Supplement to "Solution methods for models with rare disasters"

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In this Online Appendix, we present the Euler conditions of the model, we develop the pricing Calvo block, we introduce the stationary representation of the model, we define the variables that we include in our simulation, and we develop a simple example of how to implement Taylor projection in comparison with perturbation and projection.

Appendix A: Euler conditions

Define the household's maximization problem as follows:

$$\max_{c_t,k_t^*,x_t,l_t}\left\{U_t^{1-\psi}+\beta\mathbb{E}_t\big(V_{t+1}^{1-\gamma}\big)^{\frac{1-\psi}{1-\gamma}}\right\}$$

$$\text{s.t.}\quad c_t+x_t-w_tl_t-r_tk_t-F_t-T_t=0,$$

$$k_t^*-(1-\delta)k_t-\mu_t\left(1-S\left[\frac{x_t}{x_{t-1}}\right]\right)x_t=0,$$

$$k_{t+1}=k_t^*\exp(-d_{t+1}\theta_{t+1}).$$

The value function V_t depends on the household's actual stock of capital k_t and on past investment x_{t-1}, as well as on aggregate variables and shocks that the household takes as given. Thus, let us use $V_{k,t}$ and $V_{x,t}$ to denote the derivatives of V_t with respect to k_t and x_{t-1} (assuming differentiability). These derivatives are obtained by the envelope theorem:

$$(1-\psi)V_t^{-\psi}V_{k,t}=\lambda_tr_t+Q_t(1-\delta),\tag{27}$$

$$(1-\psi)V_t^{-\psi}V_{x,t-1}=Q_t\mu_tS'\left[\frac{x_t}{x_{t-1}}\right]\left(\frac{x_t}{x_{t-1}}\right)^2,\tag{28}$$

where λ_t and Q_t are the Lagrange multipliers associated with the budget constraint and the evolution law of capital (they enter the Lagrangian in negative sign). We exclude the

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third constraint from the Lagrangian and substitute it directly in the value function or the other constraints, whenever necessary.

Differentiating the Lagrangian with respect to c_t, k_t^*, x_t, and l_t yields the first-order conditions:

$$(1-\psi)U_t^{-\psi}U_{c,t} = \lambda_t, \tag{29}$$

$$(1-\psi)\beta\mathbb{E}_t\big(V_{t+1}^{1-\gamma}\big)^{\frac{\gamma-\psi}{1-\gamma}}\mathbb{E}_t\big(V_{t+1}^{-\gamma}V_{k,t+1}\exp(-d_{t+1}\theta_{t+1})\big) = Q_t, \tag{30}$$

$$\begin{aligned}\lambda_t = Q_t\mu_t\bigg[\bigg(1 - S\bigg[\frac{x_t}{x_{t-1}}\bigg]\bigg) - S'\bigg[\frac{x_t}{x_{t-1}}\bigg]\frac{x_t}{x_{t-1}}\bigg] \\ + (1-\psi)\beta\mathbb{E}_t\big(V_{t+1}^{1-\gamma}\big)^{\frac{\gamma-\psi}{1-\gamma}}\mathbb{E}_t\big(V_{t+1}^{-\gamma}V_{x,t+1}\big),\end{aligned} \tag{31}$$

$$(1-\psi)U_t^{-\psi}U_{l,t} = -\lambda_t w_t. \tag{32}$$

Substituting the envelope conditions (27)–(28) and defining

$$q_t = \frac{Q_t}{\lambda_t}$$

yields equations (6)–(8) in the main text.

Appendix B: The calvo block

The intermediate good producer that is allowed to adjust prices maximizes the discounted value of its profits. Fernández-Villaverde and Rubio-Ramírez (2009, pp. 12–13) derive the first-order conditions of this problem for expected utility preferences, which yield the recursion:

$$\bar{g}_t^1 = \lambda_t \mathrm{mc}_t y_t + \beta\theta_p\mathbb{E}_t\bigg(\frac{\Pi_t^\chi}{\Pi_{t+1}}\bigg)^{-\epsilon}\bar{g}_{t+1}^1,$$

$$\bar{g}_t^2 = \lambda_t\Pi_t^* y_t + \beta\theta_p\mathbb{E}_t\bigg(\frac{\Pi_t^\chi}{\Pi_{t+1}}\bigg)^{1-\epsilon}\bigg(\frac{\Pi_t^*}{\Pi_{t+1}^*}\bigg)\bar{g}_{t+1}^2.$$

