inequality holds due to Montgomery-Smith (1993, Theorem 1). To derive the last equality of (A.2), we consider the class of functions

$$\mathcal{F} = \left\{\int_0^{\frac{u_0+u_1}{\sqrt{n}}} \left(1\left\{Y_i^s(1) - q_1(\tau) \le v\right\} - 1\left\{Y_i^s(1) - q_1(\tau) \le 0\right\}\right) dv : \tau \in \Upsilon\right\}$$

with envelope $\frac{|u_0+u_1|}{\sqrt{n}}$ and

$$\sup_{f\in\mathcal{F}} \mathbb{E}f^2 \le \sup_{\tau\in\Upsilon} \mathbb{E}\left[\frac{u_0+u_1}{\sqrt{n}} 1\left\{\left|Y_i^s(1) - q_1(\tau)\right| \le \frac{u_0+u_1}{\sqrt{n}}\right\}\right]^2 \lesssim n^{-3/2}.$$

Note that $\mathcal{F}$ is a VC-class with a fixed VC index. Therefore, by Chernozhukov, Chetverikov, and Kato (2014, Corollary 5.1),

$$\mathbb{E}\sup_{\tau\in\Upsilon} \left|\Gamma_n^s(n,\tau) - \mathbb{E}\Gamma_n^s(n,\tau)\right| = n\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}} \lesssim n\left[\sqrt{\frac{\log(n)}{n^{5/2}}} + \frac{\log(n)}{n^{3/2}}\right] = o(1).$$

Then, (A.2) implies that

$$\sup_{\tau\in\Upsilon}\left|Q_{n,1}(u,\tau)-\sum_{s\in\mathcal{S}}\mathbb{E}\big[\Gamma_n^s\big(\lfloor n\big(N(s)/n+n_1(s)/n\big)\rfloor,\tau\big)-\Gamma_n^s\big(\lfloor n\big(N(s)/n\big)\rfloor,\tau\big)\big]\right|=o_p(1),$$

where following the convention in the empirical process literature,

$$\mathbb{E}\big[\Gamma_n^s\big(\lfloor n\big(N(s)/n+n_1(s)/n\big)\rfloor,\tau\big)-\Gamma_n^s\big(\lfloor n\big(N(s)/n\big)\rfloor,\tau\big)\big]$$

is interpreted as

$$\mathbb{E}\big[\Gamma_n^s\big(\lfloor nt_2\rfloor,\tau\big)-\Gamma_n^s\big(\lfloor nt_1\rfloor,\tau\big)\big]_{t_2=\frac{N(s)}{n},t_2=\frac{N(s)+n_1(s)}{n}}.$$

In addition, $N(s)/n\xrightarrow{p}F(s)=F(S_i<s)$ and $n_1(s)/n\xrightarrow{p}\pi p(s)$. Thus, uniformly over $\tau\in\Upsilon$,

$$\begin{aligned}&\mathbb{E}\big[\Gamma_n^s\big(\lfloor n\big(N(s)/n+n_1(s)/n\big)\rfloor,\tau\big)-\Gamma_n^s\big(\lfloor n\big(N(s)/n\big)\rfloor,\tau\big)\big]\\&=n_1(s)\int_0^{\frac{u_0+u_1}{\sqrt{n}}}\big(F_1\big(q_1(\tau)+v|s\big)-F_1\big(q_1(\tau)|s\big)\big)\,dv\\&\xrightarrow{p}\frac{\pi p(s)f_1\big(q_1(\tau)|s\big)(u_0+u_1)^2}{2},\end{aligned}$$

where $F_1(\cdot|s)$ and $f_1(\cdot|s)$ are the conditional CDF and PDF of Y_1 given $S=s$, respectively. Then, uniformly over $\tau\in\Upsilon$,

$$Q_{n,1}(u,\tau)\xrightarrow{p}\sum_{s\in\mathcal{S}}\frac{\pi p(s)f_1\big(q_1(\tau)|s\big)(u_0+u_1)^2}{2}=\frac{\pi f_1\big(q_1(\tau)\big)(u_0+u_1)^2}{2}.$$

Similarly, we can show that, uniformly over $\tau\in\Upsilon$,

$$Q_{n,0}(u,\tau)\xrightarrow{p}\frac{(1-\pi)f_0\big(q_0(\tau)\big)u_0^2}{2},$$

and thus

$$Q_n(u,\tau)\xrightarrow{p}\frac{1}{2}u'Q(\tau)u,$$

where

$$Q(\tau)=\begin{pmatrix}\pi f_1\big(q_1(\tau)\big)+(1-\pi)f_0\big(q_0(\tau)\big) & \pi f_1\big(q_1(\tau)\big)\\ \pi f_1\big(q_1(\tau)\big) & \pi f_1\big(q_1(\tau)\big)\end{pmatrix}. \tag{A.3}$$

Then

$$\sup_{\tau\in\Upsilon}\big|L_n(u,\tau)-g_n(u,\tau)\big|=\sup_{\tau\in\Upsilon}\left|Q_n(u,\tau)-\frac{1}{2}u'Q(\tau)u\right|=o_p(1).$$

This concludes the first step.

Step 2. Note that $\det(Q(\tau)) = \pi(1-\pi)f_1(q_1(\tau))f_0(q_0(\tau))$, which is bounded and bounded away from zero. In addition, it can be shown that the two eigenvalues of Q are nonnegative. This leads to the desired result.

Step 3. Let $e_1 = (1,1)'$ and $e_0 = (1,0)'$. Then we have

$$W_n(\tau) = e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} A_i 1\{S_i = s\}\big(\tau - 1\{Y_i(1) \le q_1(\tau)\}\big) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (1-A_i) 1\{S_i = s\}\big(\tau - 1\{Y_i(0) \le q_0(\tau)\}\big).$$

Let $m_j(s,\tau) = \mathbb{E}(\tau - 1\{Y_i(j) \le q_j(\tau)\}|S_i = s)$ and $\eta_{i,j}(s,\tau) = (\tau - 1\{Y_i(j) \le q_j(\tau)\}) - m_j(s,\tau)$, $j = 0,1$. Then

$$\begin{aligned} W_n(\tau) = & \left[e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} A_i 1\{S_i = s\}\eta_{i,1}(s,\tau) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (1-A_i) 1\{S_i = s\}\eta_{i,0}(s,\tau) \right] \\ & + \left[e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (A_i - \pi) 1\{S_i = s\} m_1(s,\tau) \right. \\ & \left. - e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (A_i - \pi) 1\{S_i = s\} m_0(s,\tau) \right] \\ & + \left[e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \pi 1\{S_i = s\} m_1(s,\tau) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (1-\pi) 1\{S_i = s\} m_0(s,\tau) \right] \\ \equiv & W_{n,1}(\tau) + W_{n,2}(\tau) + W_{n,3}(\tau). \end{aligned} \tag{A.4}$$

By Lemma E.2, uniformly over $\tau \in \Upsilon$,

$$\big(W_{n,1}(\tau), W_{n,2}(\tau), W_{n,3}(\tau)\big) \rightsquigarrow \big(\mathcal{B}_1(\tau), \mathcal{B}_2(\tau), \mathcal{B}_3(\tau)\big),$$

where $(\mathcal{B}_1(\tau), \mathcal{B}_2(\tau), \mathcal{B}_3(\tau))$ are three independent two-dimensional Gaussian processes with covariance kernels $\Sigma_1(\tau_1,\tau_2)$, $\Sigma_2(\tau_1,\tau_2)$, and $\Sigma_3(\tau_1,\tau_2)$, respectively. Therefore, uniformly over $\tau \in \Upsilon$,

$$W_n(\tau) \rightsquigarrow \tilde{\mathcal{B}}(\tau),$$

where $\tilde{\mathcal{B}}(\tau)$ is a two-dimensional Gaussian process with covariance kernel

$$\tilde{\Sigma}(\tau_1,\tau_2) = \sum_{j=1}^{3} \Sigma_j(\tau_1,\tau_2).$$

Consequently,

$$\sqrt{n}\big(\hat{\beta}(\tau) - \beta(\tau)\big) \rightsquigarrow \big[Q(\tau)\big]^{-1} \tilde{\mathcal{B}}(\tau) \equiv \mathcal{B}(\tau),$$

where $\mathcal{B}(\tau)$ is a two-dimensional Gaussian process with covariance kernel

$$
\begin{aligned}
&\Sigma(\tau_1, \tau_2) \\
&\quad = \big[Q(\tau_1)\big]^{-1}\tilde{\Sigma}(\tau_1, \tau_2)\big[Q(\tau_2)\big]^{-1} \\
&\quad = \frac{1}{\pi f_1\big(q_1(\tau_1)\big) f_1\big(q_1(\tau_2)\big)}\big[\min(\tau_1, \tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S, \tau_1)m_1(S, \tau_2)\big]\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\qquad + \frac{1}{(1-\pi) f_0\big(q_0(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}\big[\min(\tau_1, \tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S, \tau_1)m_0(S, \tau_2)\big]\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
&\qquad + \sum_{s\in\mathcal{S}} p(s)\gamma(s)\bigg[\frac{m_1(s, \tau_1)m_1(s, \tau_2)}{\pi^2 f_1\big(q_1(\tau_1)\big) f_1\big(q_1(\tau_2)\big)}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\qquad - \frac{m_1(s, \tau_1)m_0(s, \tau_2)}{\pi(1-\pi) f_1\big(q_1(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \\
&\qquad - \frac{m_0(s, \tau_1)m_1(s, \tau_2)}{\pi(1-\pi) f_0\big(q_0(\tau_1)\big) f_1\big(q_1(\tau_2)\big)}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \\
&\qquad + \frac{m_0(s, \tau_1)m_0(s, \tau_2)}{(1-\pi)^2 f_0\big(q_0(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\bigg] \\
&\qquad + \frac{\mathbb{E}m_1(S, \tau_1)m_1(S, \tau_2)}{f_1\big(q_1(\tau_1)\big) f_1\big(q_1(\tau_2)\big)}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\mathbb{E}m_1(S, \tau_1)m_0(S, \tau_2)}{f_1\big(q_1(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \\
&\qquad + \frac{\mathbb{E}m_0(S, \tau_1)m_1(S, \tau_2)}{f_0\big(q_0(\tau_1)\big) f_1\big(q_1(\tau_2)\big)}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \frac{\mathbb{E}m_0(S, \tau_1)m_0(S, \tau_2)}{f_0\big(q_0(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\end{aligned}
$$

Focusing on the $(2,2)$-element of $\Sigma(\tau_1, \tau_2)$, we can conclude that

$$\sqrt{n}\big(\hat{\beta}_1(\tau) - q(\tau)\big) \rightsquigarrow \mathcal{B}_{\text{sqr}}(\tau),$$

where the Gaussian process $\mathcal{B}_{\text{sqr}}(\tau)$ has a covariance kernel

$$
\begin{aligned}
&\Sigma_{\text{sqr}}(\tau_1, \tau_2) \\
&\quad = \frac{\min(\tau_1, \tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S, \tau_1)m_1(S, \tau_2)}{\pi f_1\big(q_1(\tau_1)\big) f_1\big(q_1(\tau_2)\big)} \\
&\qquad + \frac{\min(\tau_1, \tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S, \tau_1)m_0(S, \tau_2)}{(1-\pi) f_0\big(q_0(\tau_1)\big) f_0\big(q_0(\tau_2)\big)} \\
&\qquad + \mathbb{E}\gamma(S)\bigg[\frac{m_1(S, \tau_1)m_1(S, \tau_2)}{\pi^2 f_1\big(q_1(\tau_1)\big) f_1\big(q_1(\tau_2)\big)} + \frac{m_1(S, \tau_1)m_0(S, \tau_2)}{\pi(1-\pi) f_1\big(q_1(\tau_1)\big) f_0\big(q_0(\tau_2)\big)} \\
&\qquad + \frac{m_0(S, \tau_1)m_1(S, \tau_2)}{\pi(1-\pi) f_0\big(q_0(\tau_1)\big) f_1\big(q_1(\tau_2)\big)} + \frac{m_0(S, \tau_1)m_0(S, \tau_2)}{(1-\pi)^2 f_0\big(q_0(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}\bigg]
\end{aligned}
$$

$$+\mathbb{E}\left[\frac{m_1(S,\tau_1)}{f_1\big(q_1(\tau_1)\big)}-\frac{m_0(S,\tau_1)}{f_0\big(q_0(\tau_1)\big)}\right]\left[\frac{m_1(S,\tau_2)}{f_1\big(q_1(\tau_2)\big)}-\frac{m_0(S,\tau_2)}{f_0\big(q_0(\tau_2)\big)}\right].$$

Appendix B: Proof of Theorem 3.2

By Knight's identity, we have

$$\sqrt{n}\big(\hat{q}_1(\tau)-q_1(\tau)\big)=\arg\min_u L_n(u,\tau),$$

where

$$\begin{aligned}L_n(u,\tau)&\equiv\sum_{i=1}^n\frac{A_i}{\hat{\pi}(S_i)}\left[\rho_\tau\left(Y_i-q_1(\tau)-\frac{u}{\sqrt{n}}\right)-\rho_\tau\big(Y_i-q_1(\tau)\big)\right]\\&=-L_{1,n}(\tau)u+L_{2,n}(u,\tau),\\L_{1,n}(\tau)&=\frac{1}{\sqrt{n}}\sum_{i=1}^n\frac{A_i}{\hat{\pi}(S_i)}\big(\tau-1\{Y_i\le q_1(\tau)\}\big)\end{aligned}$$

and

$$L_{2,n}(u,\tau)=\sum_{i=1}^n\frac{A_i}{\hat{\pi}(S_i)}\int_0^{\frac{u}{\sqrt{n}}}\big(1\{Y_i\le q_1(\tau)+v\}-1\{Y_i\le q_1(\tau)\}\big)\,dv.$$

We aim to show that there exists

$$g_{\mathrm{ipw},n}(u,\tau)=-W_{\mathrm{ipw},n}(\tau)u+\frac{1}{2}Q_{\mathrm{ipw}}(\tau)u^2 \tag{B.1}$$

such that (1) for each u,

$$\sup_{\tau\in\Upsilon}\big|L_n(u,\tau)-g_{\mathrm{ipw},n}(u,\tau)\big|\xrightarrow{p}0;$$

(2) $Q_{\mathrm{ipw}}(\tau)$ is bounded and bounded away from zero uniformly over $\tau\in\Upsilon$. In addition, as a corollary of claim (3) below, $\sup_{\tau\in\Upsilon}|W_{\mathrm{ipw},1,n}(\tau)|=O_p(1)$. Therefore, by Kato (2009, Theorem 2), we have

$$\sqrt{n}\big(\hat{q}_1(\tau)-q_1(\tau)\big)=Q^{-1}_{\mathrm{ipw},1}(\tau)W_{\mathrm{ipw},1,n}(\tau)+R_{\mathrm{ipw},1,n}(\tau),$$

where $\sup_{\tau\in\Upsilon}|R_{\mathrm{ipw},1,n}(\tau)|=o_p(1)$. Similarly, we can show that

$$\sqrt{n}\big(\hat{q}_0(\tau)-q_0(\tau)\big)=Q^{-1}_{\mathrm{ipw},0}(\tau)W_{\mathrm{ipw},0,n}(\tau)+R_{\mathrm{ipw},0,n}(\tau),$$

where $\sup_{\tau\in\Upsilon}|R_{\mathrm{ipw},0,n}(\tau)|=o_p(1)$. Then

$$\begin{aligned}&\sqrt{n}\big(\hat{q}(\tau)-q(\tau)\big)\\&\quad=Q^{-1}_{\mathrm{ipw},1}(\tau)W_{\mathrm{ipw},1,n}(\tau)-Q^{-1}_{\mathrm{ipw},0}(\tau)W_{\mathrm{ipw},0,n}(\tau)+R_{\mathrm{ipw},1,n}(\tau)-R_{\mathrm{ipw},0,n}(\tau).\end{aligned}$$

Last, we aim to show that, (3) uniformly over $\tau \in \Upsilon$,

$$Q^{-1}_{\text{ipw},1}(\tau)W_{\text{ipw},1,n}(\tau) - Q^{-1}_{\text{ipw},0}(\tau)W_{\text{ipw},0,n}(\tau) \rightsquigarrow \mathcal{B}_{\text{ipw}}(\tau),$$

where $\mathcal{B}_{\text{ipw}}(\tau)$ is a scalar Gaussian process with covariance kernel $\Sigma_{\text{ipw}}(\tau_1, \tau_2)$. We prove claims (1)–(3) in three steps.

