Supplementary Material for "Identification of Average Effects under Magnitude and Sign Restrictions on Confounding"

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A Online Appendix A

Appendix A.1 extends the results in Section 4 on the identification of coefficients in a linear structure. Section A.1.1 allows for exogenous random coefficients and accommodates conditioning on covariates. Section A.1.2 reports empirical estimates that complement those in Section 8.1. Section A.1.3 studies a panel structure. Sections A.1.4 and A.1.5 study proxies that are included in the Y equation. Appendix A.2 characterizes the OVB of nonparametric regression and IV methods when U enters the Y and W equations additively separably. Appendix A.3 characterizes the nonparametric regression bias for $\bar{\beta}(x, x^*|x^*)$ and $\bar{\beta}(x|x)$ in the nonseparable case with discrete U, yielding analogous results to Theorem 5.1 with continuous U.

A.1 Linear Specification: Extensions

Except in Sections A.1.1 and A.1.2, we leave the covariates S implicit in the remainder of Section A.1 in order to ease the exposition.

A.1.1 Exogenous Random Coefficients and Conditioning on the Covariates

Theorem A.1 generalizes Theorem 4.1 to allow for exogenous random coefficients and to accommodate the observed covariates S. In particular, Section A.1 allows the coefficients in S.2 to depend on S and either U_Y or U_W (we sometimes leave the dependence of the coefficients on (S, U_Y) or (S, U_W) implicit to simplify the notation):

$$Y = r(X, S, U, U_Y) = X'\beta(S, U_Y) + U'\delta_Y(S, U_Y) + \alpha_Y(S, U_Y), \quad \text{and}$$
$$W' = q(S, U, U_W)' = U'\delta_W(S, U_W) + \alpha_W'(S, U_W).$$

We also extend the expected value notation from Section 4 and write the deviation of a vector A from its conditional mean as follows:

$$\bar{A}(S) \equiv E(A|S) \quad \text{and} \quad \tilde{A}(S) \equiv A - \bar{A}(S).$$

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Further, for random vectors B and C of equal dimension, we denote the conditional linear IV regression estimand and residual by:

$$R_{A.B|C}(s) \equiv Cov(C, B|S = s)^{-1} Cov(C, A|S = s) \text{ and } \epsilon'_{A.B|C}(s) \equiv \tilde{A}'(s) - \tilde{B}'(s) R_{A.B|C}(s).$$

If $B = C$, we obtain the conditional linear regression estimand $R_{A.B}(s) \equiv R_{A.B|B}(s)$ and residual $\epsilon_{A.B}(s) \equiv \epsilon_{A.B|B}(s)$. Last, for a given $s \in \mathcal{S}$, we rewrite the equations for Y and W to absorb into η_Y and η_W the deviations of the slope coefficients from their conditional means:

$$\begin{aligned} Y &= X'\bar{\beta}(s) + U'\bar{\delta}_Y(s) + \eta_Y \quad &&\text{with} \quad \eta_Y \equiv \alpha_Y + X'\tilde{\beta}(s) + U'\tilde{\delta}_Y(s), \text{ and} \\ W' &= U'\bar{\delta}_W(s) + \eta'_W &&\text{with} \quad \eta'_W \equiv \alpha'_W + U'\tilde{\delta}_W(s). \end{aligned}$$

Theorem A.1 extends Theorem 4.1 to accommodate random, albeit exogenous, coefficients and the covariates S.

Theorem A.1 *Assume S.2 (allowing the coefficients to depend on either* (S, U_Y) *or* (S, U_W)*) with* $\ell = k$ *and* $m = l$*. Let* $s \in \mathcal{S}$ *with* $Cov[Z, (Y, W')'|S = s] < \infty$*.*
(i) If (i.a) $Cov(Z, X|S = s)$ *is nonsingular and (i.b)* $Cov(\eta_Y, Z|S = s) = 0$ *then*

$$B(s) \equiv R_{Y.X|Z}(s) - \bar{\beta}(s) = R_{U.X|Z}(s)\bar{\delta}_Y(s).$$

(ii) If, in addition to (i.a), (ii.a) $\bar{\delta}_W(s)$ *is nonsingular with* $\bar{\delta}(s) \equiv \bar{\delta}_W^{-1}(s)\bar{\delta}_Y(s)$ *and (ii.b)* $Cov(\eta_W, Z|S = s) = 0$ *then*

$$B(s) = R_{W.X|Z}(s)\bar{\delta}(s).$$

Theorem A.1 derives the conditional (IV) regression OVB $B(s)$ and shows how it depends on $R_{U.X|Z}(s)$ and $\bar{\delta}_Y(s)$. Conditions $(i.b)$ and $(ii.b)$ of Theorem A.1 allow the slope coefficients to be random but restrict the coefficient heterogeneity to be exogenous such that the representation from Theorem 4.1 with constant coefficients holds (at the coefficients' conditional averages). In particular, it suffices for conditions $(i.b)$ and $(ii.b)$ that[2] $Cov[(\alpha_Y, \alpha'_W)', Z] = 0$, $E(\tilde{\beta}|Z, X) = 0$, and $E[(\tilde{\delta}'_Y, \tilde{\delta}'_W)'|Z, U] = 0$ so that the random effects[3] β and $(\delta'_Y, \delta'_W)'$ are mean independent of (Z, X) and (Z, U) respectively. In this sense, $(\alpha_Y, \beta, \delta_Y)$ and (α_W, δ_W) are exogenous ("uncorrelated") random coefficients[4]. The endogeneity or "essential heterogeneity" is

[2]If, in addition to $E(\tilde{\beta}|Z, X) = 0$ and $E[(\tilde{\delta}'_Y, \tilde{\delta}'_W)'|Z, U] = 0$, one strengthens $Cov[(\alpha_Y, \alpha'_W)', Z] = 0$ to require $E[(\tilde{\alpha}_Y, \tilde{\alpha}'_W)'|Z] = 0$ then $E[(\tilde{\eta}_Y, \tilde{\eta}'_W)|Z) = 0$. In this case, $E(\tilde{Y} - \tilde{X}'\bar{\beta} - \tilde{W}'\bar{\delta}|Z) = 0$ and it may be possible to generate a sufficient number of instruments $f(Z)$ to point identify $\bar{\beta}$ via the alternative estimand $R_{Y.(X',W')'|f(Z)}$. Nevertheless, $E[(\tilde{\eta}_Y, \tilde{\eta}'_W)|Z] = 0$ is stronger than necessary to characterize the OVB of $R_{Y.X|Z}$ in Theorem A.1. Instead, the weaker condition $Cov(\eta_Y, Z) = 0$ suffices.

[3]If $X = \tilde{p}(Z, U_X)$ then $E(\tilde{\beta}|Z, U_X) = 0$ implies $E(\tilde{\beta}|Z, X) = 0$. For example, this assumes that the average return to education does not depend on the distance to school Z and the unobserved skill U_X.

[4]In the linear correlated random coefficient model, if valid instruments are available then one can also consider IV methods, e.g. Wooldridge (1997, 2003) and Heckman and Vytlacil (1998).

due to U. When a proxy is available, Theorem A.1 characterizes $\bar{\beta}(s)$ by

$$\bar{\beta}(s) = R_{Y.X|Z}(s) - R_{W.X|Z}(s)\bar{\delta}(s).$$

Analogously to the results in Section 4, magnitude and sign restrictions on $\bar{\delta}(s)$ can then be used to either point or partially identify $\bar{\beta}(s)$.

We comment on certain special cases that arise when allowing for covariates in S.2. If $\bar{\delta}_W(S)$, $\bar{\beta}(S)$, and $\bar{\delta}_Y(S)$ are constant then variation in the covariates S may point identifying $(\bar{\beta}', \bar{\delta}')'$. In particular, when Theorem A.1 holds for all $s \in \mathcal{S}$, we obtain

$$Cov(Z, Y|S) = Cov(Z, X|S)\bar{\beta} + Cov(Z, W|S)\bar{\delta},$$

and $(\bar{\beta}', \bar{\delta}')'$ is point identified if variation in S generates a system of at least $k + m$ linearly independent equations[5]. Even if this fails, applying the law of iterated expectations gives

$$\bar{\beta} = E(\tilde{Z}(S)\tilde{X}'(S))^{-1}E(\tilde{Z}(S)\tilde{Y}(S)) - E(\tilde{Z}(S)\tilde{X}'(S))^{-1}E(\tilde{Z}(S)\tilde{W}(S))\bar{\delta}.$$

In addition to $\bar{\delta}$, this expression involves two IV regression estimands. Moreover, if $\bar{Z}(S)$ and/or the conditional expectations $(\bar{X}(S)', \bar{W}(S)', \bar{Y}(S))'$ are affine in S, we obtain

$$\bar{\beta} = E(\epsilon_{Z.S}\epsilon'_{X.S})^{-1}E(\epsilon_{Z.S}\epsilon'_{Y.S}) - E(\epsilon_{Z.S}\epsilon'_{X.S})^{-1}E(\epsilon_{Z.S}\epsilon'_{W.S})\bar{\delta}.$$

Using partitioned regressions (Frisch and Waugh, 1933), the two residual-based IV estimands in this expression can be recovered as the coefficients associated with X in the linear IV regression estimands $R_{Y.(X',S')'|(Z',S')'}$ and $R_{W.(X',S')'|(Z',S')'}$.