To adjust these conditions to Epstein–Zin preferences, divide by λ_t to have

$$\frac{\bar{g}_t^1}{\lambda_t} = \mathrm{mc}_t y_t + \beta\theta_p\mathbb{E}_t\frac{\lambda_{t+1}}{\lambda_t}\bigg(\frac{\Pi_t^\chi}{\Pi_{t+1}}\bigg)^{-\epsilon}\frac{\bar{g}_{t+1}^1}{\lambda_{t+1}}, \tag{33}$$

$$\frac{\bar{g}_t^2}{\lambda_t} = \Pi_t^* y_t + \beta\theta_p\mathbb{E}_t\frac{\lambda_{t+1}}{\lambda_t}\bigg(\frac{\Pi_t^\chi}{\Pi_{t+1}}\bigg)^{1-\epsilon}\bigg(\frac{\Pi_t^*}{\Pi_{t+1}^*}\bigg)\frac{\bar{g}_{t+1}^2}{\lambda_{t+1}}. \tag{34}$$

Note that $\beta\frac{\lambda_{t+1}}{\lambda_t}$ is the stochastic discount factor in expected utility preferences. In Epstein–Zin preferences, the stochastic discount factor is given instead by the expression presented in Section 2.1 in the main text. Substituting and defining $g_t^1 = \frac{\bar{g}_t^1}{\lambda_t}$, $g_t^2 = \frac{\bar{g}_t^2}{\lambda_t}$ yields (11)–(16). The other conditions in the Calvo block follow directly from Fernández-Villaverde and Rubio-Ramírez (2009, pp. 12–13).

Appendix C: The stationary representation of the model

To stationarize the model, we define: $\tilde{c}_t = \frac{c_t}{z_t}$, $\tilde{\lambda}_t = \lambda_t z_t^{\psi}$, $\tilde{r}_t = r_t \mu_t$, $\tilde{q}_t = q_t \mu_t$, $\tilde{x}_t = \frac{x_t}{z_t}$, $\tilde{w}_t = \frac{w_t}{z_t}$, $\tilde{k}_t = \frac{k_t}{z_t \mu_t}$, $\tilde{k}_t^* = \frac{k_t^*}{z_t \mu_t}$, $\tilde{y}_t = \frac{y_t}{z_t}$, $\tilde{U}_t = \frac{U_t}{z_t}$, $\tilde{U}_{l,t} = \frac{U_{l,t}}{z_t}$, $\tilde{V}_t = \frac{V_t}{z_t}$, $\hat{A}_t = \frac{A_t}{A_{t-1}}$, $\hat{\mu}_t = \frac{\mu_t}{\mu_{t-1}}$, $\hat{z}_t = \frac{z_t}{z_{t-1}}$. Other rescaled endogenous variables will be introduced below when we list the model conditions. Last, the detrended utility variables are normalized by their steady-state value to avoid scaling problems.

We define the following exogenous state variables to make them linear in the shocks:

$$d_{t+1} = \mu^d + \left(\epsilon_{d,t+1} - \mu^d\right), \tag{35}$$

$$\log \theta_{t+1} = (1-\rho_\theta) \log \bar{\theta} + \rho_\theta \log \theta_t + \sigma_\theta \epsilon_{\theta,t+1}, \tag{36}$$

$$z_{A,t+1} = \sigma_A \epsilon_{A,t+1}, \tag{37}$$

$$\log \hat{\mu}_{t+1} = \Lambda_\mu + \sigma_\mu \epsilon_{\mu,t+1}, \tag{38}$$

$$m_{t+1} = \sigma_m \epsilon_{m,t+1}, \tag{39}$$

$$\xi_{t+1} = \rho_\xi \xi_t + \sigma_\xi \epsilon_{\xi,t+1}. \tag{40}$$

The disaster state variable, d_t, is determined by the disaster shock $\epsilon_{d,t+1}$, which takes the values 1 or 0. The mean of this shock is μ^d. Since the mean is nonzero, the shock is demeaned in (35). The state variable $\log \theta_t$ is the log disaster size. The state variable $z_{A,t}$ is introduced to capture Gaussian productivity innovations to $\log \hat{A}_t$. The state variable $\log \hat{\mu}_t$ denotes the growth of investment technology. Finally, m_t and ξ_t are the monetary shock and the time preference shock, respectively.