Step 1. For $L_{1,n}(\tau)$, we have

$$\begin{aligned}
L_{1,n}(\tau) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\frac{A_i}{\pi}1\{S_i = s\}\big(\tau - 1\big\{Y_i(1) \le q_1(\tau)\big\}\big) \\
&\quad - \sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\frac{A_i 1\{S_i = s\}\big(\hat{\pi}(s) - \pi\big)}{\sqrt{n}\hat{\pi}(s)\pi}\big(\tau - 1\big\{Y_i(1) \le q_1(\tau)\big\}\big) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\frac{A_i}{\pi}1\{S_i = s\}\big(\tau - 1\big\{Y_i(1) \le q_1(\tau)\big\}\big) \\
&\quad - \sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\frac{A_i 1\{S_i = s\}D_n(s)}{n(s)\sqrt{n}\hat{\pi}(s)\pi}\eta_{i,1}(s,\tau) - \sum_{s\in\mathcal{S}}\frac{D_n(s)m_1(s,\tau)}{n(s)\sqrt{n}\hat{\pi}(s)\pi}D_n(s) \\
&\quad - \sum_{s\in\mathcal{S}}\frac{D_n(s)m_1(s,\tau)}{\sqrt{n}\hat{\pi}(s)} \\
&= \sum_{s\in\mathcal{S}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{A_i 1\{S_i = s\}}{\pi}\eta_{i,1}(s,\tau) + \sum_{s\in\mathcal{S}}\frac{D_n(s)}{\sqrt{n}\pi}m_1(s,\tau) + \sum_{i=1}^{n}\frac{m_1(S_i,\tau)}{\sqrt{n}} \\
&\quad - \sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\frac{A_i 1\{S_i = s\}D_n(s)}{n(s)\sqrt{n}\hat{\pi}(s)\pi}\eta_{i,1}(s,\tau) - \sum_{s\in\mathcal{S}}\frac{D_n(s)m_1(s,\tau)}{n(s)\sqrt{n}\hat{\pi}(s)\pi}D_n(s) \\
&\quad - \sum_{s\in\mathcal{S}}\frac{D_n(s)m_1(s,\tau)}{\sqrt{n}\hat{\pi}(s)} \\
&= W_{\text{ipw},1,n}(\tau) + R_{\text{ipw}}(\tau),
\end{aligned}$$

where

$$W_{\text{ipw},1,n}(\tau) = \sum_{s\in\mathcal{S}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{A_i 1\{S_i = s\}}{\pi}\eta_{i,1}(s,\tau) + \sum_{i=1}^{n}\frac{m_1(S_i,\tau)}{\sqrt{n}} \tag{B.2}$$

and

$$\begin{aligned}
&R_{\text{ipw}}(\tau) \\
&\quad = -\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\frac{A_i 1\{S_i = s\}D_n(s)}{n(s)\sqrt{n}\hat{\pi}(s)\pi}\eta_{i,1}(s,\tau) - \sum_{s\in\mathcal{S}}\frac{D_n(s)m_1(s,\tau)}{n(s)\sqrt{n}\hat{\pi}(s)\pi}D_n(s) \\
&\qquad + \sum_{s\in\mathcal{S}}\frac{D_n(s)m_1(s,\tau)}{\sqrt{n}}\left(\frac{1}{\pi} - \frac{1}{\hat{\pi}(s)}\right)
\end{aligned}$$

$$= -\sum_{i=1}^{n}\sum_{s\in\mathcal{S}} \frac{A_i 1\{S_i = s\} D_n(s)}{n(s)\sqrt{n}\hat{\pi}(s)\pi}\eta_{i,1}(s,\tau),$$

where we use the fact that $\hat{\pi}(s) - \pi = \frac{D_n(s)}{n(s)}$. By the same argument in Claim (1) of the proof of Lemma E.2, we have, for every $s \in \mathcal{S}$,

$$\sup_{\tau\in\Upsilon}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} A_i 1\{S_i = s\}\eta_{i,1}(s,\tau)\right| \stackrel{d}{=} \sup_{\tau\in\Upsilon}\left|\frac{1}{\sqrt{n}}\sum_{i=N(s)+1}^{N(s)+n(s)} \tilde{\eta}_{i,1}(s,\tau)\right| = O_p(1), \tag{B.3}$$

where $\tilde{\eta}_{i,j}(s,\tau) = \tau - 1\{Y_i^s(j) \le q_j(\tau)\} - m_j(s,\tau)$, for $j = 0, 1$, where $\{Y_i^s(0), Y_i^s(1)\}_{i\ge1}$ are the same as defined in Step 1 in the proof of Theorem 3.1.

Because of (B.3) and the fact that $\frac{D_n(s)}{n(s)} = o_p(1)$, we have

$$\sup_{\tau\in\Upsilon}\left|R_{\mathrm{ipw}}(\tau)\right| = o_p(1).$$

For $L_{2,n}(u,\tau)$, we have

$$\begin{aligned} L_{2,n}(u,\tau) &= \sum_{s\in\mathcal{S}} \frac{1}{\hat{\pi}(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \int_0^{\frac{u}{\sqrt{n}}} \left(1\left\{Y_i^s(1) \le q_1(\tau)+v\right\} - 1\left\{Y_i^s(1) \le q_1(\tau)+v\right\}\right) dv \\ &= \sum_{s\in\mathcal{S}} \frac{1}{\hat{\pi}(s)}\left[\Gamma_n^s\left(N(s)+n_1(s),\tau\right) - \Gamma_n^s\left(N(s),\tau\right)\right], \end{aligned}$$

where

$$\Gamma_n^s(k,\tau) = \sum_{i=1}^{k} \int_0^{\frac{u}{\sqrt{n}}} \left(1\left\{Y_i^s(1) \le q_1(\tau)+v\right\} - 1\left\{Y_i^s(1) \le q_1(\tau)+v\right\}\right) dv.$$

By the same argument in (A.2), we can show that

$$\sup_{t\in(0,1),\tau\in\Upsilon} \left|\Gamma_n^s\left(\lfloor nt\rfloor,\tau\right) - \mathbb{E}\Gamma_n^s\left(\lfloor nt\rfloor,\tau\right)\right| = o_p(1).$$

In addition,

$$\mathbb{E}\Gamma_n^s\left(N(s)+n_1(s),\tau\right) - \mathbb{E}\Gamma_n^s\left(N(s),\tau\right) \xrightarrow{p} \frac{\pi p(s) f_1\left(q_1(\tau)|s\right)u^2}{2}.$$

Therefore,

$$\sup_{\tau\in\Upsilon}\left|L_{2,n}(u,\tau) - \frac{f_1\left(q_1(\tau)\right)u^2}{2}\right| = o_p(1),$$

where we use the fact that $\hat{\pi}(s) - \pi = \frac{D_n(s)}{n(s)} = o_p(1)$ and

$$\sum_{s\in\mathcal{S}} p(s) f_1\left(q_1(\tau)|s\right) = f_1\left(q_1(\tau)\right).$$

This establishes (B.1) with $Q_{\mathrm{ipw},1}(\tau) = f_1(q_1(\tau))$ and $W_{\mathrm{ipw},n}(\tau)$ defined in (B.2).

Step 2. Statement (2) holds by Assumption 2.

Step 3. By a similar argument in Step 1, we have

$$W_{\text{ipw},0,n}(\tau) = \sum_{s\in\mathcal{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(1-A_i)1\{S_i = s\}}{1-\pi} \eta_{i,0}(s,\tau) + \sum_{i=1}^{n} \frac{m_0(S_i,\tau)}{\sqrt{n}}$$

and $Q_{\text{ipw},0}(\tau) = f_0(q_0(\tau))$. Therefore,

$$\begin{aligned}\sqrt{n}(\hat{q}-q) &= \frac{1}{\sqrt{n}} \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \left[\frac{A_i 1\{S_i = s\}\eta_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)} - \frac{(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau)}{(1-\pi) f_0\big(q_0(\tau)\big)} \right] \\ &\quad + \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{m_1(S_i,\tau)}{f_1\big(q_1(\tau)\big)} - \frac{m_0(S_i,\tau)}{f_0\big(q_0(\tau)\big)} \right) \right] + R_{\text{ipw},n}(\tau) \\ &= \mathcal{W}_{n,1}(\tau) + \mathcal{W}_{n,2}(\tau) + R_{\text{ipw},n}(\tau), \end{aligned} \tag{B.4}$$

where $\sup_{\tau\in\Upsilon} |R_{\text{ipw},n}(\tau)| = o_p(1)$. Last, Lemma E.3 establishes that

$$\big(\mathcal{W}_{n,1}(\tau), \mathcal{W}_{n,2}(\tau)\big) \rightsquigarrow \big(\mathcal{B}_{\text{ipw},1}(\tau), \mathcal{B}_{\text{ipw},2}(\tau)\big),$$

where $(\mathcal{B}_{\text{ipw},1}(\tau), \mathcal{B}_{\text{ipw},2}(\tau))$ are two mutually independent scalar Gaussian processes with covariance kernels

$$\begin{aligned}\Sigma_{\text{ipw},1}(\tau_1,\tau_2) &= \frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)}{\pi f_1\big(q_1(\tau_1)\big) f_1\big(q_1(\tau_2)\big)} \\ &\quad + \frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)}{(1-\pi) f_0\big(q_0(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}\end{aligned}$$

and

$$\Sigma_{\text{ipw},2}(\tau_1,\tau_2) = \mathbb{E}\left(\frac{m_1(S,\tau_1)}{f_1\big(q_1(\tau_1)\big)} - \frac{m_0(S,\tau_1)}{f_0\big(q_0(\tau_1)\big)} \right)\left(\frac{m_1(S,\tau_2)}{f_1\big(q_1(\tau_2)\big)} - \frac{m_0(S,\tau_2)}{f_0\big(q_0(\tau_2)\big)} \right),$$

respectively. In particular, the asymptotic variance for $\hat{q}$ is

$$\zeta_Y^2(\pi,\tau) + \zeta_S^2(\tau),$$

where $\zeta_Y^2(\pi,\tau)$ and $\zeta_S^2(\tau)$ are the same as those in the proof of Theorem 3.1.

Appendix C: Proof of Theorem 4.1

First, we consider the weighted bootstrap for the SQR estimator. Note that

$$\sqrt{n}\big(\hat{\beta}^w(\tau) - \beta(\tau)\big) = \arg\min_u L_n^w(u,\tau),$$

where

$$L_n^w(u,\tau) = \sum_{i=1}^{n} \xi_i \big[\rho_\tau\big(Y_i - \dot{A}_i'\beta(\tau) - \dot{A}_i' u/\sqrt{n}\big) - \rho_\tau\big(Y_i - \dot{A}_i'\beta(\tau)\big)\big].$$

Similar to the proof of Theorem 3.1, we can show that

$$\sup_{\tau\in\Upsilon}\left|L_n^w(u,\tau)-g_n^w(u,\tau)\right|\to 0,$$

where

$$g_n^w(u,\tau)=-u'W_n^w(\tau)+\frac{1}{2}u'Q(\tau)u,$$

$$W_n^w(\tau)=\sum_{i=1}^n\frac{\xi_i}{\sqrt{n}}\dot{A}_i\big(\tau-1\{Y_i\le\dot{A}'\beta(\tau)\}\big),$$

and $Q(\tau)$ is defined in (A.3). Therefore, by Kato (2009, Theorem 2), we have

$$\sqrt{n}\big(\hat{\beta}^w(\tau)-\beta(\tau)\big)=\big[Q(\tau)\big]^{-1}W_n^w(\tau)+r_n^w(\tau),$$

where $\sup_{\tau\in\Upsilon}\|r_n^w(\tau)\|=o_p(1)$. By Theorem 3.1,

$$\sqrt{n}\big(\hat{\beta}^w(\tau)-\hat{\beta}(\tau)\big)=\big[Q(\tau)\big]^{-1}\sum_{i=1}^n\frac{\xi_i-1}{\sqrt{n}}\dot{A}_i\big(\tau-1\{Y_i\le\dot{A}'\beta(\tau)\}\big)+o_p(1),$$

where the $o_p(1)$ term holds uniformly over $\tau\in\Upsilon$. In addition, Lemma E.4 shows that, conditionally on data, the second element of $[Q(\tau)]^{-1}\sum_{i=1}^n\frac{\xi_i-1}{\sqrt{n}}\dot{A}_i(\tau-1\{Y_i\le\dot{A}'\beta(\tau)\})$ converges to $\tilde{\mathcal{B}}_{\text{sqr}}(\tau)$ uniformly over $\tau\in\Upsilon$. This leads to the desired result for the weighted bootstrap simple quantile regression estimator.

Next, we turn to the IPW estimator. Denote $\hat{q}_j^w(\tau)$, $j=0,1$ the weighted bootstrap counterpart of $\hat{q}_j(\tau)$. We have

$$\sqrt{n}\big(\hat{q}_1^w(\tau)-q_1(\tau)\big)=\arg\min_u L_n^w(u,\tau),$$

where

$$\begin{aligned}L_n^w(u,\tau)&=\sum_{i=1}^n\frac{\xi_iA_i}{\hat{\pi}^w(S_i)}\left[\rho_\tau\left(Y_i-q_1(\tau)-\frac{u}{\sqrt{n}}\right)-\rho_\tau\big(Y_i-q_1(\tau)\big)\right]\\&\equiv -L_{1,n}^w(\tau)u+L_{2,n}^w(u,\tau),\end{aligned}$$

where

$$L_{1,n}^w(\tau)=\frac{1}{\sqrt{n}}\sum_{i=1}^n\frac{\xi_iA_i}{\hat{\pi}^w(S_i)}\big(\tau-1\{Y_i\le q_1(\tau)\}\big)$$

and

$$L_{2,n}^w(\tau)=\sum_{i=1}^n\frac{\xi_iA_i}{\hat{\pi}^w(S_i)}\int_0^{\frac{u}{\sqrt{n}}}\big(1\{Y_i\le q_1(\tau)+v\}-1\{Y_i\le q_1(\tau)\}\big)\,dv.$$

Recall

$$D_n^w(s) = \sum_{i=1}^n \xi_i (A_i - \pi) 1\{S_i = s\}, n^w(s) = \sum_{i=1}^n \xi_i 1\{S_i = s\},$$

and

$$\hat{\pi}^w(s) = \frac{\sum_{i=1}^n \xi_i A_i 1\{S_i = s\}}{n^w(s)} = \pi + \frac{D_n^w(s)}{n^w(s)}.$$

Then, for $L_{1,n}^w(\tau)$, we have

$$\begin{aligned}
L_{1,n}^w(\tau) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{s\in\mathcal{S}} \frac{\xi_i A_i}{\pi} 1\{S_i = s\}\big(\tau - 1\big\{Y_i(1) \le q_1(\tau)\big\}\big) \\
&\quad - \sum_{i=1}^n \sum_{s\in\mathcal{S}} \frac{\xi_i A_i 1\{S_i = s\}\big(\hat{\pi}^w(s) - \pi\big)}{\sqrt{n}\hat{\pi}^w(s)\pi}\big(\tau - 1\big\{Y_i(1) \le q_1(\tau)\big\}\big) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{s\in\mathcal{S}} \frac{\xi_i A_i}{\pi} 1\{S_i = s\}\big(\tau - 1\big\{Y_i(1) \le q_1(\tau)\big\}\big) \\
&\quad - \sum_{i=1}^n \sum_{s\in\mathcal{S}} \frac{\xi_i A_i 1\{S_i = s\} D_n^w(s)}{n^w(s)\sqrt{n}\hat{\pi}(s)\pi} \eta_{i,1}(s,\tau) - \sum_{s\in\mathcal{S}} \frac{D_n^w(s) m_1(s,\tau)}{n^w(s)\sqrt{n}\hat{\pi}^w(s)\pi} D_n^w(s) \\
&\quad - \sum_{s\in\mathcal{S}} \frac{D_n^w(s) m_1(s,\tau)}{\sqrt{n}\hat{\pi}^w(s)} \\
&= \sum_{s\in\mathcal{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\xi_i A_i 1\{S_i = s\}}{\pi} \eta_{i,1}(s,\tau) + \sum_{s\in\mathcal{S}} \frac{D_n^w(s)}{\sqrt{n}\pi} m_1(s,\tau) \\
&\quad + \sum_{i=1}^n \frac{\xi_i m_1(S_i,\tau)}{\sqrt{n}} \\
&\quad - \sum_{s\in\mathcal{S}} D_n^w(s) \sum_{i=1}^n \frac{\xi_i A_i 1\{S_i = s\}}{n^w(s)\sqrt{n}\hat{\pi}^w(s)\pi} \eta_{i,1}(s,\tau) - \sum_{s\in\mathcal{S}} \frac{D_n^w(s) m_1(s,\tau)}{n^w(s)\sqrt{n}\hat{\pi}^w(s)\pi} D_n^w(s) \\
&\quad - \sum_{s\in\mathcal{S}} \frac{D_n^w(s) m_1(s,\tau)}{\sqrt{n}\hat{\pi}^w(s)} \\
&= W_{\text{ipw},1,n}^w(\tau) + R_{\text{ipw}}^w(\tau),
\end{aligned}$$

where

$$W_{\text{ipw},1,n}^w(\tau) = \sum_{s\in\mathcal{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\xi_i A_i 1\{S_i = s\}}{\pi} \eta_{i,1}(s,\tau) + \sum_{i=1}^n \frac{\xi_i m_1(S_i,\tau)}{\sqrt{n}} \tag{C.1}$$

and

$$
\begin{aligned}
&R^w_{\text{ipw}}(\tau) \\
&= -\sum_{s\in\mathcal{S}} D^w_n(s)\sum_{i=1}^n \frac{\xi_i A_i 1\{S_i=s\}}{n^w(s)\sqrt{n}\hat{\pi}^w(s)\pi}\eta_{i,1}(s,\tau) - \sum_{s\in\mathcal{S}} \frac{D^w_n(s)m_1(s,\tau)}{n^w(s)\sqrt{n}\hat{\pi}^w(s)\pi} D^w_n(s) \\
&\quad + \sum_{s\in\mathcal{S}} \frac{D^w_n(s)m_1(s,\tau)}{\sqrt{n}}\left(\frac{1}{\pi} - \frac{1}{\hat{\pi}^w(s)}\right) \\
&= -\sum_{s\in\mathcal{S}} D^w_n(s)\sum_{i=1}^n \frac{\xi_i A_i 1\{S_i=s\}}{n^w(s)\sqrt{n}\hat{\pi}^w(s)\pi}\eta_{i,1}(s,\tau).
\end{aligned}
$$