A.1.2 Linear Return to Education: Results under Some Alternative Assumptions

We briefly report some empirical results on the return to education and the black white wage gap that complement those obtained in Section 8.1 under restrictions on confounding. Here, we maintain the linear return to education specification in Table 1. First, we consider the bounds that result when assuming that W measures U with classical measurement error[6] (see e.g. Klepper and Leamer (1984) and Bollinger (2003)). In this case, the perfect proxy estimate for $\frac{\bar{\delta}_Y}{\bar{\delta}_W}$ in Table 1 provides the lower bound on $\frac{\bar{\delta}_Y}{\bar{\delta}_W}$ but the estimate for the upper bound on $\frac{\bar{\delta}_Y}{\bar{\delta}_W}$ is very large, admitting values of d that correspond to unlikely $\bar{\beta}$ values (e.g. large negative return to education and a large wage gap in favor of blacks). Second, as discussed in Section A.1.1, we also study identifying $(\bar{\gamma}', \frac{\bar{\delta}_Y}{\bar{\delta}_W})'$ (and thus $\bar{\beta}$) when W is an imperfect proxy by using covariate-conditioned IV regressions of Y on $(1, G'_X, W)'$ using functions of $(G'_X, S')'$ as instruments. In

[5] In particular, we have $E[\tilde{Z}(S)(Y - X'\bar{\beta} - W'\bar{\delta})|S] = 0$ and $(\bar{\beta}', \bar{\delta}')'$ may be identified if interacting $\tilde{Z}(S)$ with functions of S generates sufficiently many instruments.

[6] This hold if $Cov(U_Y, (G', U)') = 0$ and $Cov(U_W, (G', U, U'_Y)') = 0$ in equations (13).

particular, using the W equation, we substitute for U in the Y equation. Then we consider excluding parental education variables (e.g. mother or father education) from G_S and using these as instruments for W. However, this yields unstable (e.g. varying depending on whether the mother's or father's education is used as an instrument) and imprecise estimates. Also, we estimate $(\bar{\gamma}', \frac{\bar{\delta}_Y}{\bar{\delta}_W})'$ via a two stage least squares regression of Y on $(1, G_X', W, G_S)'$ where we restrict G_S to S_1 and use the interaction[7] of G_X and S_1 as instruments for W (or for (G_X, W) instead). This estimates $\frac{\bar{\delta}_Y}{\bar{\delta}_W}$ to be 0.54 with $CI_{0.95}$ $[0.18, 0.90]$, the return to education to be 3.4% with $CI_{0.95}$ $[0.6\%, 6.1\%]$, and the black-white wage gap to be -6.6% with $CI_{0.95}$ $[-15.6\%, 2.3\%]$. Qualitatively similar results obtain when we condition on the fuller set of covariates G_S and/or augment the vector of instruments with the product of G_X and an indicator for low parental education[8].

A.1.3 Panel with Individual and Time-Varying Random Coefficients

We consider a panel structure whereby we index the observed variables M_t and the exogenous random coefficients in S.2 by t. It suffices to consider two time periods $t = 1, 2$. Here, U may denote time-invariant unobserved individual characteristics. We allow the proxy W_t for U to be an element $\underset{k_1 \times 1}{X_{1t}}$ of X_t. Thus, for $t = 1, 2$:

$$\begin{aligned} Y_t &= X_t'\beta_t + U'\delta_{Y_t} + \alpha_{Y_t} = X_t'\bar{\beta}_t + U'\bar{\delta}_{Y_t} + \eta_{Y_t}, \text{ and} \\ X_{1t}' &= U'\delta_{X_{1t}} + \alpha_{X_{1t}}' = U'\bar{\delta}_{X_{1t}} + \eta_{X_{1t}}', \end{aligned}$$

where $\eta_{Y_t} \equiv \alpha_{Y_t} + X_t'\tilde{\beta}_t + U'\tilde{\delta}_{Y_t}$ and $\eta_{X_{1t}}' \equiv \alpha_{X_{1t}}' + U'\tilde{\delta}_{X_{1t}}$. This is a panel structure with individual (we omit the index i for succinctness) and time-varying random coefficients. Note that we do not require "fixed effects" and thus δ_{Y_t} need not equal $\delta_{Y_{t'}}$.

For $t, t' = 1, 2$, $t \neq t'$, we apply Theorem A.1, using $X_{1t'}$ as proxy, to derive an expression for $\bar{\beta}_t$. In this case, the conditions in Theorem A.1 require that $(i.a)$ $Cov(Z_t, X_t)$ is nonsingular, $(i.b)$ $Cov(\eta_{Y_t}, Z_t) = 0$, $(ii.a)$ $\bar{\delta}_{X_{1t'}}$ is nonsingular and $(ii.b)$ $Cov(\eta_{X_{1t'}}, Z_t) = 0$. For example, it suffices for condition $(ii.b)$ that $Cov(\alpha_{X_{1t'}}, Z_t) = 0$ and $E(\delta_{X_{1t'}}|U, Z_t) = E(\delta_{X_{1t'}})$. Then, with $\bar{\delta}_t \equiv \bar{\delta}_{X_{1t'}}^{-1}\bar{\delta}_{Y_t}$, Theorem A.1 gives that for $t, t' = 1, 2$, $t \neq t'$, the IV regression bias is

$$B_t \equiv R_{Y_t.X_t|Z_t} - \bar{\beta}_t = R_{U.X_t|Z_t}\bar{\delta}_{Y_t} = R_{X_{1t'}.X_t|Z_t}\bar{\delta}_t.$$

Here, $\bar{\beta}_t$ is point identified (1) under exogeneity with $\bar{\beta}_t = R_{Y_t.X_t|Z_t}$, which holds if $Cov(U, Z_t) = 0$, and thus $R_{X_{1t'}.X_t|Z_t} = 0$, or $\bar{\delta}_t = 0$, (2) when $X_{1t'}$ is a perfect proxy with degenerate $(\alpha_{X_{1t'}}, \delta_{X_{1t'}})$ and $Cov(\eta_{Y_t}, (Z_t', U')') = 0$, so that $(\bar{\beta}_t', \bar{\delta}_t)' = R_{Y_t.(X_t', X_{1t'}')'|(Z_t', X_{1t'}')'}$, or (3) under

[7]The product of G_X with each of the indicators $1\{S_1 = s_1\}$ for $s_1 = (0, 1), (1, 0)$, and $(1, 1)$

[8]This indicator takes the value 1 if neither parent has 12 or more years of education and is 0 otherwise.

proportional confounding when $\bar{\delta}_t = d_t$ is known. Alternatively, weaker restrictions on $\bar{\delta}_t$ partially identify $\bar{\beta}_{jt}$, as in Corollary 4.3, in the region $\mathcal{B}_{jt}(\times_{h=1}^{k_1}\mathcal{D}_{ht})$ for $j = 1, ..., k$ and $t = 1, 2$.

A.1.4 "Under-Identification" Using Valid Instruments

It is possible that a proxy directly impacts the response Y. In this case, $W = X_1$, a component of X. While Theorem A.1 does not rule out that W and X have common elements, conditions $(i.a)$ and $(ii.b)$ imply that Z must be endogenous (i.e. correlated with U) in this case since $Cov(Z, X)$ is singular otherwise. Nevertheless, a vector Z_1 of one or a few valid instruments may be available and the dimension of X may exceed that of Z_1. A researcher may wish to employ the exogenous instruments Z_1. The next Theorem studies this possibility and provides an expression for $\bar{\beta}$ which depends on the average direct effect of U on Y and the average effect of U on X_1. For this, we write

$$X_1' = U'\delta_{X_1} + \alpha_{X_1}' = U'\bar{\delta}_{X_1} + \eta_{X_1}' \quad \text{with} \quad \eta_{X_1}' \equiv \alpha_{X_1}' + U'\tilde{\delta}_{X_1}.$$

Theorem A.2 *Assume S.2 (allowing the coefficients to depend on U_Y or U_W) with* $\underset{\ell\times 1}{Z} \equiv (\underset{\ell_1\times 1}{Z_1}', \underset{\ell_2\times 1}{Z_2}')'$, $\underset{k\times 1}{X} \equiv (\underset{k_1\times 1}{X_1}', \underset{k_2\times 1}{X_2}')'$, $W = X_1$, *with* $\ell_1, \ell_2 \geq 0$, $\ell = k$, $k_1 = l$.
(i) If (i.a) $Cov(Z, X)$ *is nonsingular, (i.b)* $Cov(\eta_Y, Z) = 0$, *and (i.c)* $Cov(U, Z_1) = 0$ *then*

$$B \equiv R_{Y.X|Z} - \bar{\beta} = Cov(Z, X)^{-1} \begin{bmatrix} 0 \\ Cov(Z_2, U) \end{bmatrix} \bar{\delta}_Y.$$

(ii) If (ii.a) $\bar{\delta}_{X_1}$ *is nonsingular with* $\bar{\delta} \equiv \bar{\delta}_{X_1}^{-1}\bar{\delta}_Y$ *and (ii.b)* $Cov(\eta_{X_1}, Z_2) = 0$ *then*

$$B = Cov(Z, X)^{-1} \begin{bmatrix} 0 \\ Cov(Z_2, X_1) \end{bmatrix} \bar{\delta}.$$

The conditions in Theorem A.2 are analogous to those in Theorem A.1, except that they assume that Z_1 is uncorrelated with U and let Z_1 freely depend on η_{X_1}. Thus, if $Z = Z_2$, Theorem A.2 reduces to Theorem A.1 with $W = X_1$ and no covariates. Instead, if $Z = Z_1$ then exogeneity holds. Here, (1) $\bar{\beta} = R_{Y.X|Z}$ under exogeneity which holds if $Cov(U, Z_2) = 0$, and thus $Cov(Z_2, X_1) = 0$, or $\bar{\delta} = 0$, (2) $(\bar{\beta}^{(1)\prime} + \bar{\delta}', \bar{\beta}^{(2)\prime})' = R_{Y.X|Z}$ when[9] X_1 is a perfect proxy with degenerate $(\alpha_{X_1}, \delta_{X_1})$, and (3) $\bar{\beta}$ is point identified under proportional confounding ($\bar{\delta} = d$). Otherwise, $\bar{\beta}_j$, $j = 1, ..., k$, is partially identified in the region $\mathcal{B}_j(\times_{h=1}^{k_1}\mathcal{D}_h)$ under assumptions on how the average direct effect of U on Y compares in magnitude and/or sign to the average effect of U on X_1.