The following variables depend only on the exogenous variables:

$$\log \hat{A}_t = \Lambda_A + z_{A,t} - (1-\alpha) d_t \theta_t,$$

$$\log \hat{z}_t = \frac{1}{1-\alpha} \log \hat{A}_t + \frac{\alpha}{1-\alpha} \log \hat{\mu}_t.$$

The model conditions are given by the following equations:

$$\left(\frac{\tilde{V}_t}{\tilde{V}^{ss}}\right)^{1-\psi} = \left(\frac{\tilde{U}_t}{\tilde{U}^{ss}}\right)^{1-\psi} \left(\frac{\tilde{U}^{ss}}{\tilde{V}^{ss}}\right)^{1-\psi} + \beta \mathbb{E}_t \left(\left(\frac{\tilde{V}_{t+1}}{\tilde{V}^{ss}}\right)^{1-\gamma} \hat{z}_{t+1}^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}}, \tag{41}$$

$$\tilde{U}_t = \tilde{c}_t (1-l_t)^\nu e^{\xi_t}, \tag{42}$$

$$U_{c,t} = (1-l_t)^\nu e^{\xi_t}, \tag{43}$$

$$\tilde{U}_{l,t} = -\nu \tilde{c}_t (1-l_t)^{\nu-1} e^{\xi_t}, \tag{44}$$

$$(1-\psi)(\tilde{U}_t)^{-\psi} \tilde{U}_{l,t} = -\tilde{\lambda}_t \tilde{w}_t, \tag{45}$$

$$(1-\psi)(\tilde{U}_t)^{-\psi} U_{c,t} = \tilde{\lambda}_t, \tag{46}$$

$$M_{t+1} = \beta \frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t} (\hat{z}_{t+1})^{-\psi} \frac{\left(\tilde{V}_{t+1}/\tilde{V}^{ss}\right)^{\psi-\gamma} (\hat{z}_{t+1})^{\psi-\gamma}}{\mathbb{E}_t \left(\left(\tilde{V}_{t+1}/\tilde{V}^{ss}\right)^{1-\gamma} (\hat{z}_{t+1})^{1-\gamma} \right)^{\frac{\psi-\gamma}{1-\gamma}}}, \tag{47}$$

$$\mathbb{E}_t\left(M_{t+1}\exp(-d_{t+1}\theta_{t+1})\frac{1}{\hat{\mu}_{t+1}}\left[\tilde{r}_{t+1}+\tilde{q}_{t+1}(1-\delta)\right]\right)=\tilde{q}_t, \tag{48}$$

$$\begin{aligned}1=\tilde{q}_t&\left[1-S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]-S'\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]\\&+\mathbb{E}_t\left(M_{t+1}\tilde{q}_{t+1}S'\left[\frac{\tilde{x}_{t+1}}{\tilde{x}_t}\hat{z}_{t+1}\right]\left(\frac{\tilde{x}_{t+1}}{\tilde{x}_t}\hat{z}_{t+1}\right)^2\right),\end{aligned} \tag{49}$$

$$\tilde{y}_t=\tilde{c}_t+\tilde{x}_t, \tag{50}$$

$$\tilde{k}_t^*-(1-\delta)\tilde{k}_t-\left(1-S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]\right)\tilde{x}_t=0, \tag{51}$$

$$\tilde{k}_t=\frac{\tilde{k}_{t-1}^*}{\hat{z}_t\hat{\mu}_t}\exp(-d_t\theta_t), \tag{52}$$

$$\tilde{q}_t^e=\mathbb{E}_t\left(M_{t+1}\hat{z}_{t+1}\left(\widetilde{\mathrm{div}}_{t+1}+\tilde{q}_{t+1}^e\right)\right), \tag{53}$$