In the following, we aim to show $D^w_n(s)/n^w(s) = o_p(1)$ and

$$
\sup_{\tau\in\Upsilon, s\in\mathcal{S}} \left| \sum_{i=1}^n \xi_i A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)\right| = O_p(\sqrt{n}).
$$

For the first claim, we note that $n^w(s)/n(s) \xrightarrow{p} 1$ and $D_n(s)/n(s) \xrightarrow{p} 0$. Therefore, we only need to show

$$
\frac{D^w_n(s) - D_n(s)}{n(s)} = \sum_{i=1}^n \frac{(\xi_i - 1)(A_i - \pi)1\{S_i=s\}}{n(s)} \xrightarrow{p} 0.
$$

As $n(s)\to\infty$ a.s., given data,

$$
\begin{aligned}
\frac{1}{n(s)}\sum_{i=1}^n (A_i-\pi)^2 1\{S_i=s\} &= \frac{1}{n}\sum_{i=1}^n \big(A_i - \pi - 2\pi(A_i-\pi) + \pi - \pi^2\big)1\{S_i=s\} \\
&= \frac{D_n(s) - 2\pi D_n(s)}{n(s)} + \pi(1-\pi) \xrightarrow{p} \pi(1-\pi).
\end{aligned}
$$

Then, by the Lindeberg CLT, conditionally on data,

$$
\frac{1}{\sqrt{n(s)}}\sum_{i=1}^n (\xi_i - 1)(A_i-\pi)1\{S_i=s\} \rightsquigarrow N\big(0, \pi(1-\pi)\big) = O_p(1),
$$

and thus

$$
\frac{D^w_n(s) - D_n(s)}{n(s)} = O_p\big(n^{-1/2}(s)\big) = o_p(1).
$$

This leads to the first claim. For the second claim, we note that

$$
\sum_{i=1}^n \xi_i A_i 1\{S_i=s\}\eta_{i,1}(s,\tau) = \sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi_i \tilde{\eta}_{i,1}(s,\tau).
$$

We can show the RHS of the above display is $O_p(\sqrt{n})$ for all $s \in \mathcal{S}$ following the same argument used in Claim (1) of the proof of Lemma E.2. Given these two claims and by noticing that

$$\hat{\pi}^w(s) - \pi = \frac{D_n^w(s)}{n^w(s)} = o_p(1),$$

we have

$$\sup_{\tau \in \Upsilon} \left| R_{\text{ipw}}^w(\tau) \right| = o_p(1).$$

Similar to the argument used to derive the limit of $L_{2,n}(\tau)$ in the proof of Theorem 3.2, we can show that

$$\sup_{\tau \in \Upsilon} \left| L_{2,n}^w(u, \tau) - \frac{f_1\big(q_1(\tau)\big)u^2}{2} \right| = o_p(1).$$

Therefore,

$$\sqrt{n}\big(\hat{q}_1^w(\tau) - q_1(\tau)\big) = \frac{W_{\text{ipw},1,n}^w(\tau)}{f_1\big(q_1(\tau)\big)} + R_1^w(\tau),$$

where $\sup_{\tau \in \Upsilon} |R_1^w(\tau)| = o_p(1)$. Similarly,

$$\sqrt{n}\big(\hat{q}_0^w(\tau) - q_0(\tau)\big) = \frac{W_{\text{ipw},0,n}^w(\tau)}{f_0\big(q_0(\tau)\big)} + R_0^w(\tau),$$

where

$$W_{\text{ipw},0,n}^w(\tau) = \sum_{s \in \mathcal{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\xi_i(1 - A_i)1\{S_i = s\}}{1 - \pi} \eta_{i,0}(s, \tau) + \sum_{i=1}^n \frac{\xi_i m_0(S_i, \tau)}{\sqrt{n}}$$

and $\sup_{\tau \in \Upsilon} |R_0^w(\tau)| = o_p(1)$. Therefore,

$$\begin{aligned}
&\sqrt{n}\big(\hat{q}^w(\tau) - \hat{q}(\tau)\big) \\
&= \sum_{s \in \mathcal{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^n (\xi_i - 1) \Bigg\{ \frac{A_i 1\{S_i = s\}\eta_{i,1}(s, \tau)}{\pi f_1\big(q_1(\tau)\big)} - \frac{(1 - A_i)1\{S_i = s\}\eta_{i,0}(s, \tau)}{(1 - \pi) f_0\big(q_0(\tau)\big)} \\
&\quad + \left[\frac{m_1(s, \tau)}{f_1\big(q_1(\tau)\big)} - \frac{m_0(s, \tau)}{f_0\big(q_0(\tau)\big)} \right] 1\{S_i = s\} \Bigg\} + o_p(1),
\end{aligned}$$

where the $o_p(1)$ term holds uniformly over $\tau \in \Upsilon$. In order to show the conditional weak convergence, we only need to show the conditionally stochastic equicontinuity and finite-dimensional convergence. The former can be shown in the same manner as Lemma E.4. For the latter, we note that

$$\frac{1}{n} \sum_{s \in \mathcal{S}} \sum_{i=1}^n \Bigg\{ \frac{A_i 1\{S_i = s\}\eta_{i,1}(s, \tau)}{\pi f_1\big(q_1(\tau)\big)} - \frac{(1 - A_i)1\{S_i = s\}\eta_{i,0}(s, \tau)}{(1 - \pi) f_0\big(q_0(\tau)\big)}$$

$$+\left[\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right]1\{S_i=s\}\Bigg\}^2$$

$$=\sum_{s\in\mathcal{S}}\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right\}^2+\sum_{s\in\mathcal{S}}\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{A_i1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}\right\}^2$$

$$+\sum_{s\in\mathcal{S}}\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right]1\{S_i=s\}\right\}^2$$

$$+\sum_{s\in\mathcal{S}}\frac{2}{n}\sum_{i=1}^{n}\left\{\frac{A_i1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}\right\}\left[\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right]$$

$$-\sum_{s\in\mathcal{S}}\frac{2}{n}\sum_{i=1}^{n}\left\{\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right\}\left[\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right]$$

$$\xrightarrow{p}\zeta_Y^2(\pi,\tau)+\zeta_S^2(\tau).$$

Note that the RHS of the above display is the same as the asymptotic variance of the original estimator $\hat{q}(\tau)$. By the CLT conditional on data, we can establish the one-dimensional weak convergence. Then, by the Cramér–Wold theorem, we can extend such result to any finite dimension. This concludes the proof.

Appendix D: Proof of Theorem 5.1

It suffices to prove the theorem with $\hat{q}(\tau)$ replaced by

$$\tilde{q}(\tau)=q(\tau)+\left[\sum_{s\in\mathcal{S}}\sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor}\frac{\tilde{\eta}_{i,1}(s,\tau)}{n\pi f_1(q_1(\tau))}-\sum_{s\in\mathcal{S}}\sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor}\frac{\tilde{\eta}_{i,0}(s,\tau)}{n(1-\pi)f_0(q_0(\tau))}\right]$$
$$+\left[\sum_{i=1}^{n}\frac{1}{n}\left(\frac{m_1(S_i,\tau)}{f_1(q_1(\tau))}-\frac{m_0(S_i,\tau)}{f_0(q_0(\tau))}\right)\right],$$

as we have shown in Theorem 3.2 that

$$\sup_{\tau\in\Upsilon}\left|\tilde{q}(\tau)-\hat{q}(\tau)\right|=o_p(1/\sqrt{n}).$$

We first consider the SQR estimator. Note that

$$\sqrt{n}\left(\hat{\beta}^*(\tau)-\beta(\tau)\right)=\arg\min_u L_n^*(u,\tau),$$

where $L_n^*(u,\tau)=\sum_{i=1}^n[\rho_\tau(Y_i^*-\dot{A}_i^{*\prime}\beta(\tau)-\dot{A}_i^{*\prime}u/\sqrt{n})-\rho_\tau(Y_i^*-\dot{A}_i^{*\prime}\beta(\tau))]$. Then, $\hat{\beta}_1^*(\tau)$, the bootstrap counterpart of the SQR estimator, is just the second element of $\hat{\beta}^*(\tau)$. Similar to the proof of Theorem 3.1,

$$L_n^*(u,\tau)=-u'W_n^*(\tau)+Q_n^*(u,\tau),$$

where

$$W_n^*(\tau) = \sum_{i=1}^n \frac{1}{\sqrt{n}} \dot{A}_i^* \big(\tau - 1\{Y_i^* \le \dot{A}_i^{*\prime} \beta(\tau)\}\big)$$

and

$$\begin{aligned}
Q_n^*(u,\tau) &= \sum_{i=1}^n \int_0^{\frac{\dot{A}_i^{*\prime} u}{\sqrt{n}}} \big(1\{Y_i^* - \dot{A}_i^{*\prime}\beta(\tau) \le v\} - 1\{Y_i^* - \dot{A}_i^{*\prime}\beta(\tau) \le 0\}\big)\, dv \\
&= \sum_{i=1}^n A_i^* \int_0^{\frac{u_0+u_1}{\sqrt{n}}} \big(1\{Y_i^*(1) - q_1(\tau) \le v\} - 1\{Y_i^*(1) - q_1(\tau) \le 0\}\big)\, dv \\
&\quad + \sum_{i=1}^n (1 - A_i^*) \int_0^{\frac{u_0}{\sqrt{n}}} \big(1\{Y_i^*(0) - q_0(\tau) \le v\} - 1\{Y_i^*(0) - q_0(\tau) \le 0\}\big)\, dv \\
&\equiv Q_{n,1}^*(u,\tau) + Q_{n,0}^*(u,\tau). \qquad \text{(D.1)}
\end{aligned}$$

Define $\eta_{i,j}^*(s,\tau) = (\tau - 1\{Y_i^*(j) \le q_j(\tau)\}) - m_j(s,\tau)$ and $\tilde{\eta}_{i,j}(s,\tau) = \tau - 1\{Y_i^s(j) \le q_j(\tau)\} - m_j(s,\tau)$, $j = 0, 1$, where $Y_i^s(j)$ is defined in the proof of Theorem 3.1. Then we have

$$\begin{aligned}
W_n^*(\tau) &= e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} A_i^* 1\{S_i^* = s\}\big(\tau - 1\{Y_i^*(1) \le q_1(\tau)\}\big) \\
&\quad + e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} (1 - A_i^*) 1\{S_i^* = s\}\big(\tau - 1\{Y_i^*(0) \le q_0(\tau)\}\big) \\
&= \Bigg[e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} A_i^* 1\{S_i^* = s\} \eta_{i,1}^*(s,\tau) \\
&\quad + e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} (1 - A_i^*) 1\{S_i^* = s\} \eta_{i,0}^*(s,\tau) \Bigg] \\
&\quad + \Bigg[e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} (A_i^* - \pi) 1\{S_i^* = s\} m_1(s,\tau) \\
&\quad - e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} (A_i^* - \pi) 1\{S_i^* = s\} m_0(s,\tau) \Bigg] \\
&\quad + \Bigg[e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} \pi 1\{S_i^* = s\} m_1(s,\tau) \\
&\quad + e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^n \frac{1}{\sqrt{n}} (1 - \pi) 1\{S_i^* = s\} m_0(s,\tau) \Bigg]
\end{aligned}$$

$$\equiv W^*_{n,1}(\tau) + W^*_{n,2}(\tau) + W^*_{n,3}(\tau).$$

By Lemma E.5, there exists a sequence of independent Poisson(1) random variables $\{\xi^s_i\}_{i\geq 1, s\in\mathcal{S}}$ such that $\{\xi^s_i\}_{i\geq 1, s\in\mathcal{S}} \perp\!\!\!\perp \{A^*_i, S^*_i, Y_i, A_i, S_i\}_{i\geq 1}$,

$$\sum_{i=1}^{n} A^*_i 1\{S^*_i = s\}\eta^*_{i,1}(s,\tau) = \sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi^s_i \tilde{\eta}_{i,1}(s,\tau) + R^*_1(s,\tau),$$

and

$$\sum_{i=1}^{n} (1-A^*_i) 1\{S^*_i = s\}\eta^*_{i,1}(s,\tau) = \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)} \xi^s_i \tilde{\eta}_{i,0}(s,\tau) + R^*_0(s,\tau),$$

where $\sup_{\tau\in\Upsilon}(|R^*_1(s,\tau)| + |R^*_0(s,\tau)|) = o_p(\sqrt{n(s)}) = o_p(\sqrt{n})$ for all $s \in \mathcal{S}$. Therefore,

$$\big(W^*_{n,1}(\tau), W^*_{n,2}(\tau), W^*_{n,3}(\tau)\big) \stackrel{d}{=} \big(\tilde{W}^*_{n,1}(\tau) + R(\tau), W^*_{n,2}(\tau), W^*_{n,3}(\tau)\big),$$

where $\sup_{\tau\in\Upsilon} \|R(\tau)\| = o_p(1)$ and

$$\tilde{W}^*_{n,1}(\tau) = e_1 \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \frac{\xi^s_i}{\sqrt{n}} \tilde{\eta}_{i,1}(s,\tau) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)} \frac{\xi^s_i}{\sqrt{n}} \tilde{\eta}_{i,0}(s,\tau)$$

In addition, following the same argument in the proof of Lemma E.2, we can further show that

$$\tilde{W}^*_{n,1}(\tau) = W^{**}_{n,1}(\tau) + R^*_n(\tau),$$

where $\sup_{\tau\in\Upsilon} \|R^*_n(\tau)\| = o_p(1)$ and

$$W^{**}_{n,1}(\tau) = e_1 \sum_{s\in\mathcal{S}} \sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor} \frac{\xi^s_i}{\sqrt{n}} \tilde{\eta}_{i,1}(s,\tau) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor} \frac{\xi^s_i}{\sqrt{n}} \tilde{\eta}_{i,0}(s,\tau).$$

By construction, $W^{**}_{n,1}(\tau) \perp\!\!\!\perp (W^*_{n,2}(\tau), W^*_{n,3}(\tau))$. Also note that $\{S^*_i\}_{i=1}^n$ are the nonparametric bootstrap draws based on the empirical CDF of $\{S_i\}_{i=1}^n$. Then, by van der Vaart and Wellner (1996, Section 3.6), there exists a sequence of independent Poisson(1) random variables $\{\tilde{\xi}_i\}_{i\geq 1}$ that is independent of data, $\{A^*_i\}$ and $\{\xi^s_i\}_{i\geq 1, s\in\mathcal{S}}$ such that

$$\sup_{\tau\in\Upsilon} \big\| W^*_{n,3}(\tau) - W^{**}_{n,3}(\tau) \big\| = o_p(1),$$

where

$$W^{**}_{n,3}(\tau) = e_1 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{\tilde{\xi}_i}{\sqrt{n}} \pi 1\{S_i = s\} m_1(s,\tau) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \frac{\tilde{\xi}_i}{\sqrt{n}} (1-\pi) 1\{S_i = s\} m_0(s,\tau)$$