[9]We partition $\bar{\beta} = (\bar{\beta}^{(1)\prime}, \bar{\beta}^{(2)\prime})'$ corresponding to $X = (X_1', X_2')'$.

A.1.5 Multiple Included Proxies

When $W = X_1$, condition $(ii.b)$ of Theorem 4.1 generally rules out that X_1 is a component of Z. We relax this requirement and let $W = (X_1', X_2')'$ with two proxy vectors X_1 and X_2 that are included in the Y equation and where X_1, and possibly X_2, is a component of Z. This allows for $Z = X$. Theorem A.3 derives an expression for $\bar{\beta}$ which depends on the unknowns $\bar{\delta}_{X_1}^{-1}\bar{\delta}_Y$ and $\bar{\delta}_{X_2}^{-1}\bar{\delta}_Y$ involving the average direct effect of U on Y and the average effects of U on X_1 and X_2. Here, we let $Z_1 = X_1$ and

$$X_g' = U'\delta_{X_g} + \alpha_{X_g}' = U'\bar{\delta}_{X_g} + \eta_{X_g}' \quad \text{where} \quad \eta_{X_g}' \equiv \alpha_{X_g}' + U'\tilde{\delta}_{X_g} \qquad \text{for } g = 1, 2.$$

Theorem A.3 *Assume S.2 (allowing the coefficients to depend on U_Y or U_W) and let* $W = (\underset{k_1\times 1}{X_1}{}', \underset{k_2\times 1}{X_2}{}')'$, $\underset{k\times 1}{X} = (W', \underset{k_3\times 1}{X_3}{}')'$, $Z_1 = X_1$, $\underset{\ell\times 1}{Z} \equiv (\underset{\ell_1\times 1}{Z_1}{}', \underset{\ell_2\times 1}{Z_2}{}')'$, $k_1 = k_2 = l$, $k_3 \geq 0$, $\ell = k$.
(i) If (i.a) $Cov(Z, X)$ is nonsingular and (i.b) $Cov(\eta_Y, Z) = 0$ then

$$B \equiv R_{Y.X|Z} - \bar{\beta} = R_{U.X|Z}\bar{\delta}_Y.$$

(ii) If (ii.a) $\bar{\delta}_{X_g}$ is nonsingular with $\bar{\delta}_g \equiv \bar{\delta}_{X_g}^{-1}\bar{\delta}_Y$, $g = 1, 2$, (ii.b) $Cov[\eta_{X_1}, (U', Z_2', X_2')'] = 0$, and (ii.c) $Cov(\eta_{X_2}, U) = 0$ then

$$B = Cov(Z, X)^{-1}\begin{bmatrix} Cov(Z_1, X_2)\bar{\delta}_2 \\ Cov(Z_2, X_1)\bar{\delta}_1 \end{bmatrix}.$$

The conditions in Theorem A.3 extend those in Theorem A.1 to allow the included proxies X_1 and X_2 to be components of Z but they restrict the dependence between X_2 and η_{X_1} as well as the dependence between U and (η_{X_1}, η_{X_2}). Here too, U may depend on X and Z.

We use the expression for $\bar{\beta}$ in Theorem A.3 to point or partially identify the elements of $\bar{\beta}$.

Corollary A.4 *Assume the conditions of Theorem A.3. (i) If $B_j = 0$ (exogeneity) then $\bar{\beta}_j = R_{Y.X|Z,j}$ for $j = 1, ..., k$. (ii) If X_1 (or symmetrically X_2) is a perfect proxy with degenerate $(\alpha_{X_1}, \delta_{X_1})$ then*[10] $R_{Y.X|Z} = (\bar{\beta}^{(1)\prime} + \bar{\delta}', \bar{\beta}^{(2)\prime}, \bar{\beta}^{(3)\prime})'$ *(iii) If $\bar{\delta}_1 = c_1$ and $\bar{\delta}_2 = c_2$ (proportional confounding) then*

$$B = R_{Y.X|Z} - \bar{\beta} = Cov(Z, X)^{-1}\begin{bmatrix} Cov(Z_1, X_2)c_2 \\ Cov(Z_2, X_1)c_1 \end{bmatrix}.$$

In particular, let $d \equiv (c_1', c_2')'$, $\bar{\delta} \equiv (\bar{\delta}_1', \bar{\delta}_2')'$, $X_{2,3} \equiv (X_2', X_3')'$, $\underset{k_1\times k_1}{P_1} \equiv Cov(\epsilon_{Z_1.Z_2|X_{2,3}}, X_1)$, $\underset{(k_2+k_3)\times(k_2+k_3)}{P_2} \equiv Cov(\epsilon_{Z_2.Z_1|X_1}, X_{2,3})$, *and*

$$\underset{k\times(k_1+k_2)}{A} \equiv \begin{bmatrix} -R_{X_{2,3}.X_1|Z_1}P_2^{-1}Cov(Z_2, X_1), & P_1^{-1}Cov(Z_1, X_2) \\ P_2^{-1}Cov(Z_2, X_1), & -R_{X_1.X_{2,3}|Z_2}P_1^{-1}Cov(Z_1, X_2) \end{bmatrix}.$$

[10] We partition $\bar{\beta} = (\bar{\beta}^{(1)\prime}, \bar{\beta}^{(2)\prime}, \bar{\beta}^{(3)\prime})'$ corresponding to $X = (X_1', X_2', X_3')'$.

Then

$$B = A\bar{\delta} \quad \text{and} \quad \bar{\beta}_j = R_{Y.X|Z,j} - \textstyle\sum_{h=1}^{2l} A_{jh} d_h \quad \text{for } j = 1, ..., k.$$

Thus, $\bar{\beta}_j$ is point identified (1) under exogeneity which holds if $Cov(U, Z) = 0$, and thus $Cov(Z_2, X_1) = 0$, or $\bar{\delta} = 0$, (2) if X_1 (or symmetrically X_2) is a perfect proxy and $\bar{\beta}_j$ is an element of $(\bar{\beta}^{(2)\prime}, \bar{\beta}^{(3)\prime})'$, or (3) under proportional confounding ($\bar{\delta} = d$). Otherwise, $\bar{\beta}_j$, $j = 1, ..., k$, is partially identified in the region $\mathcal{B}_j(\times_{h=1}^{2l} \mathcal{D}_h)$, defined analogously to Corollary 4.3, under assumptions on how the average direct effect of U on Y compares in magnitude and sign to the average effects of U on X_1 and X_2.

A.2 Additively Separable Confounders

This section studies the special case in which U enters r and q additively separably.

Assumption 4 (S.4) *Additive Separability: Assume S.1 with*

$$Y = r(X, U, U_Y) = \ddot{r}(X, U_Y) + U'\delta_Y(U_Y), \text{ and}$$
$$W' = q(U, U_W)' = U'\delta_W(U_W) + \alpha_W'(U_W).$$

The exogenous random coefficient specification from Section A.1 is a special case of S.4 (for simplicity, S.4 leaves the covariates S implicit) that further assumes that the effect of X on Y is linear.

A.2.1 Average Nonparametric Effects

Under S.4 (separability) and condition (6a), the law of iterated expectations gives:

$$\bar{\beta}(x, x^*|x^*) = E[\ddot{r}(x^*, U_Y) - \ddot{r}(x, U_Y)] = \bar{\beta}(x, x^*).$$

Similarly, under S.4 and condition (6b), we obtain:

$$\bar{\beta}(x|x) = E[\frac{\partial}{\partial x}\ddot{r}(x, U_Y)] = \bar{\beta}(x).$$

Theorem A.5 characterizes the OVB of the nonparametric regression estimands $R^N_{Y.X}(x, x^*)$ or $R^N_{Y.X}(x)$ for $\bar{\beta}(x, x^*)$ or $\bar{\beta}(x)$ under S.4.