$$\widetilde{\mathrm{div}}_t=\tilde{y}_t-\tilde{w}_t l_t-\tilde{x}_t, \tag{54}$$

$$q_t^f=\mathbb{E}_t M_{t+1}, \tag{55}$$

$$\tilde{g}_t^1=\mathrm{mc}_t\tilde{y}_t^d+\theta_p\mathbb{E}_t M_{t+1}\left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\epsilon}\tilde{g}_{t+1}^1\hat{z}_{t+1}, \tag{56}$$

$$\tilde{g}_t^2=\Pi_t^*\tilde{y}_t^d+\theta_p\mathbb{E}_t M_{t+1}\left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\epsilon}\left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right)\tilde{g}_{t+1}^2\hat{z}_{t+1}, \tag{57}$$

$$\epsilon\tilde{g}_t^1=(\epsilon-1)\tilde{g}_t^2, \tag{58}$$

$$1=\theta_p\left(\frac{\Pi_{t-1}^\chi}{\Pi_t}\right)^{1-\epsilon}+(1-\theta_p)\left(\Pi_t^*\right)^{1-\epsilon}, \tag{59}$$

$$\mathrm{mc}_t=\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\tilde{w}_t^{1-\alpha}\tilde{r}_t^{\alpha}, \tag{60}$$

$$\frac{\tilde{k}_t}{l_t}=\frac{\alpha}{1-\alpha}\frac{\tilde{w}_t}{\tilde{r}_t}, \tag{61}$$

$$\tilde{y}_t=\frac{\frac{\hat{A}_t}{\hat{z}_t}\left(\tilde{k}_{t-1}^*\exp(-d_t\theta_t)\right)^{\alpha}l_t^{1-\alpha}-\phi}{v_t^p}, \tag{62}$$

$$v_t^p=\theta_p\left(\frac{\Pi_{t-1}^\chi}{\Pi_t}\right)^{-\epsilon}v_{t-1}^p+(1-\theta_p)\left(\Pi_t^*\right)^{-\epsilon}, \tag{63}$$

$$\frac{R_t}{R}=\left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi}\left(\frac{\frac{\tilde{y}_t}{\tilde{y}_{t-1}}\hat{z}_t}{\exp(\Lambda_y)}\right)^{\gamma_y}\right)^{1-\gamma_R}e^{m_t}, \tag{64}$$

$$1=\mathbb{E}_t M_{t+1}\frac{R_t}{\Pi_{t+1}}. \tag{65}$$

We define the state of the economy by the endogenous variables $\log \tilde{k}^*_{t-1}\times^{\mathrm{a}} \log \tilde{x}_{t-1}$, $\log \Pi_{t-1}$, $\log v^p_{t-1}$, $\log \tilde{y}_{t-1}$, and $\log R_{t-1}$, and the exogenous variables d_t, $\log \theta_t$, $z_{A,t}$, $\log \hat{\mu}_t$, m_t, and ξ_t.

In the flexible price versions of the model, we use the following pricing conditions instead of (56)–(60):

$$\tilde{r}_t = \alpha \hat{A}_t \hat{\mu}_t \left(\tilde{k}^*_{t-1} \exp(-d_t \theta_t)\right)^{\alpha-1} l_t^{1-\alpha}, \tag{66}$$

$$\tilde{w}_t = (1-\alpha)\frac{\hat{A}_t}{\hat{z}_t}\left(\tilde{k}^*_{t-1} \exp(-d_t \theta_t)\right)^{\alpha} l_t^{-\alpha}, \tag{67}$$

$$\tilde{y}_t = \frac{\hat{A}_t}{\hat{z}_t}\left(\tilde{k}^*_{t-1} \exp(-d_t \theta_t)\right)^{\alpha} l_t^{1-\alpha} - \phi. \tag{68}$$

Appendix D: Simulation variables

The benchmark version of the model approximates the endogenous control variables: $\log \mathbb{E}_t((\frac{\tilde{V}_{t+1}}{\tilde{V}^{SS}}\hat{z}_{t+1})^{1-\gamma})$, $\log \frac{l_t}{1-l_t}$, $\log \tilde{q}^e_t$, $\log q^f_t$, and $\log \tilde{k}^*_t$. The first variable is an auxiliary variable introduced into the system. The other model variables can be expressed as functions of the approximated variables and the given state variables. We apply a change of variables to ensure that variables are bound within their natural domain. For instance, if $x > 0$, we approximate $\log x$. Similarly, labor l_t must be between 0 and 1 so we approximate $\log \frac{l_t}{1-l_t}$ instead.