By Lemma E.6,

$$Q_n^*(u,\tau) \xrightarrow{p} \frac{1}{2}u'Q(\tau)u,$$

where $Q(\tau)$ is defined in (A.3). Then, by the same argument in the proof of Theorem 3.1, we have

$$\sqrt{n}\big(\hat\beta^*(\tau)-\beta(\tau)\big) = Q^{-1}(\tau)\big(W_{n,1}^{**}(\tau)+W_{n,2}^{*}(\tau)+W_{n,3}^{**}(\tau)\big)+R^*(\tau),$$

where $\sup_{\tau\in\Upsilon}\|R^*(\tau)\| = o_p(1)$. Focusing on the second element of $\hat\beta^*(\tau)$, we have

$$\begin{aligned}
&\sqrt{n}\big(\hat\beta_1^*(\tau)-q(\tau)\big)\\
&=\Bigg[\sum_{s\in\mathcal{S}}\sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor}\frac{\xi_i^s\tilde\eta_{i,1}(s,\tau)}{\sqrt{n}\pi f_1\big(q_1(\tau)\big)}-\sum_{s\in\mathcal{S}}\sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor}\frac{\xi_i^s\tilde\eta_{i,0}(s,\tau)}{\sqrt{n}(1-\pi)f_0\big(q_0(\tau)\big)}\Bigg]\\
&\quad+\Bigg[\sum_{s\in\mathcal{S}}\frac{D_n^*(s)}{\sqrt{n}}\bigg(\frac{m_1(s,\tau)}{\pi f_1\big(q_1(\tau)\big)}+\frac{m_0(s,\tau)}{\pi f_0\big(q_0(\tau)\big)}\bigg)\Bigg]\\
&\quad+\Bigg[\sum_{i=1}^{n}\frac{\tilde\xi_i}{\sqrt{n}}\bigg(\frac{m_1(S_i,\tau)}{f_1\big(q_1(\tau)\big)}-\frac{m_0(S_i,\tau)}{f_0\big(q_0(\tau)\big)}\bigg)\Bigg]+R_1^*(\tau),
\end{aligned}$$

where $\sup_{\tau\in\Upsilon}|R_1^*(\tau)| = o_p(1)$. In addition, by definition, we have

$$\begin{aligned}
&\sqrt{n}\big(\tilde q(\tau)-q(\tau)\big)\\
&=\Bigg[\sum_{s\in\mathcal{S}}\sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor}\frac{\tilde\eta_{i,1}(s,\tau)}{\sqrt{n}\pi f_1\big(q_1(\tau)\big)}-\sum_{s\in\mathcal{S}}\sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor}\frac{\tilde\eta_{i,0}(s,\tau)}{\sqrt{n}(1-\pi)f_0\big(q_0(\tau)\big)}\Bigg]\\
&\quad+\Bigg[\sum_{i=1}^{n}\frac{1}{\sqrt{n}}\bigg(\frac{m_1(S_i,\tau)}{f_1\big(q_1(\tau)\big)}-\frac{m_0(S_i,\tau)}{f_0\big(q_0(\tau)\big)}\bigg)\Bigg].
\end{aligned}$$

By taking difference of the two displays above, we have

$$\begin{aligned}
&\sqrt{n}\big(\hat\beta_1^*(\tau)-\tilde q(\tau)\big)\\
&=\Bigg[\sum_{s\in\mathcal{S}}\sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor}\frac{\big(\xi_i^s-1\big)\tilde\eta_{i,1}(s,\tau)}{\sqrt{n}\pi f_1\big(q_1(\tau)\big)}-\sum_{s\in\mathcal{S}}\sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor}\frac{\big(\xi_i^s-1\big)\tilde\eta_{i,0}(s,\tau)}{\sqrt{n}(1-\pi)f_0\big(q_0(\tau)\big)}\Bigg]\\
&\quad+\Bigg[\sum_{s\in\mathcal{S}}\frac{D_n^*(s)}{\sqrt{n}}\bigg(\frac{m_1(s,\tau)}{\pi f_1\big(q_1(\tau)\big)}+\frac{m_0(s,\tau)}{\pi f_0\big(q_0(\tau)\big)}\bigg)\Bigg]\\
&\quad+\Bigg[\sum_{i=1}^{n}\frac{\tilde\xi_i-1}{\sqrt{n}}\bigg(\frac{m_1(S_i,\tau)}{f_1\big(q_1(\tau)\big)}-\frac{m_0(S_i,\tau)}{f_0\big(q_0(\tau)\big)}\bigg)\Bigg]+R_1^*(\tau). \qquad (D.2)
\end{aligned}$$

Note that, conditionally on data, the first and third brackets on the RHS of the above display converge to Gaussian processes with covariance kernels

$$\frac{\min(\tau_1,\tau_2)-\tau_1\tau_2-\mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)}{\pi f_1\big(q_1(\tau_1)\big)f_1\big(q_1(\tau_2)\big)}+\frac{\min(\tau_1,\tau_2)-\tau_1\tau_2-\mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)}{(1-\pi)f_0\big(q_0(\tau_1)\big)f_0\big(q_0(\tau_2)\big)}$$

and

$$\mathbb{E}\left[\frac{m_1(S,\tau_1)}{f_1\big(q_1(\tau_1)\big)}-\frac{m_0(S,\tau_1)}{f_0\big(q_0(\tau_1)\big)}\right]\left[\frac{m_1(S,\tau_2)}{f_1\big(q_1(\tau_2)\big)}-\frac{m_0(S,\tau_2)}{f_0\big(q_0(\tau_2)\big)}\right],$$

uniformly over $\tau\in\Upsilon$, respectively. In addition, by Assumption 4(i), conditionally data (and thus $\{S_i\}_{i=1}^n$), the second bracket on the RHS of (D.2) converges to a Gaussian process with a covariance kernel

$$\mathbb{E}\gamma(S)\left[\frac{m_1(S,\tau_1)m_1(S,\tau_2)}{\pi^2 f_1\big(q_1(\tau_1)\big)f_1\big(q_1(\tau_2)\big)}+\frac{m_1(S,\tau_1)m_0(S,\tau_2)}{\pi(1-\pi)f_1\big(q_1(\tau_1)\big)f_0\big(q_0(\tau_2)\big)}\right],$$

uniformly over $\tau\in\Upsilon$. Furthermore, we notice that these three Gaussian processes are independent. Therefore, we have, conditionally on data and uniformly over $\tau\in\Upsilon$,

$$\sqrt{n}\big(\hat\beta_1^*(\tau)-\tilde q(\tau)\big)\rightsquigarrow\mathcal{B}_{\mathrm{sqr}}(\tau),$$

where $\mathcal{B}_{\mathrm{sqr}}(\tau)$ is defined in Theorem 3.1. This leads to the desired result for the simple quantile regression estimator.

Next, we briefly describe the derivation for the IPW estimator. Following the proof of Theorem 3.2, we have

$$\sqrt{n}\big(\hat q_1^*(\tau)-q_1(\tau)\big)=\arg\min_u L_n^*(u,\tau),$$

where

$$\begin{aligned}L_n^*(u,\tau)&\equiv\sum_{i=1}^n\frac{A_i^*}{\hat\pi^*(S_i^*)}\left[\rho_\tau\left(Y_i^*-q_1(\tau)-\frac{u}{\sqrt{n}}\right)-\rho_\tau\big(Y_i^*-q_1(\tau)\big)\right]\\&=-L_{1,n}^*(\tau)u+L_{2,n}^*(u,\tau),\end{aligned}$$

and $\hat\pi^*(s)=\frac{n_1^*(s)}{n^*(s)}$. Then, we have

$$L_{1,n}^*(\tau)=W_{\mathrm{ipw},1,n}^*(\tau)+R_{\mathrm{ipw},1}^*(\tau),$$

where

$$W_{\mathrm{ipw},1,n}^*(\tau)=\sum_{s\in\mathcal{S}}\frac{1}{\sqrt{n}}\sum_{i=1}^n\frac{A_i^*1\{S_i^*=s\}\eta_{i,1}^*(s,\tau)}{\pi}+\sum_{i=1}^n\frac{m_1\big(S_i^*,\tau\big)}{\sqrt{n}},$$

and

$$R_{\mathrm{ipw},1}^*(\tau)=-\sum_{i=1}^n\sum_{s\in\mathcal{S}}\frac{A_i^*1\{S_i^*=s\}D_n^*(s)}{n^*(s)\sqrt{n}\hat\pi^*(s)\pi}\eta_{i,1}^*(s,\tau).$$

By Lemma E.5, $\sup_{\tau\in\Upsilon}|R^{*}_{\text{ipw},1}(\tau)|=o_p(1)$. In addition, same as above, we can show that

$$\sup_{\tau\in\Upsilon}\left|W^{*}_{\text{ipw},1,n}(\tau)-W^{**}_{\text{ipw},1,n}(\tau)\right|=o_p(1),$$

where

$$W^{**}_{\text{ipw},1,n}(\tau)=\sum_{s\in\mathcal{S}}\sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor}\frac{\xi_i^s\tilde{\eta}_{i,1}(s,\tau)}{\sqrt{n}\pi}+\sum_{i=1}^{n}\frac{\tilde{\xi}_i m_1(S_i,\tau)}{\sqrt{n}}.$$

Similar to Lemma E.6, we can show that, uniformly over $\tau\in\Upsilon$,

$$L^{*}_{2,n}(\tau)\xrightarrow{p}\frac{f_1\big(q_1(\tau)\big)u^2}{2}.$$

Therefore,

$$\sqrt{n}\big(\hat{q}_1^{*}(\tau)-q_1(\tau)\big)=\frac{W^{**}_{\text{ipw},1,n}(\tau)}{f_1\big(q_1(\tau)\big)}+R^{**}_{\text{ipw},1}(\tau),$$

where $\sup_{\tau\in\Upsilon}|R^{**}_{\text{ipw},1}(\tau)|=o_p(1)$. Similarly, we can show

$$\sqrt{n}\big(\hat{q}_0^{*}(\tau)-q_0(\tau)\big)=\frac{W^{**}_{\text{ipw},0,n}(\tau)}{f_0\big(q_0(\tau)\big)}+R^{**}_{\text{ipw},0}(\tau),$$

where $\sup_{\tau\in\Upsilon}|R^{**}_{\text{ipw},0}(\tau)|=o_p(1)$ and

$$W^{**}_{\text{ipw},0,n}(\tau)=\sum_{s\in\mathcal{S}}\sum_{i=\lfloor n(F(s)\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor}\frac{\xi_i^s\tilde{\eta}_{i,0}(s,\tau)}{\sqrt{n}\pi}+\sum_{i=1}^{n}\frac{\tilde{\xi}_i m_0(S_i,\tau)}{\sqrt{n}}.$$

Therefore,

$$\begin{aligned}
&\sqrt{n}\big(\hat{q}^{*}(\tau)-\tilde{q}(\tau)\big)\\
&=\left[\sum_{s\in\mathcal{S}}\sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor}\frac{\big(\xi_i^s-1\big)\tilde{\eta}_{i,1}(s,\tau)}{\sqrt{n}\pi f_1\big(q_1(\tau)\big)}-\sum_{s\in\mathcal{S}}\sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor}\frac{\big(\xi_i^s-1\big)\tilde{\eta}_{i,0}(s,\tau)}{\sqrt{n}(1-\pi)f_0\big(q_0(\tau)\big)}\right]\\
&\quad+\left[\sum_{i=1}^{n}\frac{\tilde{\xi}_i-1}{\sqrt{n}}\left(\frac{m_1(S_i,\tau)}{f_1\big(q_1(\tau)\big)}-\frac{m_0(S_i,\tau)}{f_0\big(q_0(\tau)\big)}\right)\right]+R^{*}_{\text{ipw}}(\tau),
\end{aligned}$$

where $\sup_{\tau\in\Upsilon}|R^{*}_{\text{ipw}}(\tau)|=o_p(1)$. Last, we can show that, conditionally on data and uniformly over $\tau\in\Upsilon$, the RHS of the above display weakly converges to the Gaussian process $\mathcal{B}_{\text{ipw}}(\tau)$, where $\mathcal{B}_{\text{ipw}}(\tau)$ is defined in Theorem 3.2.

Appendix E: Technical lemmas

Lemma E.1. *Let S_k be the kth partial sum of Banach space valued independent identically distributed random variables, then*

$$\mathbb{P}\Big(\max_{1\le k\le n}\|S_k\|\ge\varepsilon\Big)\le 3\max_{1\le k\le n}\mathbb{P}\big(\|S_k\|\ge\varepsilon/3\big).$$

When S_k takes values on $\Re$, Lemma E.1 is Peña, Lai, and Shao (2008, Exercise 2.3).

Proof. First suppose $\max_k \mathbb{P}(\|S_n-S_k\|\ge 2\varepsilon/3)\le 2/3$. In addition, define

$$A_k=\big\{\|S_k\|\ge\varepsilon, \|S_j\|<\varepsilon, 1\le j<k\big\}.$$

Then

$$\begin{aligned}\mathbb{P}\Big(\max_k\|S_k\|\ge\varepsilon\Big)&\le\mathbb{P}\big(\|S_n\|\ge\varepsilon/3\big)+\sum_{k=1}^n\mathbb{P}\big(\|S_n\|\le\varepsilon/3, A_k\big)\\&\le\mathbb{P}\big(\|S_n\|\ge\varepsilon/3\big)+\sum_{k=1}^n\mathbb{P}\big(\|S_n-S_k\|\ge 2\varepsilon/3\big)\mathbb{P}(A_k)\\&\le\mathbb{P}\big(\|S_n\|\ge\varepsilon/3\big)+\frac{2}{3}\mathbb{P}\Big(\max_k\|S_k\|\ge\varepsilon\Big).\end{aligned}$$

This implies

$$\mathbb{P}\Big(\max_k\|S_k\|\ge\varepsilon\Big)\le 3\mathbb{P}\big(\|S_n\|\ge\varepsilon/3\big).$$

On the other hand, if $\max_k \mathbb{P}(\|S_n-S_k\|\ge 2\varepsilon/3)>2/3$, then there exists k_0 such that $\mathbb{P}(\|S_n-S_{k_0}\|\ge 2\varepsilon/3)>2/3$. Thus,

$$\mathbb{P}\big(\|S_n\|\ge\varepsilon/3\big)+\mathbb{P}\big(\|S_{k_0}\|\ge\varepsilon/3\big)\ge 2/3.$$

This implies

$$3\max_{1\le k\le n}\mathbb{P}\big(\|S_k\|\ge\varepsilon/3\big)\ge 3\max\big(\mathbb{P}\big(\|S_n\|\ge\varepsilon/3\big),\mathbb{P}\big(\|S_{k_0}\|\ge\varepsilon/3\big)\big)\ge 1\ge\mathbb{P}\Big(\max_{1\le k\le n}\|S_k\|\ge\varepsilon\Big).$$

This concludes the proof. □

Lemma E.2. *Let $W_{n,j}(\tau)$, $j=1,2,3$ be defined as in* (A.4). *If Assumptions in Theorem* 3.1 *hold, then uniformly over $\tau\in\Upsilon$,*

$$\big(W_{n,1}(\tau),W_{n,2}(\tau),W_{n,3}(\tau)\big)\rightsquigarrow\big(\mathcal{B}_1(\tau),\mathcal{B}_2(\tau),\mathcal{B}_3(\tau)\big),$$

where $(\mathcal{B}_1(\tau),\mathcal{B}_2(\tau),\mathcal{B}_3(\tau))$ are three independent two-dimensional Gaussian processes with covariance kernels $\Sigma_1(\tau_1,\tau_2)$, $\Sigma_2(\tau_1,\tau_2)$, and $\Sigma_3(\tau_1,\tau_2)$, respectively. The expressions for the three kernels are derived in the proof below.

Proof. We follow the general argument in the proof of Bugni, Canay, and Shaikh (2018, Lemma B.2). We divide the proof into two steps. In the first step, we show that

$$\big(W_{n,1}(\tau), W_{n,2}(\tau), W_{n,3}(\tau)\big) \stackrel{d}{=} \big(W^{\star}_{n,1}(\tau), W_{n,2}(\tau), W_{n,3}(\tau)\big) + o_p(1),$$

where the $o_p(1)$ term holds uniformly over $\tau \in \Upsilon$, $W^{\star}_{n,1}(\tau) \perp\!\!\!\perp (W_{n,2}(\tau), W_{n,3}(\tau))$, and, uniformly over $\tau \in \Upsilon$,

$$W^{\star}_{n,1}(\tau) \rightsquigarrow \mathcal{B}_1(\tau).$$

In the second step, we show that

$$\big(W_{n,2}(\tau), W_{n,3}(\tau)\big) \rightsquigarrow \big(\mathcal{B}_2(\tau), \mathcal{B}_3(\tau)\big)$$

uniformly over $\tau \in \Upsilon$ and $\mathcal{B}_2(\tau) \perp\!\!\!\perp \mathcal{B}_3(\tau)$.