Theorem A.5 *Assume S.4 with $m = l$ and let $x, x^* \in \mathcal{X}$.*
(i.a) If conditions B.1(i.a) and (6a) hold then

$$B(x, x^*) \equiv R^N_{Y.X}(x, x^*) - \bar{\beta}(x, x^*) = R^N_{U.X}(x, x^*)\bar{\delta}_Y.$$

(i.b) If (i.b.1) $\bar{\delta}_W$ is nonsingular with $\bar{\delta} \equiv \bar{\delta}_W^{-1}\bar{\delta}_Y$ and (i.b.2) conditions B.1(i.e) and (7) hold then

$$B(x, x^*) = R^N_{W.X}(x, x^*)\bar{\delta}.$$

(ii.a) Set $k = 1$ and assume the conditions in (i.a). If (ii.a.1) $\frac{\partial}{\partial x}E(U'|X = x)$ exists and is finite and (ii.a.2) for all $x^\dagger \in \mathcal{N}(x) \subseteq \mathcal{X}$, a nonempty open neighborhood of x, $E[\ddot{r}(x^\dagger, U_Y)] < \infty$, $\frac{\partial}{\partial x}\ddot{r}(x^\dagger, u_y)$ exists for a.e. u_y, and there is a function $\Delta(U_Y)$ with $E[\Delta(U_Y)] < \infty$ such that $\left|\frac{\partial}{\partial x}\ddot{r}(x^\dagger, u_y)\right| < \Delta(u_y)$ for a.e. u_y then

$$B(x) \equiv R^N_{Y.X}(x) - \bar{\beta}(x) = R^N_{U.X}(x)\bar{\delta}_Y.$$

(ii.b) If the conditions in (i.b) and (ii.a.1) hold then

$$B(x) = R^N_{W.X}(x)\bar{\delta}.$$

The conditions in $(ii.a)$ ensure that the moments and derivatives exist and that $\frac{\partial}{\partial x}E[\ddot{r}(x, U_Y)] = E[\frac{\partial}{\partial x}\ddot{r}(x, U_Y)]$. The expressions in $(i.a)$ and $(ii.a)$ for the biases $B(x, x^*)$ and $B(x)$ show how the OVB depends on $R^N_{U.X}(x, x^*)$ and $\bar{\delta}_Y$. As in the linear case, under conditional exogeneity $\bar{\beta}(x, x^*) = R^N_{Y.X}(x, x^*)$ is point identified. This obtains if $E(U|X) = E(U)$, in which case $R^N_{W.X}(x, x^*) = 0$, or $\bar{\delta}_Y = 0$. Alternatively, if W is a perfect proxy with (δ_W, α_W) degenerate then, provided $U_Y \perp (U, X)$, we have that $\bar{\beta}(x, x^*)$ and $\bar{\delta}$ are point identified by:

$$E(Y|X = x^*, W = w) - E(Y|X = x, W = w) = \bar{\beta}(x, x^*) \quad \text{and}$$
$$\frac{1}{(w^* - w)}E(Y|X = x, W = w^*) - E(Y|X = x, W = w) = \bar{\delta}.$$

When W is an imperfect proxy, Theorem A.5 shows that $\bar{\beta}(x, x^*)$ and $\bar{\beta}(x)$ depend on the average direct effect of U on Y and the average effect of U on W via $\bar{\delta} \equiv \bar{\delta}_W^{-1}\bar{\delta}_Y$:

$$\bar{\beta}(x, x^*) = R^N_{Y.X}(x, x^*) - R^N_{W.X}(x, x^*)\bar{\delta} \quad \text{and} \quad \bar{\beta}(x) = R^N_{Y.X}(x) - R^N_{W.X}(x)\bar{\delta}.$$

In this case, $\bar{\beta}(x, x^*)$ is point identified under proportional confounding, when $\bar{\delta} = d$ is known. Alternatively, restrictions $\bar{\delta}_h \in \mathcal{D}_h \equiv [d_{L,h}, d_{H,h}]$ on the magnitude and/or sign of confounding can partially identify $\bar{\beta}(x, x^*)$ in the region $\mathcal{B}(\times_{h=1}^m \mathcal{D}_h)$ given by:

$$\bar{\beta}(x, x^*) \in \mathcal{B}(\times_{h=1}^m \mathcal{D}_h) \equiv \{R^N_{Y.X}(x, x^*) - R^N_{W.X}(x, x^*)d : d_h \in \mathcal{D}_h, h = 1, ..., m\}.$$

These restrictions are weaker than requiring $\bar{\delta} = 0$ (which ensures exogeneity when $R^N_{W.X}(x) \neq 0$ and U depends on X), a perfect proxy estimate for $\bar{\delta}$, or proportional confounding ($\bar{\delta} = d$). Arguments similar to those in the proofs of corollaries 4.3 and 5.2 show that the region $\mathcal{B}(\times_{h=1}^m \mathcal{D}_h)$, defined above, is sharp under the assumptions in Theorem A.5 and the restrictions $\mathcal{D}_h$, $h = 1, ..., m$, on confounding. Analogous results hold for $\bar{\beta}(x)$.

Last, Theorem A.5 shows that imposing m restrictions on $\bar{\beta}(\cdot,\cdot)$ or $\bar{\beta}(\cdot)$ provides another avenue to point identify $\bar{\beta}(x,x^*)$ or $\bar{\beta}(x)$. For example, if $m = l = 1$ and one assumes that $\bar{\beta}(x^\dagger, x^\ddagger) = 0$ for $x^\dagger, x^\ddagger \in \mathcal{X}$, as occurs if a nondegenerate component of X is excluded from r and thus the Y equation, then, provided $R^N_{W.X}(x^\dagger, x^\ddagger) \neq 0$, $\bar{\delta} = \frac{R^N_{Y.X}(x^\dagger,x^\ddagger)}{R^N_{W.X}(x^\dagger,x^\ddagger)}$ and $\bar{\beta}(x,x^*)$ is therefore point identified. Analogous restrictions can point identify $\bar{\beta}(x)$.

A.2.2 Local and Marginal Treatment Effects

Suppose, in addition to S.4, that X is generated as in S.3 so that:

$$Y = \ddot{r}(X, U_Y) + U'\delta_Y(U_Y), \quad X = \mathbf{1}\{U_X \leq \nu(Z)\}, \text{ and } W' = U'\delta_W(U_W) + \alpha'_W(U_W).$$

Given separability in S.4 and condition (10), the law of iterated expectations gives:

$$\bar{\beta}(\nu(z) < U_X \leq \nu(z^*), z^*) = E[\ddot{r}(1,U_Y) - \ddot{r}(0,U_Y)|\nu(z) < U_X \leq \nu(z^*)] = \bar{\beta}(\nu(z) < U_X \leq \nu(z^*))$$

and

$$\bar{\beta}(\nu(z), z) = E[\ddot{r}(1,U_Y) - \ddot{r}(0,U_Y)|U_X = \nu(z)] = \bar{\beta}(\nu(z)).$$

Theorem A.6 characterizes the OVB of the Wald or LIV estimand for LATE or MTE.

Theorem A.6 *Assume S.3 and S.4 with $m = l$. Let $z, z^* \in \mathcal{Z}$ with* $\Pr[\nu(z) < U_X \leq \nu(z^*)] > 0$. *(i.a) If conditions B.2(i.a) and (9,10) hold then*

$$B(\nu(z) < U_X \leq \nu(z^*)) \equiv R^{Wald}_{Y.X|Z}(z,z^*) - \bar{\beta}(\nu(z) < U_X \leq \nu(z^*)) = R^{Wald}_{U.X|Z}(z,z^*)\bar{\delta}_Y.$$

(i.b) If (i.b.1) $\bar{\delta}_W$ is nonsingular with $\bar{\delta} \equiv \bar{\delta}_W^{-1}\bar{\delta}_Y$ and (i.b.2) conditions B.2(i.e) and (11) hold then

$$B(\nu(z) < U_X \leq \nu(z^*)) = R^{Wald}_{W.X|Z}(z,z^*)\bar{\delta}.$$

(ii.a) Set $\ell = 1$ and assume the conditions in (i.a). If (ii.a.1) $\frac{\partial}{\partial z}E(U'|Z=z)$ exists and (ii.a.2) $\nu(\cdot)$ is differentiable at z with $\frac{\partial}{\partial z}\nu(z) \neq 0$ and $\bar{\beta}(\cdot)$ and $f_{U_X}(\cdot)$ are continuous at $\nu(z)$ with $f_{U_X}(\nu(z)) > 0$ then

$$B(\nu(z)) \equiv R^{LIV}_{Y.X|Z}(z) - \bar{\beta}(\nu(z)) = R^{LIV}_{U.X|Z}(z)\bar{\delta}_Y.$$

(ii.b) If the conditions in (i.b) and (ii.a.1) hold then

$$B(\nu(z)) = R^{LIV}_{W.X|Z}(z)\bar{\delta}.$$

The regularity conditions in $(ii.a)$ enable applying theorems for the derivative of an integral. The expression for the Wald OVB shows how this depends on $R^N_{U.Z}(z,z^*)$ and $\bar{\delta}_Y$. The OVB vanishes under exogeneity, which holds if $E(U|Z) = E(U)$, and thus $R^{Wald}_{W.X|Z}(z, z^*) = 0$, or $\bar{\delta}_Y = 0$. In this case, we obtain the standard point identification result:

$$\bar{\beta}(\nu(z) < U_X \leq \nu(z^*)) = R^{Wald}_{Y.X|Z}(z, z^*).$$

If W is a perfect proxy, with (δ_W, α_W) degenerate, then provided $(U_X, U_Y) \perp (U, Z)$, we have that $\bar{\beta}(\nu(z) < U_X \leq \nu(z^*))$ and $\bar{\delta}$ are point identified:

$$\frac{E(Y|Z = z^*, W = w) - E(Y|Z = z, W = w)}{E(X|Z = z^*, W = w) - E(X|Z = z, W = w)} = \bar{\beta}(\nu(z) < U_X \leq \nu(z^*)) \text{ and}$$
$$\frac{1}{(w^* - w)} E(Y|Z = z, W = w^*) - E(Y|Z = z, W = w) = \bar{\delta}.$$

When W is an imperfect proxy, Theorem A.6 gives that

$$\bar{\beta}(\nu(z) < U_X \leq \nu(z^*)) = R^{Wald}_{Y.X|Z}(z, z^*) - R^{Wald}_{W.X|Z}(z, z^*)\bar{\delta}.$$

Thus, proportional confounding, with known $\bar{\delta} = d$, also point identifies $\bar{\beta}(\nu(z) < U_X \leq \nu(z^*))$. Alternatively, magnitude and sign restrictions on confounding partially identify LATE:

$$\bar{\beta}(\nu(z) < U_X \leq \nu(z^*)) \in \mathcal{B}(\times^m_{h=1}\mathcal{D}_h) \equiv \{R^{Wald}_{Y.X|Z}(z, z^*) - R^{Wald}_{W.X|Z}(z, z^*)d : d_h \in \mathcal{D}_h, h = 1, ..., m\}.$$

These restrictions are are weaker than requiring $\bar{\delta} = 0$ (which ensures exogeneity when $R^{Wald}_{W.X|Z}(z, z^*) \neq 0$ and U depends on Z), a perfect proxy estimate for $\bar{\delta}$, or proportional confounding ($\bar{\delta} = d$). Arguments similar to those in the proofs of corollaries 4.3 and 6.2 show that the region $\mathcal{B}(\times^m_{h=1}\mathcal{D}_h)$, defined above, is sharp under the assumptions in Theorem A.6 and the restrictions $\mathcal{D}_h$, $h = 1, ..., m$, on confounding. Analogous results obtain for $\bar{\beta}(\nu(z))$.