The second version with capital adjustment costs approximates, in addition, the variables $\log \tilde{q}_t$ and $\log \tilde{x}^{\text{back}}_{t+1}$, which are both determined in period t. The notation $^{\text{back}}$ denotes the past value of the variable, for example, $\tilde{x}^{\text{back}}_t \equiv \tilde{x}_{t-1}$. This is required when the past value of a control variable (e.g., past investment) is an endogenous state variable.

The third version with Calvo pricing approximates, in addition, the variables: $\log \tilde{w}_t$, $\log \tilde{x}_t$, $\log \tilde{g}^1_t$, $\log \Pi_t + \log \Pi^*_t$, $\log \Pi^{\text{back}}_{t+1}$, and $\log v^{p,\text{back}}_{t+1}$, all determined in period t. We approximate $\log \Pi_t + \log \Pi^*_t$ instead of approximating separately $\log \Pi_t$ and $\log \Pi^*_t$. It can be shown that this transformation ensures that Π_t is always positive, while keeping the number of approximated variables as small as possible.

The fourth version with a Taylor rule that depends on output growth approximates, in addition, $\log \tilde{y}^{\text{back}}_{t+1}$, which is determined in period t.

The fifth version with a smoothed Taylor rule approximates, in addition, the variable $\log R^{\text{back}}_{t+1}$, which is determined in period t.

The other versions add only exogenous variables, so the number of approximated variables does not change.

Appendix E: Perturbation versus Taylor projection: A simple example

As we mention in the main text, in standard perturbation, we find a solution for the variables of interest by perturbing a volatility of the shocks around zero. In comparison, in

Taylor projection (as we would do in a projection), we take account of the true volatility of the shocks.

An example should clarify this point. Imagine that we are dealing with the stochastic neoclassical growth model with fixed labor supply, full depreciation, and no persistence of the productivity shock (these two assumptions allows us to derive simple analytic expressions).

The social planner problem of this model can be written as

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t.} \quad c_t + k_{t+1} = e^{z_t} k_t^{\alpha}, \quad z_t \sim \mathcal{N}(0, \sigma),$$

where $\mathbb{E}_0$ is the conditional expectation operator, β is the discount factor, c_t is consumption, k_t is capital, and z_t is the productivity shock with volatility σ.

To ease the presentation, we will switch now to the recursive notation, where we drop the time subindex and where for an arbitrary variable x, we have that $x' = x_{t+1}$. Thus, consumption can be written in terms of the policy function $c_t = c(k, z)$ and from the resource constraint of the economy $k' = e^z k^{\alpha} - c(k, z)$.

If we substitute $c(k, z)$ and $k' = e^z k^{\alpha} - c(k, z)$ in the Euler equation of the model, we get

$$-\frac{1}{c(k, z)} + \alpha\beta\mathbb{E}_t \frac{e^{z'}\left(e^z k^{\alpha} - c(k, z)\right)^{\alpha-1}}{c\left(e^z k^{\alpha} - c(k, z), z'\right)} = 0.$$

From this Euler equation, we can find the deterministic steady state of the model:

$$k_{ss} = (\alpha\beta)^{\frac{1}{1-\alpha}},$$
$$c_{ss} = k_{ss}^{\alpha} - k_{ss}.$$

In a first-order perturbation, we postulate an approximation for the policy function of the form

$$c(k, z) = \theta_0 + \theta_1(k - k_{ss}) + \theta_2 z,$$

where we are already taking advantage of the certainty equivalence property of first-order approximations to drop the term on σ.