Step 1. Let $\tilde{\eta}_{i,j}(s,\tau) = \tau - 1\{Y_i^s(j) \le q_j(\tau)\} - m_j(s,\tau)$, for $j = 0, 1$, where $\{Y_i^s(0), Y_i^s(1)\}_{i\ge 1}$ are the same as defined in Step 1 in the proof of Theorem 3.1. In addition, denote

$$\tilde{W}_{n,1}(\tau) = e_1 \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \frac{1}{\sqrt{n}} \tilde{\eta}_{i,1}(s,\tau) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)} \frac{1}{\sqrt{n}} \tilde{\eta}_{i,0}(s,\tau).$$

Then we have

$$\big\{W_{n,1}(\tau) | \{A_i, S_i\}_{i=1}^n\big\} \stackrel{d}{=} \big\{\tilde{W}_{n,1}(\tau) | \{A_i, S_i\}_{i=1}^n\big\}.$$

Because both $W_{n,2}(\tau)$ and $W_{n,3}(\tau)$ are only functions of $\{A_i, S_i\}_{i=1}^n$, we have

$$\big(W_{n,1}(\tau), W_{n,2}(\tau), W_{n,3}(\tau)\big) \stackrel{d}{=} \big(\tilde{W}_{n,1}(\tau), W_{n,2}(\tau), W_{n,3}(\tau)\big).$$

Let

$$W^{\star}_{n,1}(\tau) = e_1 \sum_{s\in\mathcal{S}} \sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor} \frac{1}{\sqrt{n}} \tilde{\eta}_{i,1}(s,\tau) + e_0 \sum_{s\in\mathcal{S}} \sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor} \frac{1}{\sqrt{n}} \tilde{\eta}_{i,0}(s,\tau).$$

Note that $W^{\star}_{n,1}(\tau)$ is a function of $(Y_i^s(1), Y_i^s(0))_{i\ge 1}$ only, which is independent of $\{A_i, S_i\}_{i=1}^n$ by construction. Therefore, $W^{\star}_{n,1}(\tau) \perp\!\!\!\perp (W_{n,2}(\tau), W_{n,3}(\tau))$.

Furthermore, note that

$$\frac{N(s)}{n} \xrightarrow{p} F(s), \qquad \frac{n_1(s)}{n} \xrightarrow{p} \pi p(s) \quad \text{and} \quad \frac{n(s)}{n} \xrightarrow{p} p(s).$$

Denote $\Gamma_{n,j}(s,t,\tau) = \sum_{i=1}^{\lfloor nt\rfloor} \frac{1}{\sqrt{n}} \tilde{\eta}_{i,j}(s,\tau)$. In order to show $\sup_{\tau\in\Upsilon} |\tilde{W}_{n,1}(\tau) - W^{\star}_{n,1}(\tau)| = o_p(1)$ and $W^{\star}_{n,1}(\tau) \rightsquigarrow \mathcal{B}_1(\tau)$, it suffices to show that, (1) for $j = 0, 1$ and $s \in \mathcal{S}$, the stochastic processes

$$\big\{\Gamma_{n,j}(s,t,\tau) : t \in (0,1), \tau \in \Upsilon\big\}$$

in stochastically equicontinuous; and (2) $W^{\star}_{n,1}(\tau)$ converges to $\mathcal{B}_1(\tau)$ in finite dimension.

Claim (1). We want to bound

$$\sup\big|\Gamma_{n,j}(s,t_2,\tau_2)-\Gamma_{n,j}(s,t_1,\tau_1)\big|,$$

where supremum is taken over $0<t_1<t_2<t_1+\varepsilon<1$ and $\tau_1<\tau_2<\tau_1+\varepsilon$ such that $\tau_1,\tau_1+\varepsilon\in\Upsilon$. Note that,

$$\begin{aligned}
&\sup\big|\Gamma_{n,j}(s,t_2,\tau_2)-\Gamma_{n,j}(s,t_1,\tau_1)\big| \\
&\le \sup_{0<t_1<t_2<t_1+\varepsilon<1,\tau\in\Upsilon}\big|\Gamma_{n,j}(s,t_2,\tau)-\Gamma_{n,j}(s,t_1,\tau)\big| \\
&\quad+\sup_{t\in(0,1),\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\varepsilon}\big|\Gamma_{n,j}(s,t,\tau_2)-\Gamma_{n,j}(s,t,\tau_1)\big|. \qquad (E.1)
\end{aligned}$$

Let $m=\lfloor nt_2\rfloor-\lfloor nt_1\rfloor\le\lfloor n\varepsilon\rfloor+1$. Then, for an arbitrary $\delta>0$, by taking $\varepsilon=\delta^4$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{0<t_1<t_2<t_1+\varepsilon<1,\tau\in\Upsilon}\big|\Gamma_{n,j}(s,t_2,\tau)-\Gamma_{n,j}(s,t_1,\tau)\big|\ge\delta\Big) \\
&=\mathbb{P}\left(\sup_{0<t_1<t_2<t_1+\varepsilon<1,\tau\in\Upsilon}\left|\sum_{i=\lfloor nt_1\rfloor+1}^{i=\lfloor nt_2\rfloor}\tilde\eta_{i,j}(s,\tau)\right|\ge\sqrt{n}\delta\right) \\
&=\mathbb{P}\left(\sup_{0<t\le\varepsilon,\tau\in\Upsilon}\left|\sum_{i=1}^{\lfloor nt\rfloor}\tilde\eta_{i,j}(s,\tau)\right|\ge\sqrt{n}\delta\right) \\
&\le\mathbb{P}\Big(\max_{1\le k\le\lfloor n\varepsilon\rfloor}\sup_{\tau\in\Upsilon}\big|S_k(\tau)\big|\ge\sqrt{n}\delta\Big) \\
&\le\frac{270\mathbb{E}\sup_{\tau\in\Upsilon}\left|\sum_{i=1}^{\lfloor n\varepsilon\rfloor}\tilde\eta_{i,j}(s,\tau)\right|}{\sqrt{n}\delta} \\
&\lesssim\frac{\sqrt{n\varepsilon}}{\sqrt{n}\delta}\lesssim\delta,
\end{aligned}$$

where in the first inequality, $S_k(\tau)=\sum_{i=1}^k\tilde\eta_{i,j}(s,\tau)$ and the second inequality holds due to the same argument in (A.2). For the third inequality, denote

$$\mathcal{F}=\big\{\tilde\eta_{i,j}(s,\tau):\tau\in\Upsilon\big\}$$

with an envelope function $F=2$. In addition, because $\mathcal{F}$ is a VC-class with a fixed VC-index, we have

$$J(1,\mathcal{F})<\infty,$$

where

$$J(\delta,\mathcal{F})=\sup_Q\int_0^\delta\sqrt{1+\log N\big(\varepsilon\|F\|_{Q,2},\mathcal{F},L_2(Q)\big)}\,d\varepsilon,$$

$N(\varepsilon\|F\|_{Q,2}, \mathcal{F}, L_2(Q))$ is the covering number, and the supremum is taken over all discrete probability measures Q. Therefore, by van der Vaart and Wellner (1996, Theorem 2.14.1)

$$\frac{270\mathbb{E}\sup_{\tau\in Y}\left|\sum_{i=1}^{\lfloor n\varepsilon\rfloor}\tilde{\eta}_{i,j}(s,\tau)\right|}{\sqrt{n}\delta} \lesssim \frac{\sqrt{\lfloor n\varepsilon\rfloor}\left[\mathbb{E}\sqrt{\lfloor n\varepsilon\rfloor}\|\mathbb{P}_{\lfloor n\varepsilon\rfloor}-\mathbb{P}\|_{\mathcal{F}}\right]}{\sqrt{n}\delta} \lesssim \frac{\sqrt{\lfloor n\varepsilon\rfloor}J(1,\mathcal{F})}{\sqrt{n}\delta}.$$

For the second term on the RHS of (E.1), by taking $\varepsilon=\delta^4$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in(0,1),\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\varepsilon}\left|\Gamma_{n,j}(s,t,\tau_2)-\Gamma_{n,j}(s,t,\tau_1)\right|\geq\delta\Big)\\
&=\mathbb{P}\Big(\max_{1\leq k\leq n}\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\varepsilon}\left|S_k(\tau_1,\tau_2)\right|\geq\sqrt{n}\delta\Big)\\
&\leq\frac{270\mathbb{E}\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\varepsilon}\left|\sum_{i=1}^{n}\big(\tilde{\eta}_{i,j}(s,\tau_2)-\tilde{\eta}_{i,j}(s,\tau_1)\big)\right|}{\sqrt{n}\delta}\lesssim\delta\sqrt{\log\Big(\frac{C}{\delta^2}\Big)},
\end{aligned}$$

where in the first equality, $S_k(\tau_1,\tau_2)=\sum_{i=1}^{k}(\tilde{\eta}_{i,j}(s,\tau_2)-\tilde{\eta}_{i,j}(s,\tau_1))$ and the first inequality follows the same argument as in (A.2). For the last inequality, denote

$$\mathcal{F}=\left\{\tilde{\eta}_{i,j}(s,\tau_2)-\tilde{\eta}_{i,j}(s,\tau_1):\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\varepsilon\right\}$$

with a constant envelope function $F=C$ and

$$\sigma^2=\sup_{f\in\mathcal{F}}\mathbb{E}f^2\in[c_1\varepsilon,c_2\varepsilon],$$

for some constant $0<c_1<c_2<\infty$. Last, $\mathcal{F}$ is nested by some VC class with a fixed VC index. Therefore, by Chernozhukov, Chetverikov, and Kato (2014, Corollary 5.1),

$$\begin{aligned}
&\frac{270\mathbb{E}\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\varepsilon}\left|\sum_{i=1}^{n}\big(\tilde{\eta}_{i,j}(s,\tau_2)-\tilde{\eta}_{i,j}(s,\tau_1)\big)\right|}{\sqrt{n}\delta}\\
&\lesssim\frac{\sqrt{n}\mathbb{E}\|\mathbb{P}_n-\mathbb{P}\|_{\mathcal{F}}}{\delta}\lesssim\sqrt{\frac{\sigma^2\log\Big(\frac{C}{\sigma}\Big)}{\delta^2}+\frac{C\log\Big(\frac{C}{\sigma}\Big)}{\sqrt{n}\delta}}\lesssim\delta\sqrt{\log\Big(\frac{C}{\delta^2}\Big)},
\end{aligned}$$

where the last inequality holds by letting n be sufficiently large. Note that $\delta\sqrt{\log(\frac{C}{\delta^2})}\to 0$ as $\delta\to 0$. This concludes the proof of Claim (1).

Claim (2). For a single τ, by the triangular CLT,

$$W_{n,1}^{\star}(\tau)\rightsquigarrow N\big(0,\Sigma_1(\tau)\big),$$

where $\Sigma_1(\tau) = \pi[\tau(1-\tau) - \mathbb{E}m_1^2(S,\tau)]e_1e_1' + (1-\pi)[\tau(1-\tau) - \mathbb{E}m_0^2(S,\tau)]e_0e_0'$. The convergence in finite dimension can be proved by using the Cramér–Wold device. In particular, we can show that the covariance kernel is

$$\begin{aligned}\Sigma_1(\tau_1,\tau_2) = {} & \pi\big[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)\big]e_1e_1' \\ & + (1-\pi)\big[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)\big]e_0e_0'.\end{aligned}$$

This concludes the proof of Claim (2), and thus leads to the desired results in Step 1.

Step 2. We first consider the marginal distributions for $W_{n,2}(\tau)$ and $W_{n,3}(\tau)$. For $W_{n,2}(\tau)$, by Assumption 1 and the fact that $m_j(s,\tau)$ is continuous in $\tau \in \Upsilon$ $j = 0, 1$, we have, conditionally on $\{S_i\}_{i=1}^n$,

$$W_{n,2}(\tau) = \sum_{s\in\mathcal{S}} \frac{D_n(s)}{\sqrt{n}}\big[e_1m_1(s,\tau) - e_0m_0(s,\tau)\big] \rightsquigarrow \mathcal{B}_2(\tau), \tag{E.2}$$

where $\mathcal{B}_2(\tau)$ is a two-dimensional Gaussian process with covariance kernel

$$\begin{aligned}&\Sigma_2(\tau_1,\tau_2) \\ &\quad = \sum_{s\in\mathcal{S}} p(s)\gamma(s)\big[e_1e_1'm_1(s,\tau_1)m_1(s,\tau_2) - e_1e_0'm_1(s,\tau_1)m_0(s,\tau_2) \\ &\qquad - e_0e_1'm_0(s,\tau_1)m_1(s,\tau_2) + e_0e_0'm_0(s,\tau_1)m_0(s,\tau_2)\big].\end{aligned}$$

For $W_{n,3}(\tau)$, by the fact that $m_j(s,\tau)$ is continuous in $\tau \in \Upsilon$ $j = 0, 1$, we have that, uniformly over $\tau \in \Upsilon$,

$$W_{n,3}(\tau) = \frac{1}{\sqrt{n}}\sum_{i=1}^n\big[e_1\pi m_1(S_i,\tau) + e_0(1-\pi)m_0(S_i,\tau)\big] \rightsquigarrow \mathcal{B}_3(\tau), \tag{E.3}$$

where $\mathcal{B}_3(\tau)$ a two-dimensional Gaussian process with covariance kernel

$$\begin{aligned}\Sigma_3(\tau_1,\tau_2) = {} & e_1e_1'\pi^2\mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2) + e_1e_0'\pi(1-\pi)\mathbb{E}m_1(S,\tau_1)m_0(S,\tau_2) \\ & + e_0e_1'\pi(1-\pi)\mathbb{E}m_0(S,\tau_1)m_1(S,\tau_2) + e_0e_0'(1-\pi)^2\mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2).\end{aligned}$$

In addition, we note that, for any fixed τ,

$$\begin{aligned}\mathbb{P}\big(W_{n,2}(\tau) \le w_1, W_{n,3}(\tau) \le w_2\big) &= \mathbb{E}\mathbb{P}\big(W_{n,2}(\tau) \le w_1 | \{S_i\}_{i=1}^n\big)1\big\{W_{n,3}(\tau) \le w_2\big\} \\ &= \mathbb{E}\mathbb{P}\big(N\big(0,\Sigma_2(\tau,\tau)\big) \le w_1\big)1\big\{W_{n,3}(\tau) \le w_2\big\} + o(1) \\ &= \mathbb{P}\big(N\big(0,\Sigma_3(\tau,\tau)\big) \le w_2\big)\mathbb{P}\big(N\big(0,\Sigma_2(\tau,\tau)\big) \le w_1\big) + o(1).\end{aligned}$$

This implies $\mathcal{B}_2(\tau) \perp\!\!\!\perp \mathcal{B}_3(\tau)$. By the Cramér–Wold device, we can show that

$$\big(W_{n,2}(\tau), W_{n,3}(\tau)\big) \rightsquigarrow \big(\mathcal{B}_2(\tau), \mathcal{B}_3(\tau)\big)$$

jointly in finite dimension, where by an abuse of notation, $\mathcal{B}_2(\tau)$ and $\mathcal{B}_3(\tau)$ have the same marginal distributions of those in (E.2) and (E.3), respectively, and $\mathcal{B}_2(\tau) \perp\!\!\!\perp \mathcal{B}_3(\tau)$.

Last, because both $W_{n,2}(\tau)$ and $W_{n,3}(\tau)$ are tight marginally, so be the joint process $(W_{n,2}(\tau), W_{n,3}(\tau))$. This concludes the proof of Step 2, and thus the whole lemma. □

LEMMA E.3. *Let $\mathcal{W}_{n,j}(\tau)$, $j = 1, 2$ be defined as in* (B.4). *If Assumptions in Theorem* 3.2 *hold, then uniformly over $\tau \in \Upsilon$,*

$$\big(\mathcal{W}_{n,1}(\tau), \mathcal{W}_{n,2}(\tau)\big) \rightsquigarrow \big(\mathcal{B}_{\text{ipw},1}(\tau), \mathcal{B}_{\text{ipw},2}(\tau)\big),$$

where $(\mathcal{B}_{\text{ipw},1}(\tau), \mathcal{B}_{\text{ipw},2}(\tau))$ are two independent two-dimensional Gaussian processes with covariance kernels $\Sigma_{\text{ipw},1}(\tau_1,\tau_2)$ and $\Sigma_{\text{ipw},2}(\tau_1,\tau_2)$, respectively. The expressions for $\Sigma_{\text{ipw},1}(\tau_1,\tau_2)$ and $\Sigma_{\text{ipw},2}(\tau_1,\tau_2)$ are derived in the proof below.

PROOF. The proofs of weak convergence and the independence between $(\mathcal{B}_{\text{ipw},1}(\tau), \mathcal{B}_{\text{ipw},2}(\tau))$ are similar to that in Lemma E.2, and thus, are omitted. Next, we focus on deriving the covariance kernels.