Building on the results in Heckman and Vytlacil (2005), the bounds on MTE can be used to point or partially identify the average treatment effects for the population, treated, and untreated under restrictions on confounding. Similar to when exogeneity holds, this requires support conditions for Z. In particular, since U_X is absolutely continuous, we obtain the convenient representation

$$X = \mathbf{1}\{U_X \leq \nu(Z)\} = \mathbf{1}\{F_{U_X}(U_X) \leq F_{U_X}(\nu(Z))\} = \mathbf{1}\{V \leq P\}$$

where $V \backsim Unif[0, 1]$ and $P \equiv p(Z)$ equals the propensity score $\Pr(X = 1|Z)$ when $U_X \perp Z$. For instance, we can rewrite the MTE $\bar{\beta}(\nu(z))$ as $\bar{\beta}(p) \equiv E[\beta(U, U_Y)|V = p]$, with $p = p(z)$. Applying Theorem A.6 for almost every p using the representation $X = \mathbf{1}\{V \leq P\}$ with potential instrument $P \equiv p(Z)$, gives that $\bar{\beta}(p) = R^{LIV}_{Y.X|P}(p) - R^{LIV}_{W.X|P}(p)\bar{\delta}$. If P has the unit interval for support then the average treatment effect is characterized by:

$$\bar{\beta} = \int_0^1 \bar{\beta}(p)dp = \int_0^1 [R^{LIV}_{Y.X|P}(p) - R^{LIV}_{W.X|P}(p)\bar{\delta}]dp.$$

Similarly, the average treatment effects for the treated and untreated are characterized respectively by

$$\int_0^1 [R^{LIV}_{Y.X|P}(p) - R^{LIV}_{W.X|P}(p)\bar{\delta}]\frac{(1-F_P(p))}{E(P(Z))}dp \quad \text{and} \quad \int_0^1 [R^{LIV}_{Y.X|P}(p) - R^{LIV}_{W.X|P}(p)\bar{\delta}]\frac{F_P(p)}{E(1-P(Z))}dp.$$

As these expressions show, these average effects are point identified under conditional exogeneity, e.g. $B(p) = 0$, using a perfect proxy to point identify $\bar{\delta}$, or under proportional confounding with $\bar{\delta} = d$. Moreover, they are partially identify under sign and magnitude restrictions on $\bar{\delta}$.

Last, Theorem A.6 also shows how imposing m restrictions on $\bar{\beta}(\nu(z) < U_X \leq \nu(z^*))$ or $\bar{\beta}(\nu(z))$ can point identify this effect. For example, if $m = l = 1$ and one assumes that $\bar{\beta}(\nu(z^\dagger) < U_X \leq \nu(z^\ddagger)) = \bar{\beta}(\nu(\dot{z}) < U_X \leq \nu(\ddot{z}))$ for $z^\dagger, z^\ddagger, \dot{z}, \ddot{z} \in \mathcal{Z}$, as occurs if a nondegenerate component of Z is excluded from ν and thus the X equation, then, provided $R^{Wald}_{W.X|Z}(z^\dagger, z^\ddagger) \neq R^{Wald}_{W.X|Z}(\dot{z}, \ddot{z})$, $\bar{\delta} = \frac{R^{Wald}_{Y.X|Z}(\dot{z},\ddot{z}) - R^{Wald}_{Y.X|Z}(z^\dagger,z^\ddagger)}{R^{Wald}_{W.X|Z}(\dot{z},\ddot{z}) - R^{Wald}_{W.X|Z}(z^\dagger,z^\ddagger)}$ and $\bar{\beta}(\nu(z) < U_X \leq \nu(z^*))$ is therefore point identified. Analogous restrictions can point identify $\bar{\beta}(\nu(z))$.

A.3 Nonseparable Discrete Confounder

Let r and q be nonseparable functions, with Y and W generated as in S.1:

$$Y = r(X, U, U_Y) \quad \text{and} \quad W = q(U, U_W).$$

Theorem A.7 characterizes the nonparametric bias $B(x, x^*|x^*)$ or $B(x|x)$ of $R^N_{Y.X}(x, x^*)$ or $R^N_{Y.X}(x)$ in recovering the average effects $\bar{\beta}(x, x^*|x^*)$ or $\bar{\beta}(x|x)$ in the case of discrete U. This complements the results in Theorem 5.1 for continuous U. As in Corollary 5.2, $\bar{\beta}(x, x^*|x^*)$ or $\bar{\beta}(x|x)$ are partially identified by imposing restrictions on the average effects of U, when changing u to u^*, on Y at x and on the scalar W, denoted by:

$$\begin{aligned}
\bar{\delta}_Y(u, u^*; x) &\equiv \bar{r}(x, u^*) - \bar{r}(x, u) \equiv E[r(x, u^*, U_Y) - r(x, u, U_Y)] \quad \text{and} \\
\bar{\delta}_W(u, u^*) &\equiv \bar{q}(u^*) - \bar{q}(u) \equiv E[q(u^*, U_W) - q(u, U_W)].
\end{aligned}$$

Theorem A.7 *Assume S.1 with $m = l = 1$ and $x, x^* \in \mathcal{X}$. Suppose that $\mathcal{U}_x \cup \mathcal{U}_{x^*} = \{u_0, u_1, ..., u_L\}$ with $u_{g-1} < u_g$ for $g = 1, ..., L$ and that $\bar{r}(\ddot{x}, u) < \infty$ and $\bar{q}(u) < \infty$ for $\ddot{x} = x, x^*$ and all $u \in \{u_0, u_1, ..., u_L\}$.*
(i.a) If condition (6a) holds then

$$\begin{aligned}
B(x, x^*|x^*) &\equiv R^N_{Y.X}(x, x^*) - \bar{\beta}(x, x^*|x^*) \\
&= -\sum_{g=1}^{L} \bar{\delta}_Y(u_{g-1}, u_g; x)[F_{U|X}(u_{g-1}|x^*) - F_{U|X}(u_{g-1}|x)].
\end{aligned}$$

(i.b) If condition (7) holds then

$$R^N_{W.X}(x, x^*) = -\sum_{g=1}^{L} \bar{\delta}_W(u_{g-1}, u_g)[F_{U|X}(u_{g-1}|x^*) - F_{U|X}(u_{g-1}|x)].$$

(ii) Set $k = 1$ *and suppose that* $\frac{\partial}{\partial x} f_{U|X}(u_g|x)$ *exists and is finite for all* $u_g \in \{u_0, u_1, ..., u_L\}$.
(ii.a) If conditions B.1(ii.c) and (6a,6b) hold then

$$B(x|x) \equiv R^N_{Y.X}(x) - \bar{\beta}(x|x) = -\sum_{g=1}^{L} \bar{\delta}_Y(u_{g-1}, u_g; x)\frac{\partial}{\partial x} F_{U|X}(u_{g-1}|x).$$

(ii.b) If condition (7) holds then

$$R^N_{W.X}(x) = -\sum_{g=1}^{L} \bar{\delta}_W(u_{g-1}, u_g)\frac{\partial}{\partial x} F_{U|X}(u_{g-1}|x)$$

B Online Appendix B: Mathematical Proofs

Proof of Theorem 4.1 (i) By $(i.b)$ we have

$$Cov(Z, Y) = Cov(Z, X)\bar{\beta} + Cov(Z, U)\bar{\delta}_Y,$$

and thus, by $(i.a)$,

$$R_{Y.X|Z} = \bar{\beta} + R_{U.X|Z}\bar{\delta}_Y.$$

(ii) By $(ii.b)$ we have

$$Cov(Z, W) = Cov(Z, U)\bar{\delta}_W,$$

and thus, by $(i.a)$ and $(ii.a)$,

$$R_{W.X|Z}\bar{\delta} = R_{U.X|Z}\bar{\delta}_Y.$$

Proof of Corollary 4.2 (i, iii) The proof is immediate. (ii) Substitute for $U' = (W' - U'_W\bar{\alpha}_W)\bar{\delta}_W^{-1}$ in the equation for Y and apply Theorem 4.1(i) after relabeling the variables.