If we plug this policy function into the equilibrium conditions before, we get

$$-\frac{1}{\theta_0 + \theta_1(k - k_{ss}) + \theta_2 z} + \alpha\beta\mathbb{E}_t \frac{e^{z'}\left(e^z k^{\alpha} - \theta_0 - \theta_1(k - k_{ss}) - \theta_2 z\right)^{\alpha-1}}{\theta_0 + \theta_1\left(e^z k^{\alpha} - \theta_0 - \theta_1(k - k_{ss}) - \theta_2 z - k_{ss}\right) + \theta_2 z'} = 0.$$

To find θ_0, we first evaluate the previous expression at the deterministic steady-state value of the state variables ($k = k_{ss}$ and $z = 0$):

$$-\frac{1}{\theta_0} + \alpha\beta\mathbb{E}_t \frac{e^{z'}\left(k_{ss}^{\alpha} - \theta_0\right)^{\alpha-1}}{\theta_0 + \theta_1\left(k_{ss}^{\alpha} - \theta_0 - k_{ss}\right) + \theta_2 z'} = 0,$$

and then take $\sigma \to 0$ to get

$$-\frac{1}{\theta_0} + \alpha\beta \frac{\left(k_{ss}^{\alpha} - \theta_0\right)^{\alpha-1}}{\theta_0 + \theta_1\left(k_{ss}^{\alpha} - \theta_0 - k_{ss}\right)} = 0.$$

This equation has a zero at $\theta_0 = c_{ss} = k_{ss}^{\alpha} - k_{ss}$. This result is natural: the leading constant term of a perturbation around the deterministic steady state of the policy function of an endogenous variable is just the steady-state value of such a variable (in practice, this result is just assumed without solving for it explicitly).

To find θ_1 and θ_2, we take derivatives of the Euler equation with respect to capital and productivity, evaluate them at the deterministic steady state, take $\sigma \to 0$, and solve for the unknown coefficients. The algebra is straightforward, but tedious. Note, however, that the procedure is recursive: we solve first for θ_0, and when this coefficient is known, for θ_1 and θ_2.

In a Taylor projection, up to first order, we also postulate:

$$c = \theta_0 + \theta_1(k - k_{ss}) + \theta_2 z.$$

In this Taylor projection, we will take our approximation around $(k_{ss}, 0)$ to make the comparison with perturbation easier, but other approximation points are possible.

As before, we substitute in the equilibrium condition:

$$\begin{aligned} &-\frac{1}{\theta_0 + \theta_1(k - k_{ss}) + \theta_2 z} \\ &\quad + \alpha\beta\mathbb{E}_t \frac{e^{z'}\left(e^{z}k^{\alpha} - \theta_0 - \theta_1(k - k_{ss}) - \theta_2 z\right)^{\alpha-1}}{\theta_0 + \theta_1\left(e^{z}k^{\alpha} - \theta_0 - \theta_1(k - k_{ss}) - \theta_2 z - k_{ss}\right) + \theta_2 z'} = 0, \end{aligned} \tag{69}$$

and evaluate it at the deterministic steady-state value of the state variables ($k = k_{ss}$ and $z = 0$):

$$-\frac{1}{\theta_0} + \alpha\beta\mathbb{E}_t \frac{e^{z'}\left(k_{ss}^{\alpha} - \theta_0\right)^{\alpha-1}}{\theta_0 + \theta_1\left(k_{ss}^{\alpha} - \theta_0 - k_{ss}\right) + \theta_2 z'} = 0.$$

But now we do not let $\sigma \to 0$. Note, in particular, that this means we still have an expectation operator $\mathbb{E}_t$ and a z'. Furthermore, it also means that we must simultaneously solve for θ_0, θ_1, and θ_2, and not recursively as in perturbation. To do so, we take derivatives of equation (69) with respect to k and z and evaluate them at the deterministic steady state. This operation gives us three equations ((69) and the two derivatives) on three unknowns (θ_0, θ_1, and θ_2) that can be solved with a standard Newton algorithm.