First, similar to the argument in the proof of Lemma E.2,

$$\mathcal{W}_{n,1}(\tau) \stackrel{d}{=} \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \frac{1}{\sqrt{n} f_1\big(q_1(\tau)\big)} \tilde{\eta}_{i,1}(s,\tau) - \sum_{s\in\mathcal{S}} \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)} \frac{1}{\sqrt{n} f_0\big(q_0(\tau)\big)} \tilde{\eta}_{i,0}(s,\tau).$$

Because $(\tilde{\eta}_{i,1}(s,\tau), \tilde{\eta}_{i,0}(s,\tau))$ are independent across i, $n_1(s)/n \xrightarrow{p} \pi p(s)$, and $(n(s) - n_1(s))/n \xrightarrow{p} (1-\pi)p(s)$, we have

$$\Sigma_{\text{ipw},1}(\tau_1,\tau_2) = \frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E} m_1(S,\tau_1) m_1(S,\tau_2)}{\pi f_1\big(q_1(\tau_1)\big) f_1\big(q_1(\tau_2)\big)} + \frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E} m_0(S,\tau_1) m_0(S,\tau_2)}{(1-\pi) f_0\big(q_0(\tau_1)\big) f_0\big(q_0(\tau_2)\big)}.$$

Obviously,

$$\Sigma_{\text{ipw},2}(\tau_1,\tau_2) = \mathbb{E}\left(\frac{m_1(S,\tau_1)}{f_1\big(q_1(\tau_1)\big)} - \frac{m_0(S,\tau_1)}{f_0\big(q_0(\tau_1)\big)}\right)\left(\frac{m_1(S,\tau_2)}{f_1\big(q_1(\tau_2)\big)} - \frac{m_0(S,\tau_2)}{f_0\big(q_0(\tau_2)\big)}\right),$$ □

LEMMA E.4. *If Assumptions* 1 *and* 2 *hold, then conditionally on data, the second element of $[Q(\tau)]^{-1} \sum_{i=1}^n \frac{\xi_i - 1}{\sqrt{n}} \dot{A}_i(\tau - 1\{Y_i \le \dot{A}'\beta(\tau)\})$ weakly converges to $\tilde{\mathcal{B}}_{\text{sqr}}(\tau)$, where $\tilde{\mathcal{B}}_{\text{sqr}}(\tau)$ is a Gaussian process with covariance kernel $\tilde{\Sigma}_{\text{sqr}}(\cdot,\cdot)$ defined in Theorem* 4.1.

PROOF. We denote the second element of $[Q(\tau)]^{-1} \sum_{i=1}^n \frac{\xi_i - 1}{\sqrt{n}} \dot{A}_i(\tau - 1\{Y_i \le \dot{A}'\beta(\tau)\})$ as

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (\xi_i - 1)\mathcal{J}_i(s,\tau),$$

where

$$\mathcal{J}_i(s,\tau) = \mathcal{J}_{i,1}(s,\tau) + \mathcal{J}_{i,2}(s,\tau) + \mathcal{J}_{i,3}(s,\tau),$$

$$\mathcal{J}_{i,1}(s,\tau) = \frac{A_i 1\{S_i = s\}\eta_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)} - \frac{(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau)}{(1-\pi) f_0\big(q_0(\tau)\big)},$$
$$\mathcal{J}_{i,2}(s,\tau) = F_1(s,\tau)(A_i - \pi)1\{S_i = s\},$$
$$F_1(s,\tau) = \frac{m_1(s,\tau)}{\pi f_1\big(q_1(\tau)\big)} + \frac{m_0(s,\tau)}{(1-\pi) f_0\big(q_0(\tau)\big)},$$

and

$$\mathcal{J}_{i,3}(s,\tau) = \left(\frac{m_1(s,\tau)}{f_1\big(q_1(\tau)\big)} - \frac{m_0(s,\tau)}{f_0\big(q_0(\tau)\big)}\right)1\{S_i = s\}.$$

In order to show the weak convergence, we only need to show (1) conditionally stochastic equicontinuity and (2) conditional convergence in finite dimension. We divide the proof into two steps accordingly.

Step 1. In order to show the conditionally stochastic equicontinuity, it suffices to show that, for any $\varepsilon > 0$, as $n \to \infty$ followed by $\delta \to 0$,

$$\mathbb{P}_\xi\left(\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i - 1)\big(\mathcal{J}_i(s,\tau_2) - \mathcal{J}_i(s,\tau_1)\big)\right| \geq \varepsilon\right) \xrightarrow{p} 0,$$

where $\mathbb{P}_\xi(\cdot)$ means that the probability operator is with respect to $\xi_1,\ldots,\xi_n$ and conditional on data. Note

$$\begin{aligned}
&\mathbb{E}\mathbb{P}_\xi\left(\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i - 1)\big(\mathcal{J}_i(s,\tau_1) - \mathcal{J}_i(s,\tau_1)\big)\right| \geq \varepsilon\right)\\
&= \mathbb{P}\left(\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i - 1)\big(\mathcal{J}_i(s,\tau_2) - \mathcal{J}_i(s,\tau_1)\big)\right| \geq \varepsilon\right)\\
&\leq \mathbb{P}\left(\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i - 1)\big(\mathcal{J}_{i,1}(s,\tau_2) - \mathcal{J}_{i,1}(s,\tau_1)\big)\right| \geq \varepsilon/3\right)\\
&\quad + \mathbb{P}\left(\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i - 1)\big(\mathcal{J}_{i,2}(s,\tau_2) - \mathcal{J}_{i,2}(s,\tau_1)\big)\right| \geq \varepsilon/3\right)\\
&\quad + \mathbb{P}\left(\sup_{\tau_1,\tau_2\in Y,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i - 1)\big(\mathcal{J}_{i,3}(s,\tau_2) - \mathcal{J}_{i,3}(s,\tau_1)\big)\right| \geq \varepsilon/3\right).
\end{aligned}$$

Further note that

$$\sum_{i=1}^n(\xi_i - 1)\mathcal{J}_{i,1}(s,\tau) \stackrel{d}{=} \sum_{i=N(s)+1}^{N(s)+n_1(s)}\frac{(\xi_i - 1)\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)} - \sum_{i=n(s)+n_1(s)+1}^{N(s)+n(s)}\frac{(\xi_i - 1)\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi) f_0\big(q_0(\tau)\big)}$$

By the same argument in Claim (1) in the proof of Lemma E.2, we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)\big(\mathcal{J}_{i,1}(s,\tau_2)-\mathcal{J}_i(s,\tau_1)\big)\right|\geq\varepsilon/3\right)\\&\leq\frac{3\mathbb{E}\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)\big(\mathcal{J}_{i,1}(s,\tau_2)-\mathcal{J}_{i,1}(s,\tau_1)\big)\right|}{\varepsilon}\\&\leq\frac{3\sqrt{c_2\delta\log\left(\frac{C}{c_1\delta}\right)}+\frac{3C\log\left(\frac{C}{c_1\delta}\right)}{\sqrt{n}}}{\varepsilon},\end{aligned}$$

where C, $c_1<c_2$ are some positive constants that are independent of (n,ε,δ). By letting $n\to\infty$ followed by $\delta\to 0$, the RHS vanishes.

For $\mathcal{J}_{i,2}$, we note that $F_1(s,\tau)$ is Lipschitz in τ. Therefore,

$$\begin{aligned}&\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)\big(\mathcal{J}_{i,2}(s,\tau_2)-\mathcal{J}_{i,2}(s,\tau_1)\big)\right|\geq\varepsilon/3\right)\\&\leq\sum_{s\in\mathcal{S}}\mathbb{P}\left(C\delta\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(A_i-\pi)1\{S_i=s\}\right|\geq\varepsilon/3\right)\to 0\end{aligned}$$

as $n\to\infty$ followed by $\delta\to 0$, where we use the fact that

$$\sup_{s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(A_i-\pi)1\{S_i=s\}\right|=O_p(1).$$

To see this claim, we note that, conditionally on data,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n(A_i-\pi)^2 1\{S_i=s\}&=\frac{1}{n}\sum_{i=1}^n\big(A_i-\pi-2\pi(A_i-\pi)+\pi-\pi^2\big)1\{S_i=s\}\\&=\frac{D_n(s)-2\pi D_n(s)}{n}+\pi(1-\pi)\frac{n(s)}{n}\xrightarrow{p}\pi(1-\pi)p(s).\end{aligned}$$

Then, by the Lindeberg CLT, conditionally on data,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(A_i-\pi)1\{S_i=s\}\rightsquigarrow N\big(0,\pi(1-\pi)p(s)\big)=O_p(1).$$

Last, by the standard maximal inequality (e.g., van der Vaart and Wellner (1996, Theorem 2.14.1)) and the fact that

$$\left(\frac{m_1(s,\tau)}{f_1\big(q_1(\tau)\big)}-\frac{m_0(s,\tau)}{f_0\big(q_0(\tau)\big)}\right)$$

is Lipschitz in τ, we have, as $n \to \infty$ followed by $\delta \to 0$,

$$\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\xi_i-1)\big(\mathcal{J}_{i,3}(s,\tau_2)-\mathcal{J}_{i,3}(s,\tau_1)\big)\right|\geq\varepsilon/3\right)\to 0.$$

This concludes the proof of the conditionally stochastic equicontinuity.

Step 2. We focus on the one-dimension case and aim to show that, conditionally on data, for fixed $\tau \in \Upsilon$,

$$\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}(\xi_i-1)\mathcal{J}_i(s,\tau)\rightsquigarrow\mathcal{N}\big(0,\tilde{\Sigma}_{\text{sqr}}(\tau,\tau)\big).$$

The finite-dimensional convergence can be established similarly by the Cramér–Wold device. In view of Lindeberg–Feller central limit theorem, we only need to show that (1)

$$\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{s\in\mathcal{S}}\mathcal{J}_i(s,\tau)\right]^2\xrightarrow{p}\zeta_Y^2(\pi,\tau)+\tilde{\xi}_A^2(\pi,\tau)+\xi_S^2(\pi,\tau)$$

and (2)

$$\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{s\in\mathcal{S}}\mathcal{J}_i(s,\tau)\right]^2\mathbb{E}_\xi(\xi-1)^2 1\left\{\left|\sum_{s\in\mathcal{S}}(\xi_i-1)\mathcal{J}_i(s,\tau)\right|\geq\varepsilon\sqrt{n}\right\}\to 0.$$

(2) is obvious as $|\mathcal{J}_i(s,\tau)|$ is bounded and $\max_i|\xi_i-1|\lesssim\log(n)$ as ξ_i is sub-exponential. Next, we focus on (1). We have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{s\in\mathcal{S}}\mathcal{J}_i(s,\tau)\right]^2\\
&=\frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\Bigg\{\left[\frac{A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0\big(q_0(\tau)\big)}\right]\\
&\quad+F_1(s,\tau)(A_i-\pi)1\{S_i=s\}+\left[\left(\frac{m_1(s,\tau)}{f_1\big(q_1(\tau)\big)}-\frac{m_0(s,\tau)}{f_0\big(q_0(\tau)\big)}\right)1\{S_i=s\}\right]\Bigg\}^2\\
&\equiv\sigma_1^2+\sigma_2^2+\sigma_3^2+2\sigma_{12}+2\sigma_{13}+2\sigma_{23},
\end{aligned}$$

where

$$\sigma_1^2=\frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\left[\frac{A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0\big(q_0(\tau)\big)}\right]^2,$$

$$\sigma_2^2=\frac{1}{n}\sum_{s\in\mathcal{S}}F_1^2(s,\tau)\sum_{i=1}^{n}(A_i-\pi)^2 1\{S_i=s\},$$

$$\sigma_3^2=\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{m_1(S_i,\tau)}{f_1\big(q_1(\tau)\big)}-\frac{m_0(S_i,\tau)}{f_0\big(q_0(\tau)\big)}\right)\right]^2,$$

$$\sigma_{12} = \frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\Big[\frac{A_i 1\{S_i = s\}\eta_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)} - \frac{(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0\big(q_0(\tau)\big)}\Big]$$
$$\times F_1(s,\tau)(A_i - \pi)1\{S_i = s\},$$
$$\sigma_{13} = \frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\Big[\frac{A_i 1\{S_i = s\}\eta_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)} - \frac{(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0\big(q_0(\tau)\big)}\Big]$$
$$\times \Big[\Big(\frac{m_1(s,\tau)}{f_1\big(q_1(\tau)\big)} - \frac{m_0(s,\tau)}{f_0\big(q_0(\tau)\big)}\Big)\Big],$$

and

$$\sigma_{23} = \sigma_{12} = \frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}} F_1(s,\tau)(A_i-\pi)1\{S_i = s\}\Big[\Big(\frac{m_1(s,\tau)}{f_1\big(q_1(\tau)\big)} - \frac{m_0(s,\tau)}{f_0\big(q_0(\tau)\big)}\Big)\Big].$$

For σ_1^2, we have

$$\sigma_1^2 = \frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\Big[\frac{A_i 1\{S_i = s\}\eta_{i,1}^2(s,\tau)}{\pi^2 f_1^2\big(q_1(\tau)\big)} - \frac{(1-A_i)1\{S_i = s\}\eta_{i,0}^2(s,\tau)}{(1-\pi)^2 f_0^2\big(q_0(\tau)\big)}\Big]$$
$$\stackrel{d}{=} \frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\frac{\tilde{\eta}_{i,1}^2(s,\tau)}{\pi^2 f_1^2\big(q_1(\tau)\big)} + \frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}\frac{\tilde{\eta}_{i,0}^2(s,\tau)}{(1-\pi)^2 f_0^2\big(q_0(\tau)\big)}$$
$$\xrightarrow{p} \frac{\tau(1-\tau) - \mathbb{E}m_1^s(S,\tau)}{\pi f_1^2\big(q_1(\tau)\big)} + \frac{\tau(1-\tau) - \mathbb{E}m_0^s(S,\tau)}{(1-\pi) f_0^2\big(q_0(\tau)\big)} = \zeta_Y^2(\pi,\tau),$$

where the second equality holds due to the rearrangement argument in Lemma E.2 and the convergence in probability holds due to uniform convergence of the partial sum process.

For σ_2^2, by Assumption 1,

$$\sigma_2^2 = \frac{1}{n}\sum_{s\in\mathcal{S}} F_1^2(s,\tau)\big(D_n(s) - 2\pi D_n(s) + \pi(1-\pi)1\{S_i = s\}\big)$$
$$\xrightarrow{p} \pi(1-\pi)\mathbb{E}F_1^2(S_i,\tau) = \tilde{\xi}_A^2(\pi,\tau).$$

For σ_3^2, by the law of large number,

$$\sigma_3^2 \xrightarrow{p} \mathbb{E}\Big[\Big(\frac{m_1(S_i,\tau)}{f_1\big(q_1(\tau)\big)} - \frac{m_0(S_i,\tau)}{f_0\big(q_0(\tau)\big)}\Big)\Big]^2 = \xi_S^2(\pi,\tau).$$

For σ_{12}, we have

$$\sigma_{12} = \frac{1}{n}\sum_{s\in\mathcal{S}}(1-\pi)F_1(s,\tau)\sum_{i=1}^{n}\frac{A_i 1\{S_i = s\}\eta_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)}$$

$$- \frac{1}{n} \sum_{s \in \mathcal{S}} \pi F_1(s,\tau) \sum_{i=1}^{n} \frac{(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau)}{(1-\pi) f_0\big(q_0(\tau)\big)}$$

$$\stackrel{d}{=} \frac{1}{n} \sum_{s \in \mathcal{S}} (1-\pi) F_1(s,\tau) \sum_{i=N(s)+1}^{N(s)+n_1(s)} \frac{\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1\big(q_1(\tau)\big)}$$

$$- \frac{1}{n} \sum_{s \in \mathcal{S}} \pi F_1(s,\tau) \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)} \frac{\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi) f_0\big(q_0(\tau)\big)} \xrightarrow{p} 0,$$

where the last convergence holds because by Lemma E.2,

$$\frac{1}{n} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \tilde{\eta}_{i,1}(s,\tau) \xrightarrow{p} 0 \quad \text{and} \quad \frac{1}{n} \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)} \tilde{\eta}_{i,0}(s,\tau) \xrightarrow{p} 0.$$

By the same argument, we can show that

$$\sigma_{13} \xrightarrow{p} 0.$$

Last, for σ_{23}, by Assumption 1,

$$\sigma_{23} = \sum_{s \in \mathcal{S}} F_1(s,\tau) \left[\left(\frac{m_1(s,\tau)}{f_1\big(q_1(\tau)\big)} - \frac{m_0(s,\tau)}{f_0\big(q_0(\tau)\big)} \right) \right] \frac{D_n(s)}{n} \xrightarrow{p} 0.$$

Therefore, conditionally on data,

$$\frac{1}{n} \sum_{i=1}^{n} \left[\sum_{s \in \mathcal{S}} \mathcal{J}_i(s,\tau) \right]^2 \xrightarrow{p} \zeta_Y^2(\pi,\tau) + \tilde{\xi}_A^2(\pi,\tau) + \xi_S^2(\pi,\tau).$$

□

Lemma E.5. *If Assumptions* 1(i) *and* 1(ii) *hold,* $\sup_{s \in \mathcal{S}} \frac{|D_n^*(s)|}{\sqrt{n^*(s)}} = O_p(1)$, $\sup_{s \in \mathcal{S}} \frac{|D_n(s)|}{\sqrt{n(s)}} = O_p(1)$, *and* $n(s) \to \infty$ *for all* $s \in \mathcal{S}$, *a.s., then there exists a sequence of Poisson*(1) *random variables* $\{\xi_i^s\}_{i \geq 1, s \in \mathcal{S}}$ *independent of* $\{A_i^*, S_i^*, Y_i, A_i, S_i\}_{i \geq 1}$ *such that*

$$\sum_{i=1}^{n} A_i^* 1\{S_i^* = s\} \eta_{i,1}^*(s,\tau) = \sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi_i^s \tilde{\eta}_{i,1}(s,\tau) + R_1^*(s,\tau),$$

where $\sup_{\tau \in \Upsilon, s \in \mathcal{S}} |R_1^*(s,\tau)/\sqrt{n(s)}| = o_p(1)$. *In addition,*

$$\sup_{s \in \mathcal{S}, \tau \in \Upsilon} \left| \sum_{i=1}^{n} A_i^* 1\{S_i^* = s\} \eta_{i,1}^*(s,\tau) \right| / \sqrt{n(s)} = O_p(1). \tag{E.4}$$