Proof of Corollary 4.3 The bounds obtain directly from $\bar{\beta} = R_{Y.X|Z} - R_{W.X|Z}\bar{\delta}$. We have that $\mathcal{B}_j(\times_{h=1}^m \mathcal{D}_h)$ is sharp since for each $\bar{b}_j \in \mathcal{B}_j(\times_{h=1}^m \mathcal{D}_h)$ there exist random variables (V, V_Y, V_W) and constant vectors $\bar{b}$ (whose j^{th} element is $\bar{b}_j$) and $\bar{d}_Y$ and matrix $\bar{d}_W$ with $\bar{d}_W^{-1}\bar{d}_Y = \bar{d} \in \times_{h=1}^m \mathcal{D}_h$ such that $Cov[(V_Y, V'_W)', Z] = 0$ and

$$Y = X'\bar{b} + V'\bar{d}_Y + V_Y \quad \text{and} \quad W' = V'\bar{d}_W + V_W.$$

In particular, consider the linear mapping $L_j : \mathcal{D}_1 \times ... \times \mathcal{D}_m \to \mathcal{B}_j$ given by $\bar{b}_j = R_{Y.X|Z,j} - R_{W.X|Z,j}\bar{d}$. Since $\mathcal{D}_1 \times ... \times \mathcal{D}_m$ is connected, $\mathcal{B}_j$ is totally ordered, and L_j is continuous, the generalized intermediate value theorem ensures that for every $\bar{b}_j \in \mathcal{B}_j(\times_{h=1}^m \mathcal{D}_h)$, there exists a vector $\bar{d} \in \times_{h=1}^m \mathcal{D}_h$ of constants such that $L_j(\bar{d}) = \bar{b}_j$ (see e.g. Pugh, 2002, p. 83). Let $\bar{d}_W$ and $\bar{d}_Y$ be any constant matrix and vector such that $\bar{d} = \bar{d}_W^{-1}\bar{d}_Y$ (e.g. $\bar{d}_W = I$ and $\bar{d}_Y = \bar{d}$) and set

$$\begin{aligned}
Y &\equiv X'R_{Y.X|Z} + \epsilon_{Y.X|Z} + E(Y - X'R_{Y.X|Z}) \\
&= X'(R_{Y.X|Z} - R_{W.X|Z}\bar{d}) + (X'R_{W.X|Z}\bar{d}_W^{-1})\bar{d}_Y + [\epsilon_{Y.X|Z} + E(Y - X'R_{Y.X|Z})] \\
&\equiv X'\bar{b} + V'\bar{d}_Y + V_Y \qquad \text{and} \\
W &\equiv X'R_{W.X|Z} + \epsilon_{W.X|Z} + E(W - X'R_{W.X|Z}) \\
&= (X'R_{W.X|Z}\bar{d}_W^{-1})\bar{d}_W + [\epsilon_{W.X|Z} + E(W - X'R_{W.X|Z})] \\
&\equiv V'\bar{d}_W + V_W,
\end{aligned}$$

where the above identities and $Cov[(V_Y, V_W')', Z] = 0$ hold by the definition of $\epsilon_{A.B|C}$.

We make use of the following regularity conditions in the proof of Theorem 5.1. For this, we let $\bar{r}(x,u) \equiv E[r(x,u,U_Y)]$ and $\bar{q}(u) \equiv E[q(u,U_W)]$. It is implicitly assumed that the referenced derivatives exist. We view the case in which $U|X = x$ (or $U|X = x^*$) is degenerate as a limiting case as $\tau \to 0$ for a sequence of absolutely continuous $F_{U|X}^{\tau}(u|x)$ that satisfy the regularity conditions in B.1 (see e.g., Bracewell, 1986).

Assumption B.1 *Let $x, x^* \in \mathcal{X}$, and denote by $\mathcal{N}(u) \subseteq \mathcal{U}$ and $\mathcal{N}(x) \subseteq \mathcal{X}$ nonempty open neighborhoods of u and x respectively.*

(i.a) $E[r(x,U,U_Y)|X = x^*] < \infty$ and $E(Y|X = \ddot{x}) < \infty$ for $\ddot{x} = x, x^*$,

(i.b) $\mathcal{U}_{x^*} = \mathcal{U}_x$,

(i.c) $\bar{r}(x,\cdot)$ is absolutely continuous on $\mathcal{U}_x$,

(i.d) for a.e. u and all $u^\dagger \in \mathcal{N}(u)$, $\bar{r}(x,u^\dagger) < \infty$ and there is a function $\Delta_{1,u}(u_y)$ with $E[\Delta_{1,u}(U_Y)] < \infty$ such that $\left|\frac{\partial}{\partial u} r(x,u^\dagger,u_y)\right| \leq \Delta_{1,u}(u_y)$ for a.e. u_y,

(i.e) $E(W|X = \ddot{x}) < \infty$ for $\ddot{x} = x, x^*$,

(i.f) $\bar{q}(\cdot)$ is absolutely continuous on $\mathcal{U}_x$,

(i.g) for a.e. u and all $u^\dagger \in \mathcal{N}(u)$, $\bar{q}(u^\dagger) < \infty$ and there is a function $\Gamma_{1,u}(u_w)$ with $E[\Gamma_{1,u}(U_W)] < \infty$ such that $\left|\frac{\partial}{\partial u} q(u^\dagger,u_w)\right| \leq \Gamma_{1,u}(u_w)$ for a.e. u_w,

(ii.a) for all $x^\dagger \in \mathcal{N}(x)$, $\mathcal{U}_{x^\dagger} = \mathcal{U}_x$ and $F_{U|X}(\cdot|x^\dagger)$ is absolutely continuous on $\mathcal{U}_x$,

(ii.b) for all $x^\dagger \in \mathcal{N}(x)$, $\int_{\mathcal{U}_x} \bar{r}(x^\dagger,u) f_{U|X}(u|x^\dagger)du < \infty$ and there is a function $\Delta_2(u)$ with $\int_{\mathcal{U}_x} \Delta_2(u)du < \infty$ such that $\left|\frac{\partial}{\partial x}\{\bar{r}(x^\dagger,u) f_{U|X}(u|x^\dagger)\}\right| \leq \Delta_2(u)$ for a.e. u,

(ii.c) for a.e. u and all $x^\dagger \in \mathcal{N}(x)$, $\bar{r}(x^\dagger,u) < \infty$ and there is a function $\Delta_{3,u}(u_y)$ with $E[\Delta_{3,u}(U_Y)] < \infty$ such that $\left|\frac{\partial}{\partial x} r(x^\dagger,u,u_y)\right| \leq \Delta_{3,u}(u_y)$ for a.e. u_y,

(ii.d) for all $x^\dagger \in \mathcal{N}(x)$, there is a function $\Delta_4(u)$ with $\int_{\mathcal{U}_x} \Delta_4(u)du < \infty$ such that $\left|\frac{\partial}{\partial x} f_{U|X}(u|x^\dagger)\right| \leq \Delta_4(u)$ for a.e. u,

(ii.e) for all $x^\dagger \in \mathcal{N}(x)$*,* $\int_{\mathcal{U}_x} \bar{q}(u) f_{U|X}(u|x^\dagger)du < \infty$ *and there is a function* $\Gamma_2(u)$ *with* $\int_{\mathcal{U}_x} \Gamma_2(u)du < \infty$ *such that* $\left|\bar{q}(u)\frac{\partial}{\partial x} f_{U|X}(u|x^\dagger)\right| \leq \Gamma_2(u)$ *for a.e.* u*.*

The absolute continuity of $\bar{r}(x,\cdot)$ and $\bar{q}(\cdot)$ on $\mathcal{U}_x$ in B.1 ensures that $\frac{\partial}{\partial u}\bar{r}(x,\cdot)$ and $\frac{\partial}{\partial u}\bar{q}(\cdot)$ exist for a.e. u and are integrable. Assuming that derivatives are bounded almost everywhere by an integrable function justifies the interchange of derivative and integral.

Proof of Theorem 5.1: $(i.a)$ By B.1$(i.a)$, we have

$$\bar{\beta}(x, x^*|x^*) = R^N_{Y.X}(x, x^*) - \{E[r(x, U, U_Y)|X = x^*] - E[r(x, U, U_Y)|X = x]\},$$

where the second term is $B(x, x^*|x^*)$. Using equation (6a), we have for $\ddot{x} = x, x^*$:

$$E[r(x, U, U_Y)|X = \ddot{x}] = E[\bar{r}(x, U) \ |X = \ddot{x}].$$

By B.1.$i(b, c)$ and absolutely continuity of $F_{U|X}(\cdot|\ddot{x})$, integration by parts gives:

$$\begin{aligned} B(x, x^*|x^*) &= \int_{\mathcal{U}_x} \bar{r}(x, u) \ [f_{U|X}(u|x^*) - f_{U|X}(u|x)]du \\ &= \bar{r}(x, u)[F_{U|X}(u|x^*) - F_{U|X}(u|x)]\big|_{\underline{u}}^{\bar{u}} - \int_{\mathcal{U}_x} \frac{\partial}{\partial u}\bar{r}(x, u)[F_{U|X}(u|x^*) - F_{U|X}(u|x)]du, \end{aligned}$$

with $\underline{u}$ and $\bar{u}$ the (possibly infinite) infimum and supremum over $\mathcal{U}_x$. The first term vanishes and the result obtains since, by B.1$(i.d)$, $\frac{\partial}{\partial u}\bar{r}(x, u) = \bar{\delta}_Y(u; x)$ for a.e. u (see e.g. Corbae, Stinchcombe, and Zeman (2009, Theorem 7.5.17) or Bartle (1966, corollary 5.9)).
$(i.b)$ Similarly, equation (7) and B.1.$i(b, e, f, g)$ give

$$R^N_{W.X}(x, x^*) = \int_{\mathcal{U}_x} \bar{q}(u) \ [f_{U|X}(u|x^*) - f_{U|X}(u|x)]du = -\int_{\mathcal{U}_x} \bar{\delta}_W(u)[F_{U|X}(u|x^*) - F_{U|X}(u|x)]du.$$