In general, the presence of the expectation operator will imply that the θ_0 from first-order perturbation and the θ_0 from Taylor projection will be different. To see this, we can plug $\theta_0 = c_{ss}$ after equation (69) and verify that

$$-\frac{1}{c_{ss}} + \alpha\beta k_{ss}^{\alpha-1}\mathbb{E}_t \frac{e^{z'}}{c_{ss} + \theta_2 z'} \neq 0$$

unless $\sigma = 0$.

Table 27. Solution for θ's.

Parameter	Taylor Projection	Perturbation
θ_0	0.4872	0.4176
θ_1	0.9326	0.7417
θ_2	0.5252	0.4176

To further illustrate this point, we will implement a simple calibration of the model with $\alpha = 0.3$ and $\beta = 0.96$. Productivity, instead of being a normal distribution as before, is now a two-point process:

$$z = \left[\log(0.4), -\frac{0.1}{0.9} * \log(0.4)\right],$$

$$\text{Prob} = [0.1, 0.9].$$

This calibration assumes 10% probability of a 60% fall in TFP and 90% probability of a 10.7% increase (the mean of z is still zero).

The solutions for the θ's are reported in Table 27. Note the difference between the Taylor projection θ's and the perturbation θ's.

Appendix F: Taylor projection versus projection: A simple example

We can continue our previous example with the stochastic neoclassical growth model with full depreciation, but now comparing Taylor projection with a standard projection.

The first steps of a Taylor projection and a standard projection are the same. In both cases, we postulate a policy function:

$$c = \theta_0 + \theta_1(k - k_{ss}) + \theta_2 z.$$

For this example and to make the comparison with perturbation easier, we center the policy function around k_{ss}, even if other approximation points are possible.

As we did in previous cases, we substitute in the equilibrium condition to get a residual function:

$$\begin{aligned} &R(k, z, \theta_0, \theta_1, \theta_2) \\ &\quad = -\frac{1}{\theta_0 + \theta_1(k - k_{ss}) + \theta_2 z} \\ &\qquad + \alpha\beta\mathbb{E}_t \frac{e^{z'}\left(e^z k^\alpha - \theta_0 - \theta_1(k - k_{ss}) - \theta_2 z\right)^{\alpha-1}}{\theta_0 + \theta_1\left(e^z k^\alpha - \theta_0 - \theta_1(k - k_{ss}) - \theta_2 z - k_{ss}\right) + \theta_2 z'}, \end{aligned} \tag{70}$$

but we do not impose that this residual function is zero. Instead, we express it as an explicit function of k, z, θ_0, θ_1, and θ_2.

In Taylor projection, we find the values of θ_0, θ_1, and θ_2 that solve

$$R(k_{ss}, 0, \theta_0, \theta_1, \theta_2) = 0, \tag{71}$$

$$\left.\frac{\partial R(k, z, \theta_0, \theta_1, \theta_2)}{\partial k}\right|_{k_{ss},0} = 0, \tag{72}$$

$$\left.\frac{\partial R(k, z, \theta_0, \theta_1, \theta_2)}{\partial z}\right|_{k_{ss},0} = 0. \tag{73}$$

In comparison, projection selects three points (k_1, z_1), (k_2, z_2), and (k_3, z_3) (one of these points can be $(k_{ss}, 0)$; there are different choices of how to undertake this selection) and finds the values of θ_0, θ_1, and θ_2 that solve

$$R(k_1, z_1, \theta_0, \theta_1, \theta_2) = 0, \tag{74}$$

$$R(k_2, z_2, \theta_0, \theta_1, \theta_2) = 0, \tag{75}$$

$$R(k_3, z_3, \theta_0, \theta_1, \theta_2) = 0. \tag{76}$$

In both cases, we have three equations (71)–(73) for Taylor projection and (74)–(76) for projection in three unknowns $(\theta_0, \theta_1, \theta_2)$ that come from the residual function $R(k, z, \theta_0, \theta_1, \theta_2)$, but in the former case we deal with the level and two partial derivatives of the function at one point, and in the latter, we deal with the level of the function at three different points.

Reference

Fernández-Villaverde, J. and J. F. Rubio-Ramírez (2009), "A baseline DSGE model." Discussion paper, University of Pennsylvania. [2]

Co-editor Karl Schmedders handled this manuscript.

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