Proof. Recall $\{Y_i^s(0), Y_i^s(1)\}_{i=1}^n$ as defined in the proof of Theorem 3.1 and

$$\tilde{\eta}_{i,j}(s,\tau) = \tau - 1\{Y_i^s(j) \leq q_j(\tau)\} - m_j(s,\tau),$$

$j = 0, 1$. In addition, let $\Psi_n = \{\eta_{i,1}(s,\tau)\}_{i=1}^n$,

$$\mathbb{N}_n = \{n(s)/n, n_1(s)/n, n^*(s)/n, n_1^*(s)/n\}_{s \in \mathcal{S}}$$

and given $\mathbb{N}_n$, $\{M_{ni}\}_{i=1}^n$ be a sequence of random variables such that the $n_1(s) \times 1$ vector

$$M_n^1(s) = (M_{n,N(s)+1}, \ldots, M_{n,N(s)+n_1(s)})$$

and the $(n(s) - n_1(s)) \times 1$ vector

$$M_n^0(s) = (M_{n,N(s)+n_1(s)+1}, \ldots, M_{n,N(s)+n(s)})$$

satisfy:

1. $M_n^1(s) = \sum_{i=1}^{n_1^*(s)} m_i$ and $M_n^0(s) = \sum_{i=1}^{n^*(s)-n_1^*(s)} m_i'$, where $\{m_i\}_{i=1}^{n_1^*(s)}$ and $\{m_i'\}_{i=1}^{n^*(s)-n_1^*(s)}$ are $n_1^*(s)$ i.i.d. multinomial$(1, n_1^{-1}(s), \ldots, n_1^{-1}(s))$ random vectors and $n^*(s) - n_1^*(s)$ i.i.d. multinomial $(1, (n(s) - n_1(s))^{-1}, \ldots, (n(s) - n_1(s))^{-1})$ random vectors, respectively;
2. $M_n^0(s) \perp\!\!\!\perp M_n^1(s) | \mathbb{N}_n$; and
3. $\{M_n^0(s), M_n^1(s)\}_{s\in\mathcal{S}}$ are independent across s given $\mathbb{N}_n$ and are independent of Ψ_n.

Recall that, by Bugni, Canay, and Shaikh (2018), the original observations can be rearranged according to $s \in \mathcal{S}$ and then within strata, treatment group first and then the control group. Then, given $\mathbb{N}_n$, Step 3 in Section 5 implies that the bootstrap observations $\{Y_i^*\}_{i=1}^n$ can be generated by drawing with replacement from the empirical distribution of the outcomes in each (s, a) cell for $(s, a) \in \mathcal{S} \times \{0, 1\}$, $n_a^*(s)$ times, $a = 0, 1$, where $n_0^*(s) = n^*(s) - n_1^*(s)$. Therefore,

$$\sum_{i=1}^{n} A_i^* 1\{S_i^* = s\} \eta_{i,1}^*(s, \tau) = \sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni} \tilde{\eta}_{i,1}(s, \tau). \tag{E.5}$$

Following the standard approach in dealing with the nonparametric bootstrap, we want to approximate

$$M_{ni}, i = N(s) + 1, \ldots, N(s) + n_1(s)$$

by a sequence of i.i.d. Poisson(1) random variables. We construct this sequence as follows. Let $\widetilde{M}_n^1(s) = \sum_{i=1}^{\widetilde{N}(n_1(s))} m_i$, where $\widetilde{N}(k)$ is a Poisson number with mean k and is independent of $\mathbb{N}_n$. The $n_1(s)$ elements of vector $\widetilde{M}_n^1(s)$ are denoted as $\{\widetilde{M}_{ni}\}_{i=N(s)+1}^{N(s)+n_1(s)}$, which is a sequence of i.i.d. Poisson(1) random variables, given $\mathbb{N}_n$. Therefore,

$$\{\widetilde{M}_{ni}, i = N(s) + 1, \ldots, N(s) + n_1(s) | \mathbb{N}_n\} \equiv \{\xi_i^s, i = N(s) + 1, \ldots, N(s) + n_1(s) | \mathbb{N}_n\}$$

where $\{\xi_i^s\}_{i=1}^n$, $s \in \mathcal{S}$ are i.i.d. sequences of Poisson(1) random variables such that $\{\xi_i^s\}_{i=1}^n$ are independent across $s \in \mathcal{S}$ and against $\mathbb{N}_n$.

Following the argument in van der Vaart and Wellner (1996, Section 3.6), given $n_1(s)$, $n_1^*(s)$, and $\widetilde{N}(n_1(s)) = k$, $|\xi_i^s - M_{ni}|$ is binomially $(|k - n_1^*(s)|, n_1(s)^{-1})$-distributed. In addition, there exists a sequence $\ell_n = O(\sqrt{n(s)})$ such that

$$\begin{aligned}
&\mathbb{P}\big(\big|\widetilde{N}\big(n_1(s)\big) - n_1^*(s)\big| \geq \ell_n\big) \\
&\quad \leq \mathbb{P}\big(\big|\widetilde{N}\big(n_1(s)\big) - n_1(s)\big| \geq \ell_n/3\big) + \mathbb{P}\big(\big|n_1^*(s) - n_1(s)\big| \geq 2\ell_n/3\big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\mathbb{P}\big(\big|N\big(n_1(s)\big) - n_1(s)\big| \geq \ell_n/3|n_1(s)\big) + \mathbb{P}\big(\big|n_1^*(s) - n_1(s)\big| \geq 2\ell_n/3\big) \\
&\leq \varepsilon/3 + \mathbb{P}\big(\big|n_1^*(s) - n_1(s)\big| \geq 2\ell_n/3\big) \\
&\leq \varepsilon/3 + \mathbb{P}\big(\big|D_n^*(s)\big| + \big|D_n(s)\big| + \pi\big|n^*(s) - n(s)\big| \geq 2\ell_n/3\big) \\
&\leq 2\varepsilon/3 + \mathbb{P}\big(\pi\big|n^*(s) - n(s)\big| \geq \ell_n/3\big) \\
&\leq \varepsilon,
\end{aligned}
$$

where the first inequality holds due to the union bound inequality, the second inequality holds by the law of iterated expectation, the third inequality holds because (1) conditionally on data, $\widetilde{N}(n_1(s)) - n_1(s) = O_p(\sqrt{n_1(s)})$ and (2) $n_1(s)/n(s) = \pi + \frac{D_n(s)}{n(s)} \to \pi > 0$ as $n(s) \to \infty$, the fourth inequality holds by the fact that

$$
n_1^*(s) - n_1(s) = D_n^*(s) - D_n(s) + \pi\big(n^*(s) - n(s)\big),
$$

the fifth inequality holds because by Assumptions 1 and 4, $|D_n^*(s)| + |D_n(s)| = O_p(\sqrt{n(s)})$, and the sixth inequality holds because $\{S_i^*\}_{i=1}^n$ is generated from $\{S_i\}_{i=1}^n$ by the standard bootstrap procedure, and thus, by van der Vaart and Wellner (1996, Theorem 3.6.1),

$$
n^*(s) - n(s) = \sum_{i=1}^{n}\big(M_{ni}^w - 1\big)\big(1\{S_i = s\} - p(s)\big) = O_p\big(\sqrt{n(s)}\big),
$$

where $(M_{n1}^w, \dots, M_{nn}^w)$ is independent of $\{S_i\}_{i=1}^n$ and multinomially distributed with parameters n and (probabilities) $1/n, \dots, 1/n$. Therefore, by direct calculation, as $n \to \infty$,

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{N(s)+1 \leq i \leq N(s)+n_1(s)} \big|\xi_i^s - M_{ni}\big| > 2\Big) \\
&\quad \leq \mathbb{P}\Big(\max_{N(s)+1 \leq i \leq N(s)+n_1(s)} \big|\xi_i^s - M_{ni}\big| > 2, n_1(s) \geq n(s)\varepsilon\Big) + \mathbb{P}\big(n_1(s) \leq n(s)\varepsilon\big) \\
&\quad \leq \varepsilon + \mathbb{E}\sum_{i=N(s)+1}^{N(s)+n_1(s)} \mathbb{P}\big(\big|\xi_i^s - M_{ni}\big| > 2, \big|N\big(n_1(s)\big) - n_1^*(s)\big| \leq \ell_n, \\
&\qquad n_1(s) \geq n(s)\varepsilon|n_1(s), n_1^*(s), n(s)\big) + \varepsilon \\
&\quad \leq 2\varepsilon + \mathbb{E}n_1(s)\mathbb{P}\big(\text{bin}\big(\ell_n, n_1^{-1}(s)\big) > 2|n_1(s), n_1^*(s), n(s)\big)1\big\{n_1(s) \geq n(s)\varepsilon\big\} \to 2\varepsilon,
\end{aligned}
$$

where we use the fact that

$$
\begin{aligned}
&n_1(s)\mathbb{P}\big(\text{bin}\big(\ell_n, n_1^{-1}(s)\big) > 2|n_1(s), n_1^*(s), n(s)\big)1\big\{n_1(s) \geq n(s)\varepsilon\big\} \\
&\quad \lesssim n_1(s)\bigg(\frac{\ell_n}{n(s)}\bigg)^3\bigg(\frac{n(s)}{n_1(s)}\bigg)^3 1\big\{n_1(s) \geq n(s)\varepsilon\big\} \lesssim \frac{1}{\sqrt{n(s)}\varepsilon^3} \to 0.
\end{aligned}
$$

Because ε is arbitrary, we have

$$
\mathbb{P}\Big(\max_{N(s)+1 \leq i \leq N(s)+n_1(s)} \big|\xi_i^s - M_{ni}\big| > 2\Big) \to 0. \tag{E.6}
$$

Note that $|\xi_i^s - M_{ni}| = \sum_{j=1}^{\infty} 1\{|\xi_i^s - M_{ni}| \geq j\}$. Let $I_n^j(s)$ be the set of indexes $i \in \{N(s)+1, \ldots, N(s)+n_1(s)\}$ such that $|\xi_i^s - M_{ni}| \geq j$. Then, $\xi_i^s - M_{ni} = \text{sign}(\widetilde{N}(n_1(s)) - n_1^*(s)) \sum_{j=1}^{\infty} 1\{i \in I_n^j(s)\}$. Thus,

$$\frac{1}{\sqrt{n(s)}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \left(\xi_i^s - M_{ni}\right) \tilde{\eta}_{i,1}(s,\tau)$$
$$= \text{sign}\left(\widetilde{N}\left(n_1(s)\right) - n_1^*(s)\right) \sum_{j=1}^{\infty} \left[\frac{\# I_n^j(s)}{\sqrt{n(s)}} \frac{1}{\# I_n^j(s)} \sum_{i \in I_n^j(s)} \tilde{\eta}_{i,1}(s,\tau) \right]. \tag{E.7}$$

In the following, we aim to show that the RHS of (E.7) converges to zero in probability uniformly over $s \in \mathcal{S}$, $\tau \in \Upsilon$. First, note that, by (E.6), $\max_{N(s)+1 \leq i \leq N(s)+n_1(s)} |\xi_i^s - M_{ni}| \leq 2$ occurs with probability approaching one. In the event set that $\max_{N(s)+1 \leq i \leq N(s)+n_1(s)} |\xi_i^s - M_{ni}| \leq 2$, only the first two terms of the first summation on the RHS of (E.7) can be nonzero. In addition, for any j, we have $j(\# I_n^j(s)) \leq |\widetilde{N}(n_1(s)) - n_1(s)| = O_p(\sqrt{n(s)})$, and thus, $\frac{\# I_n^j(s)}{\sqrt{n(s)}} = O_p(1)$ for $j = 1, 2$. Therefore, it suffices to show that, for $j = 1, 2$,

$$\sup_{s \in \mathcal{S}, \tau \in \Upsilon} \left| \frac{1}{\# I_n^j(s)} \sum_{i \in I_n^j(s)} \tilde{\eta}_{i,1}(s,\tau) \right| = o_p(1).$$

Note that

$$\frac{1}{\# I_n^j(s)} \sum_{i \in I_n^j(s)} \tilde{\eta}_{i,1}(s,\tau) = \sum_{i=N(s)+1}^{N(s)+n_1(s)} \omega_{ni} \tilde{\eta}_{i,1}(s,\tau), \tag{E.8}$$

where $\omega_{ni} = \frac{1\{|\xi_i^s - M_{ni}| \geq j\}}{\# I_n^j(s)}$, $i = N(s)+1, \ldots, N(s)+n_1(s)$ and by construction, $\{\omega_{ni}\}_{i=N(s)+1}^{N(s)+n_1(s)}$ is independent of $\{\eta_{i,1}(s,\tau)\}_{i=1}^n$. In addition, because $\{\omega_{ni}\}_{i=N(s)+1}^{N(s)+n_1(s)}$ is exchangeable conditional on $\mathbb{N}_n$, so be it unconditionally. Third, $\sum_{i=N(s)+1}^{N(s)+n_1(s)} \omega_{ni} = 1$ and $\max_{i=N(s)+1,\ldots,N(s)+n_1(s)} |\omega_{ni}| \leq 1/\# I_n^j(s) \xrightarrow{p} 0$. Then, by the same argument in the proof of van der Vaart and Wellner (1996, Lemma 3.6.16), for some $r \in (0,1)$ and any $n_0 = N(s)+1, \ldots, N(s)+n_1(s)$, we have

$$\mathbb{E}\left(\sup_{\tau \in \Upsilon, s \in \mathcal{S}} \left| \sum_{i=N(s)+1}^{N(s)+n_1(s)} \omega_{ni} \tilde{\eta}_{i,1}(s,\tau) \right|^r |\Psi_n, \mathbb{N}_n \right)$$
$$\leq (n_0 - 1)\mathbb{E}\left[\max_{N(s)+n_0 \leq i \leq N(s)+n_1(s)} \omega_{ni}^r | \mathbb{N}_n \right] \left[\frac{1}{n_1(s)} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \sup_{\tau \in \Upsilon, s \in \mathcal{S}} \left| \tilde{\eta}_{i,1}^r(s,\tau) \right| \right]$$
$$+ \left(n_1(s)\mathbb{E}(\omega_{ni}|\mathbb{N}_n)\right)^r$$
$$\times \max_{n_0 \leq k \leq n_1(s)} \mathbb{E}\left[\sup_{\tau \in \Upsilon, s \in \mathcal{S}} \left| \frac{1}{k} \sum_{j=N(s)+n_0}^{N(s)+k} \tilde{\eta}_{R_j(N(s),n_1(s)),1}(s,\tau) \right|^r |\mathbb{N}_n, \Psi_n \right], \tag{E.9}$$

where $(R_{k_1+1}(k_1,k_2),\dots,R_{k_1+k_2}(k_1,k_2))$ is uniformly distributed on the set of all permutations of $k_1+1,\dots,k_1+k_2$ and independent of $\mathbb{N}_n$ and Ψ_n. First, note that $\sup_{s\in\mathcal{S},\tau\in Y}|\eta_{i,1}(s,\tau)|$ is bounded and

$$\max_{N(s)+1\le i\le N(s)+n_1(s)} \omega_{ni}^r \le 1/\big(\#I_n^j(s)\big)^r \xrightarrow{p} 0.$$

Therefore, the first term on the RHS of (E.9) converges to zero in probability for every fixed n_0. For the second term, because $\omega_{ni}|\mathbb{N}_n$ is exchangeable,

$$n_1(s)\mathbb{E}(\omega_{ni}|\mathbb{N}_n) = \sum_{i=N(s)+1}^{N(s)+n_1(s)} \mathbb{E}(\omega_{ni}|\mathbb{N}_n) = 1.$$