$(ii.a)$ Using equation (6a) and interchanging the derivative and integral by B.1.$ii(a, b)$:

$$\begin{aligned} R^N_{Y.X}(x) &= \frac{\partial}{\partial x}E[\bar{r}(x, U)|X = x] = \frac{\partial}{\partial x}\int_{\mathcal{U}_x} \bar{r}(x, u) \ f_{U|X}(u|x)du \\ &= \int_{\mathcal{U}_x} [\frac{\partial}{\partial x}\bar{r}(x, u)]f_{U|X}(u|x)du + \int_{\mathcal{U}_x} \bar{r}(x, u)[\frac{\partial}{\partial x}f_{U|X}(u|x)]du \equiv T_1 + T_2, \end{aligned}$$

where the product rule derivatives exist by B.1.$ii(c, d)$. By B.1$(ii.c)$ and equation (6b):

$$\begin{aligned} T_1 &= \int_{\mathcal{U}_x} \frac{\partial}{\partial x}E[r(x, u, U_Y)] \ f_{U|X}(u|x) \ du = \int_{\mathcal{U}_x} E[\frac{\partial}{\partial x}r(x, u, U_Y)] \ f_{U|X}(u|x) \ du \\ &= \int_{\mathcal{U}_x} E[\frac{\partial}{\partial x}r(x, u, U_Y)|U = u, X = x] \ f_{U|X}(u|x) \ du = \bar{\beta}(x|x). \end{aligned}$$

By B.1$ii(a,d)$ and B.1 $i(c,d)$, integration by parts gives

$$T_2 = \bar{r}(x,u)\ \frac{\partial}{\partial x}F_{U|X}(u|x)\bigg|_{\underline{u}}^{\bar{u}} - \int_{\mathcal{U}_x} \frac{\partial}{\partial u}\bar{r}(x,u)\frac{\partial}{\partial x}F_{U|X}(u|x)du$$
$$= -\int_{\mathcal{U}_x} \bar{\delta}_Y(u;x)\frac{\partial}{\partial x}F_{U|X}(u|x)du = B(x|x).$$

$(ii.b)$ Similarly, equation (7), B.1.$i(f,g)$, and B.1.$ii(a,d,e)$ give

$$R^N_{W.X}(x) = \frac{\partial}{\partial x}\int_{\mathcal{U}_x} \bar{q}(u)\ f_{U|X}(u|x)\ du = -\int_{\mathcal{U}_x} \bar{\delta}_W(u)\frac{\partial}{\partial x}F_{U|X}(u|x)du$$

Proof of Corollary 5.2: (i) Since $\bar{\delta}_Y(u;x) = d(u,x)\bar{\delta}_W(u)$ for a.e. $u \in \mathcal{U}_x$, we have:

$$B(x,x^*|x^*) = -\int_{\mathcal{U}_x} d(u,x)\bar{\delta}_W(u)[F_{U|X}(u|x^*) - F_{U|X}(u|x)]du.$$

Since $\bar{\delta}_W(u)[F_{U|X}(u|x^*) - F_{U|X}(u|x)]$ does not change sign for a.e. $u \in \mathcal{U}_x$, and

$$R^N_{W.X}(x,x^*) = -\int_{\mathcal{U}_x} \bar{\delta}_W(u)[F_{U|X}(u|x^*) - F_{U|X}(u|x)]du,$$

we have that $d_L(x) \leq d(u,x) \leq d_H(x)$ gives

$$B(x,x^*|x^*) \in \{R^N_{W.X}(x,x^*)d : d \in \mathcal{D}(x)\},$$

The bounds then follow from $\bar{\beta}(x,x^*|x^*) = R^N_{Y.X}(x,x^*) - B(x,x^*|x^*)$.

$\mathcal{B}(\mathcal{D}(x))$ is sharp since for a given $x, x^* \in \mathcal{X}$ and each effect $\bar{b}(x,x^*|x^*) \in \mathcal{B}(\mathcal{D}(x))$ there exist random variables (V, V_Y, V_W) and functions r^* and q^*, such that $Y = r^*(X,V,V_Y)$ and $W = q^*(V,V_W)$, that satisfy the conditions in Theorem 5.1 and Corollary 5.2. For this, let

$$d(x) = \frac{1}{R^N_{W.X}(x,x^*)}(R^N_{Y.X}(x,x^*) - \bar{b}(x,x^*|x^*))$$

(recall that if $R^N_{W.X}(x,x^*) = 0$ then $\mathcal{B}(\mathcal{D}(x))$ is a singleton) so that $d(x) \in \mathcal{D}(x)$. Further, let d_W^{-1} and $d_Y(x)$ be any constants such that $d(x) = d_W^{-1}d_Y(x)$ (e.g. $d_W = 1$ and $d_Y(x) = d(x)$). Then it suffices to define V, V_Y, V_W, r^* and q^* as follows

$$\begin{aligned}
Y &\equiv E(Y|X) + [Y - E(Y|X)] \\
&= \{E(Y|X) - E(W|X)d(x)\} + [E(W|X)d_W^{-1}]d_Y(x) + [Y - E(Y|X)] \\
&\equiv r_1^*(X) + V d_Y(x) + V_Y \equiv r^*(X,V,V_Y), \\
W &\equiv E(W|X) + [W - E(W|X)] = V d_W + V_W \equiv q^*(V,V_W),
\end{aligned}$$

so that

$$\bar{b}(x,x^*|x^*) = r_1^*(x^*) - r_1^*(x) = R^N_{Y.X}(x,x^*) - R^N_{W.X}(x,x^*)d(x).$$

By construction of V, V_Y, and V_W, the analogues of equations (6a) and (7) hold since for $x^\dagger, \ddot{x} \in \{x, x^*\}$ and all $v \in \mathcal{V}_{\ddot{x}}$, we have:

$$E[r^*(x^\dagger, v, V_Y)|V = v, X = \ddot{x}] = E[r_1^*(x^\dagger) + vd_Y(x) + V_Y|X = \ddot{x}] = r_1^*(x^\dagger) + vd_Y(x) = E[r^*(x^\dagger, v, V_Y)],$$
$$E[q^*(v, V_W)|V = v, X = \ddot{x}] = E[vd_W + V_W|X = \ddot{x}] = vd_W = E[q^*(v, V_W)].$$

Also, $E[\frac{\partial}{\partial v} q^*(v, V_W)] = d_W$ clearly does not change sign. Last, $F_{V|X}(v|\ddot{x})$ is degenerate:

$$F_{V|X}(v|\ddot{x}) = \Pr[E(W|X = \ddot{x})d_W^{-1} \leq v|X = \ddot{x}] = H(v - E(W|X = \ddot{x})d_W^{-1}),$$

where $H(t) \begin{cases} 1 & \text{if } 0 \leq t \\ 0 & \text{if } t < 0 \end{cases}$ is the Heaviside step function and we have that $F_{V|X}(v|x) - F_{V|X}(v|x^*)$ is either nonnegative or nonpositive for all v. Here, we view $F_{V|X}(v|\ddot{x})$ as a limiting case as $\tau \to 0$ for a sequence $F^\tau_{V|X}(v|\ddot{x})$ that, along with the additively separable functions r^* and q^*, satisfy the regularity conditions in B.1 to interchange the order of well defined derivative and integral (see e.g., Bracewell, 1986).

(ii) The bounds obtain using similar arguments to (i). The sharpness proof constructs V, V_Y, V_W, r^*, and q^* analogously to (i). In particular, for a given $x \in \mathcal{X}$ and each $\bar{b}(x|x) \in \mathcal{B}(\mathcal{D}(x))$, set $d_W^{-1} d_Y(x) = d(x) = \frac{1}{R^N_{W.X}(x)}(R^N_{Y.X}(x) - \bar{b}(x|x))$ so that $\bar{b}(x, |x) = \frac{\partial}{\partial x} r_1^*(x) = R^N_{Y.X}(x) - R^N_{W.X}(x) d(x)$. The analogue of equation (6b) holds since $E[\frac{\partial}{\partial x} r^*(x, v, V_Y)|V = v, X = x] = \frac{\partial}{\partial x} r_1^*(x)$ for $v = E(W|X = x)d_W^{-1}$. Further, we have $\frac{\partial}{\partial x} F_{V|X}(v|x) = -R^N_{W.X}(x) d_W^{-1} \delta(v - E(W|X = x)d_W^{-1})$ where δ is the Dirac delta function with an impulse concentrated at $E(W|X = x)d_W^{-1}$.

Analogously to Theorem 5.1, Theorem 6.1 employs regularity conditions that we collect in Assumption B.2. In what follows, we slightly abuse the previous notation and write $\bar{r}(z, u) \equiv E[r(\mathbf{1}\{U_X \leq \nu(z)\}, u, U_Y)]$. It is implicitly assumed that the referenced derivatives exist. We view the case in which $U|Z = z$ (or $U|Z = z^*$) is degenerate as a limiting case as $\tau \to 0$ for a sequence of absolutely continuous $F^\tau_{U|Z}(u|z)$ that satisfy the regularity conditions in B.2 (see e.g., Bracewell, 1986).