In addition, let $\mathbb{S}_n(k_1,k_2)$ be the σ-field generated by all functions of $\{\tilde{\eta}_{i,1}(s,\tau)\}_{i\ge1}$ that are symmetric in their k_1+1 to k_1+k_2 arguments. Then

$$\begin{aligned}
&\max_{n_0\le k\le n_1(s)} \mathbb{E}\left[\sup_{\tau\in Y,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=N(s)+n_0}^{N(s)+k} \tilde{\eta}_{R_j(N(s),n_1(s)),1}(s,\tau)\right|^r \Big|\mathbb{N}_n,\Psi_n\right]\\
&= \max_{n_0\le k\le n_1(s)} \mathbb{E}\left[\sup_{\tau\in Y,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=N(s)+n_0}^{N(s)+k} \tilde{\eta}_{j,1}(s,\tau)\right|^r \Big|\mathbb{N}_n,\mathbb{S}_n\big(N(s),n_1(s)\big)\right]\\
&\le 2\mathbb{E}\left\{\max_{n_0\le k}\left[\sup_{\tau\in Y,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=N(s)+1}^{N(s)+k} \tilde{\eta}_{j,1}(s,\tau)\right|^r\right]\Big|\mathbb{N}_n,\mathbb{S}_n\big(N(s),n_1(s)\big)\right\}\\
&= 2\mathbb{E}\left\{\max_{n_0\le k}\left[\sup_{\tau\in Y,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k} \tilde{\eta}_{j,1}(s,\tau)\right|^r\right]\Big|\mathbb{N}_n,\mathbb{S}_n\big(0,n_1(s)\big)\right\},
\end{aligned}$$

where the inequality holds by the Jansen's inequality and the triangle inequality and the last equality holds because $\{\tilde{\eta}_{j,1}(s,\tau)\}_{j\ge1}$ is an i.i.d. sequence. Apply expectation on both sides, we obtain that

$$\begin{aligned}
&\mathbb{E}\max_{n_0\le k\le n_1(s)} \mathbb{E}\left[\sup_{\tau\in Y,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=N(s)+n_0}^{N(s)+k} \tilde{\eta}_{R_j(N(s),n_1(s)),1}(s,\tau)\right|^r \Big|\mathbb{N}_n,\Psi_n\right]\\
&\quad\le 2\mathbb{E}\max_{n_0\le k\le n}\left[\sup_{\tau\in Y,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\tilde{\eta}_{j,1}(s,\tau)\right|^r\right].
\end{aligned}\tag{E.10}$$

By the usual maximal inequality, as $k\to\infty$,

$$\sup_{\tau\in Y,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\tilde{\eta}_{j,1}(s,\tau)\right| \xrightarrow{a.s.} 0,$$

which implies that as $n_0 \to \infty$

$$\max_{n_0 \le k \le n} \left[\sup_{\tau \in Y, s \in \mathcal{S}} \left| \frac{1}{k} \sum_{j=1}^{k} \tilde{\eta}_{j,1}(s,\tau) \right|^r \right] \le \max_{n_0 \le k} \left[\sup_{\tau \in Y, s \in \mathcal{S}} \left| \frac{1}{k} \sum_{j=1}^{k} \tilde{\eta}_{j,1}(s,\tau) \right|^r \right] \xrightarrow{a.s.} 0.$$

In addition, $\sup_{\tau \in Y, s \in \mathcal{S}} |\frac{1}{k} \sum_{j=1}^{k} \tilde{\eta}_{j,1}(s,\tau)|$ is bounded. Then, by the bounded convergence theorem, we have, as $n_0 \to \infty$,

$$\mathbb{E} \max_{n_0 \le k \le n} \left[\sup_{\tau \in Y, s \in \mathcal{S}} \left| \frac{1}{k} \sum_{j=1}^{k} \tilde{\eta}_{j,1}(s,\tau) \right|^r \right] \to 0,$$

which implies that

$$\mathbb{E} \max_{n_0 \le k \le n_1(s)} \mathbb{E} \left[\sup_{\tau \in Y, s \in \mathcal{S}} \left| \frac{1}{k} \sum_{j=N(s)+n_0}^{N(s)+k} \tilde{\eta}_{R_j(N(s),n_1(s)),1}(s,\tau) \right|^r |\mathbb{N}_n, \Psi_n \right] \xrightarrow{p} 0.$$

Therefore, the second term on the RHS of (E.9) converges to zero in probability as $n_0 \to \infty$. Then, as $n \to \infty$ followed by $n_0 \to \infty$,

$$\mathbb{E}\left(\sup_{\tau \in Y, s \in \mathcal{S}} \left| \sum_{i=N(s)+1}^{N(s)+n_1(s)} \omega_{ni} \tilde{\eta}_{i,1}(s,\tau) \right|^r |\Psi_n, \mathbb{N}_n \right) \xrightarrow{p} 0.$$

Hence, by the Markov inequality and (E.8), we have

$$\sup_{s \in \mathcal{S}, \tau \in Y} \left| \frac{1}{\# I_n^j(s)} \sum_{i \in I_n^j(s)} \tilde{\eta}_{i,1}(s,\tau) \right| \xrightarrow{p} 0.$$

Consequently, following (E.7)

$$\sup_{s \in \mathcal{S}, \tau \in Y} \left| \sum_{i=N(s)+1}^{N(s)+n_1(s)} \left(\xi_i^s - M_{ni} \right) \tilde{\eta}_{i,1}(s,\tau) \right| = o_p\left(\sqrt{n(s)}\right). \tag{E.11}$$

This concludes the first part of this lemma. For the second part, we note

$$\sum_{i=N(s)+1}^{N(s)+n_1(s)} \widetilde{M}_{ni} \tilde{\eta}_{i,1}(s,\tau) \stackrel{d}{=} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi_i^s \tilde{\eta}_{i,1}(s,\tau) \stackrel{d}{=} \sum_{i=1}^{n_1(s)} \xi_i^s \tilde{\eta}_{i,1}(s,\tau),$$

where the second equality holds because $\{\xi_i^s, \tilde{\eta}_{i,1}(s,\tau)\}_{i \ge 1} \perp\!\!\!\perp \{N(s), n_1(s), n(s)\}$. Then, conditionally on $\{N(s), n_1(s), n(s)\}$ and uniformly over $s \in \mathcal{S}$, the usual maximal inequality (van der Vaart and Wellner (1996, Theorem 2.14.1)) implies

$$\sup_{\tau \in Y} \left| \sum_{i=N(s)+1}^{N(s)+n_1(s)} \widetilde{M}_{ni} \tilde{\eta}_{i,1}(s,\tau) \right| \stackrel{d}{=} \sup_{\tau \in Y} \left| \sum_{i=1}^{n_1(s)} \xi_i^s \tilde{\eta}_{i,1}(s,\tau) \right| = O_p\left(\sqrt{n(s)}\right). \tag{E.12}$$

Combining (E.5), (E.11), and (E.12), we establish (E.4). This concludes the proof. □

LEMMA E.6. *If Assumptions* 1(i) *and* 1(ii) *hold,* $\sup_{s\in\mathcal{S}} \frac{|D_n^*(s)|}{\sqrt{n^*(s)}} = O_p(1)$, $\sup_{s\in\mathcal{S}} \frac{|D_n(s)|}{\sqrt{n(s)}} = O_p(1)$, *and* $n(s)\to\infty$ *for all* $s\in\mathcal{S}$, *a.s., then, uniformly over* $\tau\in\Upsilon$,

$$Q_n^*(u,\tau) \xrightarrow{p} \frac{1}{2}u'Qu.$$

PROOF. Recall $Q_{n,1}^*(u,\tau)$ and $Q_{n,0}^*(u,\tau)$ defined in (D.1). We focus on $Q_{n,1}^*(u,\tau)$. Recall the definition of M_{ni} in the proof of Lemma E.5. We have

$$\begin{aligned}
Q_{n,1}^*(u,\tau) &= \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni} \int_0^{\frac{u_0+u_1}{\sqrt{n}}} \left(1\left\{Y_i^s(1)-q_1(\tau)\le v\right\} - 1\left\{Y_i^s(1)-q_1(\tau)\le 0\right\}\right) dv \\
&= \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni}\left[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\right] \\
&\quad + \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni}\mathbb{E}\phi_i(u,\tau,s),
\end{aligned} \tag{E.13}$$

where $\phi_i(u,\tau,s) = \int_0^{\frac{u_0+u_1}{\sqrt{n}}} (1\{Y_i^s(1)-q_1(\tau)\le v\} - 1\{Y_i^s(1)-q_1(\tau)\le 0\})\, dv$.

Similar to (E.11), we have

$$\begin{aligned}
&\sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni}\left[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\right] \\
&\quad = \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi_i^s\left[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\right] + \sum_{s\in\mathcal{S}} r_n(u,\tau,s),
\end{aligned} \tag{E.14}$$

where $\{\xi_i^s\}_{i=1}^n$ is a sequence of i.i.d. Poisson(1) random variables and is independent of everything else, and

$$\begin{aligned}
&r_n(u,\tau,s) \\
&\quad = \operatorname{sign}\left(\tilde{N}\left(n_1(s)\right) - n_1^*(s)\right) \sum_{j=1}^{\infty} \frac{\# I_n^j(s)}{\sqrt{n(s)}} \frac{1}{\# I_n^j(s)} \sum_{i\in I_n^j(s)} \sqrt{n(s)}\left[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\right].
\end{aligned}$$

We aim to show

$$\sup_{\tau\in\Upsilon, s\in\mathcal{S}} \left|r_n(u,\tau,s)\right| = o_p(1), \tag{E.15}$$

Recall that the proof of Lemma E.5 relies on (E.10) and the fact that

$$\mathbb{E} \sup_{n(s)\ge k\ge n_0} \sup_{\tau\in\Upsilon, s\in\mathcal{S}} \left| \frac{1}{k}\sum_{j=1}^{k} \tilde{\eta}_{j,1}(s,\tau)\right| \to 0.$$

Using the same argument and replacing $\tilde{\eta}_{j,1}(s,\tau)$ by $\sqrt{n(s)}[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)]$, in order to show (E.15), we only need to verify that, as $n \to \infty$ followed by $n_0 \to \infty$,

$$\mathbb{E} \sup_{n(s)\geq k\geq n_0} \sup_{\tau\in Y, s\in\mathcal{S}} \left| \frac{1}{k}\sum_{i=1}^{k} \sqrt{n(s)}\big[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\big]\right| \to 0.$$

Note $\sup_{\tau\in Y, s\in\mathcal{S}} |\frac{1}{k}\sum_{i=1}^{k} \sqrt{n(s)}[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)]|$ is bounded by $|u_0| + |u_1|$. It suffices to show that, for any $\varepsilon > 0$, as $n(s) \to \infty$ followed by $n_0 \to \infty$,

$$\mathbb{P}\left(\sup_{n(s)\geq k\geq n_0} \sup_{\tau\in Y, s\in\mathcal{S}} \left| \frac{1}{k}\sum_{i=1}^{k} \sqrt{n(s)}\big[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\big]\right| \geq \varepsilon\right) \to 0. \tag{E.16}$$

Define the class of functions $\mathcal{F}_n$ as

$$\mathcal{F}_n = \left\{\sqrt{n(s)}\big[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\big] : \tau \in Y, s \in \mathcal{S}\right\}.$$

Then $\mathcal{F}_n$ is nested by a VC-class with fixed VC-index. In addition, for fixed u, $\mathcal{F}_n$ has a bounded (and independent of n) envelope function $F = |u_0| + |u_1|$. Last, define $\mathcal{I}_l = \{2^l, 2^l+1, \ldots, 2^{l+1}-1\}$. Then

$$\begin{aligned}
&\mathbb{P}\left(\sup_{n(s)\geq k\geq n_0} \sup_{\tau\in Y, s\in\mathcal{S}} \left| \frac{1}{k}\sum_{i=1}^{k} \sqrt{n(s)}\big[\phi_i(u,\tau,s,e) - \mathbb{E}\phi_i(u,\tau,s,e)\big]\right| \geq \varepsilon | n(s)\right) \\
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n(s))\rfloor+1} \mathbb{P}\left(\sup_{k\in\mathcal{I}_l} \sup_{\tau\in Y, s\in\mathcal{S}} \left| \frac{1}{k}\sum_{i=1}^{k} \sqrt{n(s)}\big[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\big]\right| \geq \varepsilon | n(s)\right) \\
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n(s))\rfloor+1} \mathbb{P}\left(\sup_{k\leq 2^{l+1}} \sup_{\tau\in Y, s\in\mathcal{S}} \left| \sum_{i=1}^{k} \sqrt{n(s)}\big[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\big]\right| \geq \varepsilon 2^l | n(s)\right) \\
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n(s))\rfloor+1} 9\mathbb{P}\left(\sup_{\tau\in Y, s\in\mathcal{S}} \left| \sum_{i=1}^{2^{l+1}} \sqrt{n(s)}\big[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\big]\right| \geq \varepsilon 2^l/30 | n(s)\right) \\
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n(s))\rfloor+1} \frac{270\mathbb{E}\left(\sup_{\tau\in Y, s\in\mathcal{S}} \left| \sum_{i=1}^{2^{l+1}} \sqrt{n(s)}\big[\phi_i(u,\tau,s) - \mathbb{E}\phi_i(u,\tau,s)\big]\right| | n(s)\right)}{\varepsilon 2^l} \\
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n(s))\rfloor+1} \frac{C_1}{\varepsilon 2^{l/2}} \\
&\leq \frac{2C_1}{\varepsilon\sqrt{n_0}} \to 0,
\end{aligned}$$

where the first inequality holds by the union bound, the second inequality holds because on $\mathcal{I}_l$, $2^{l+1} \geq k \geq 2^l$, the third inequality follows the same argument in the proof of

Theorem 3.1, the fourth inequality is due to the Markov inequality, the fifth inequality follows the standard maximal inequality such as van der Vaart and Wellner (1996, Theorem 2.14.1) and the constant C_1 is independent of (l, ε, n), and the last inequality holds by letting $n \to \infty$. Because ε is arbitrary, we have established (E.16), and thus, (E.15), which further implies that

$$\sup_{\tau \in Y, s \in \mathcal{S}} \left| r_n(u, \tau, s) \right| = o_p(1).$$

In addition, for the leading term of (E.14), we have

$$\begin{aligned} &\sum_{s \in \mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi_i^s \left[\phi_i(u, \tau, s) - \mathbb{E}\phi_i(u, \tau, s) \right] \\ &\quad = \sum_{s \in \mathcal{S}} \left[\Gamma_n^{s*}\left(N(s) + n_1(s), \tau\right) - \Gamma_n^{s*}\left(N(s), \tau\right) \right], \end{aligned}$$

where

$$\begin{aligned} \Gamma_n^{s*}(k, \tau, e) = {} & \sum_{i=1}^{k} \xi_i^s \int_0^{\frac{u_0+u_1}{\sqrt{n}}} \left(1\left\{ Y_i^s(1) \le q_1(\tau) + v \right\} - 1\left\{ Y_i^s(1) \le q_1(\tau) \right\} \right) dv \\ & - k\mathbb{E}\left[\int_0^{\frac{u_0+u_1}{\sqrt{n}}} \left(1\left\{ Y_i^s(1) \le q_1(\tau) + v \right\} - 1\left\{ Y_i^s(1) \le q_1(\tau) \right\} \right) dv \right]. \end{aligned}$$

By the same argument in (A.1), we can show that

$$\sup_{0 < t \le 1, \tau \in Y} \left| \Gamma_n^{s*}(k, \tau, e) \right| = o_p(1),$$

where we need to use the fact that the Poisson(1) random variable has an exponential tail, and thus

$$\mathbb{E} \sup_{i \in \{1, \dots, n\}, s \in \mathcal{S}} \xi_i^s = O\left(\log(n)\right).$$

Therefore,

$$\sup_{\tau \in Y} \left| \sum_{s \in \mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni} \left[\phi_i(u, \tau, s) - \mathbb{E}\phi_i(u, \tau, s) \right] \right| = o_p(1). \tag{E.17}$$

For the second term on the RHS of (E.13), we have

$$\begin{aligned} \sum_{s \in \mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni} \mathbb{E}\phi_i(u, \tau, s) &= \sum_{s \in \mathcal{S}} n_1^*(s) \mathbb{E}\phi_i(u, \tau, s) \\ &= \sum_{s \in \mathcal{S}} \pi p(s) \frac{f_1\left(q_1(\tau)|s\right)}{2} (u_0 + u_1)^2 + o(1) \end{aligned}$$

$$= \frac{\pi f_1\big(q_1(\tau)\big)(u_0+u_1)^2}{2} + o(1), \tag{E.18}$$

where the $o(1)$ term holds uniformly over $\tau \in \Upsilon$, the first equality holds because $\sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni} = n_1^*(s)$ and the second equality holds by the same calculation in (A.1) and the facts that $n^*(s)/n \xrightarrow{p} p(s)$ and

$$\frac{n_1^*(s)}{n} = \frac{D_n^*(s) + \pi n^*(s)}{n} \xrightarrow{p} \pi p(s).$$

Combining (E.13)–(E.15), (E.17), and (E.18), we have

$$Q_{n,1}^*(u,\tau) \xrightarrow{p} \frac{\pi f_1\big(q_1(\tau)\big)(u_0+u_1)^2}{2},$$

uniformly over $\tau \in \Upsilon$. By the same argument, we can show that, uniformly over $\tau \in \Upsilon$,

$$Q_{n,0}^*(u,\tau) \xrightarrow{p} \frac{(1-\pi) f_0\big(q_0(\tau)\big)u_0^2}{2}.$$

This concludes the proof. □

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