Assumption B.2 *Let $z, z^* \in \mathcal{Z}$, and denote by $\mathcal{N}(u) \subseteq \mathcal{U}$ and $\mathcal{N}(z) \subseteq \mathcal{Z}$ nonempty open neighborhoods of u and z respectively.*

(i.a) $E[r(\mathbf{1}\{U_X \leq \nu(z)\}, U, U_Y)|Z = z^*] < \infty$ and $E(Y|Z = \ddot{z}) < \infty$ for $\ddot{z} = z, z^*$,

(i.b) $\mathcal{U}_{z^*} = \mathcal{U}_z$,

(i.c) $\bar{r}(z, \cdot)$ is absolutely continuous on $\mathcal{U}_z$,

(i.d) for a.e. u and all $u^\dagger \in \mathcal{N}(u)$, $\bar{r}(z, u^\dagger) < \infty$ and there is a function $\Phi_{1,u}(u_x, u_y)$ with $E[\Phi_{1,u}(U_X, U_Y)] < \infty$ such that $\left|\frac{\partial}{\partial u} r(\mathbf{1}\{u_x \leq \nu(z)\}, u^\dagger, u_y)\right| \leq \Phi_{1,u}(u_x, u_y)$ for a.e. (u_x, u_y),

(i.e) $E(W|Z = \ddot{z}) < \infty$ for $\ddot{z} = z, z^*$,

(i.f) $\bar{q}(\cdot)$ *is absolutely continuous on* $\mathcal{U}_z$,

(i.g) for a.e. u *and all* $u^\dagger \in \mathcal{N}(u)$, $\bar{q}(u^\dagger) < \infty$ *and there is a function* $\Upsilon_{1,u}(u_w)$ *with* $E[\Upsilon_{1,u}(U_W)] < \infty$ *such that* $\left|\frac{\partial}{\partial u} q(u^\dagger, u_w)\right| \leq \Upsilon_{1,u}(u_w)$ *for a.e.* u_w,

(ii.a) for all $z^\dagger \in \mathcal{N}(z)$, $\mathcal{U}_{z^\dagger} = \mathcal{U}_z$ *and* $F_{U|Z}(\cdot|z^\dagger)$ *is absolutely continuous on* $\mathcal{U}_z$,

(ii.b) for all $z^\dagger \in \mathcal{N}(z)$, $\int_{\mathcal{U}_z} \bar{r}(z^\dagger, u) f_{U|Z}(u|z^\dagger) du < \infty$ *and there is a function* $\Phi_2(u)$ *with* $\int_{\mathcal{U}_z} \Phi_2(u) du < \infty$ *such that* $\left|\frac{\partial}{\partial z}\{\bar{r}(z^\dagger, u) f_{U|Z}(u|z^\dagger)\}\right| \leq \Phi_2(u)$ *for a.e.* u,

(ii.c) $\frac{\partial}{\partial z}\nu(z) \neq 0$ *and* $f_{U_X}(\cdot)$ *is continuous at* $\nu(z)$ *with* $f_{U_X}(\nu(z)) > 0$,

(ii.d) for a.e. u, $E[\beta(u, U_Y)|U_X = \cdot]$ *is continuous at* $\nu(z)$,

(ii.e) for all $z^\dagger \in \mathcal{N}(z)$, *there is a function* $\Phi_3(u)$ *with* $\int_{\mathcal{U}_z} \Phi_3(u) du < \infty$ *such that* $\left|\frac{\partial}{\partial z} f_{U|Z}(u|z^\dagger)\right| \leq \Phi_3(u)$ *for a.e.* u,

(ii.f) for all $z^\dagger \in \mathcal{N}(z)$, $\int_{\mathcal{U}_z} \bar{q}(u) f_{U|Z}(u|z^\dagger) du < \infty$ *and there is a function* $\Upsilon_2(u)$ *with* $\int_{\mathcal{U}_z} \Upsilon_2(u) du < \infty$ *such that* $\left|\bar{q}(u) \frac{\partial}{\partial z} f_{U|Z}(u|z^\dagger)\right| \leq \Upsilon_2(u)$ *for a.e.* u.

Proof of Theorem 6.1: $(i.a)$ By B.2$(i.a)$, adding and subtracting $E(Y|Z = z)$ gives

$$\begin{aligned}\gamma(z, z^*|z^*) &\equiv E[r(\mathbf{1}\{U_X \leq \nu(z^*)\}, U, U_Y) - r(\mathbf{1}\{U_X \leq \nu(z)\}, U, U_Y)|Z = z^*] \\ &= R^N_{Y.Z}(z, z^*; s) - \{E[r(\mathbf{1}\{U_X \leq \nu(z)\}, U, U_Y)|Z = z^*] - E[r(\mathbf{1}\{U_X \leq \nu(z)\}, U, U_Y)|Z = z]\},\end{aligned}$$

where we label the second term $B_\gamma(z, z^*|z^*)$. Further, for $\ddot{z} = z, z^*$,

$$E[r(\mathbf{1}\{U_X \leq \nu(\ddot{z})\}, U, U_Y)|Z = z^*] = E[\alpha(U, U_Y)|Z = z^*] + E[\mathbf{1}\{U_X \leq \nu(\ddot{z})\}\beta(U, U_Y)\ |Z = z^*].$$

$\Pr[\nu(z) < U_X \leq \nu(z^*)] > 0$ gives that $\nu(z) < \nu(z^*)$ and thus

$$\begin{aligned}\gamma(z, z^*|z^*) &= E[\mathbf{1}\{\nu(z) < U_X \leq \nu(z^*)\}\beta(U, U_Y)\ |\ Z = z^*] \\ &= E[\beta(U, U_Y)\ |\ \nu(z) < U_X \leq \nu(z^*), Z = z^*] \times \Pr[\nu(z) < U_X \leq \nu(z^*)\ |Z = z^*].\end{aligned}$$

Further, by $U_X \perp Z$, we have

$$R^N_{X.Z}(z, z^*) = E[\mathbf{1}\{\nu(z) < U_X \leq \nu(z^*)\}] = \Pr[\nu(z) < U_X \leq \nu(z^*)|Z = z^*].$$

Dividing $\gamma(z, z^*|z^*)$ by $R^N_{X.Z}(z, z^*) > 0$ gives the Wald OVB

$$\bar{\beta}(0, 1|\nu(z) < U_X \leq \nu(z^*), z^*) = R^{Wald}_{Y.X|Z}(z, z^*) - \frac{1}{R^N_{X.Z}(z, z^*)} B_\gamma(z, z^*|z^*).$$

To derive the expression for $B_\gamma(z,z^*|z^*)$, we apply conditions (9) and (10) for $\ddot{z}=z,z^*$:

$$
\begin{aligned}
&E[r(\mathbf{1}\{U_X\le\nu(z)\},U,U_Y)|Z=\ddot{z}]\\
&\qquad=\int_{\mathcal{U}_z}E[\alpha(u,U_Y)+\mathbf{1}\{U_X\le\nu(z)\}\beta(u,U_Y)|U=u,Z=\ddot{z}]\ f_{U|Z}(u|\ddot{z})du\\
&\qquad=\int_{\mathcal{U}_z}E[\alpha(u,U_Y)]+E\{\mathbf{1}\{U_X\le\nu(z)\}E[\beta(u,U_Y)|U_X,U=u,Z=\ddot{z}]|U=u,Z=\ddot{z}\}f_{U|Z}(u|\ddot{z})du\\
&\qquad=\int_{\mathcal{U}_z}E[\alpha(u,U_Y)]+E\{E[\mathbf{1}\{U_X\le\nu(z)\}\beta(u,U_Y)|U_X]\}\ f_{U|Z}(u|\ddot{z})du\\
&\qquad=\int_{\mathcal{U}_z}E[\alpha(u,U_Y)+\mathbf{1}\{U_X\le\nu(z)\}\beta(u,U_Y)]f_{U|Z}(u|\ddot{z})du=E[\bar{r}(z,U)|Z=\ddot{z}].
\end{aligned}
$$

By B.2.$i(b,c)$ and absolutely continuity of $F_{U|Z}(\cdot|\ddot{z})$, integration by parts then gives

$$
\begin{aligned}
B_\gamma(z,z^*|z^*)&=\int_{\mathcal{U}_z}\bar{r}(z,u)[f_{U|Z}(u|z^*)-f_{U|Z}(u|z)]du\\
&=\bar{r}(z,u)[F_{U|Z}(u|z^*)-F_{U|Z}(u|z)]\big|_{\underline{u}}^{\bar{u}}-\int_{\mathcal{U}_z}\frac{\partial}{\partial u}\bar{r}(z,u)[F_{U|Z}(u|z^*)-F_{U|Z}(u|z)]du,
\end{aligned}
$$

with $\underline{u}$ and $\bar{u}$ the (possibly infinite) infimum and supremum over $\mathcal{U}_z$. The first term vanishes and the result obtains since B.2$(i.d)$ gives $\frac{\partial}{\partial u}\bar{r}(z,u)=\bar{\delta}_Y(u;z)$ for a.e. u.

$(i.b)$ Similarly, condition (11), B.2.$i(b,e,f,g)$, and integration by parts give

$$
R^N_{W.Z}(z,z^*)=\int_{\mathcal{U}_z}\bar{q}(u)[f_{U|Z}(u|z^*)-f_{U|Z}(u|z)du=-\int_{\mathcal{U}_z}\bar{\delta}_W(u)[F_{U|Z}(u|z^*)-F_{U|Z}(u|z)]du.
$$

The result then obtains from dividing by $R^N_{X.Z}(z,z^*)>0$.

$(ii.a)$ From $(i.a)$, recall that $E(Y|Z=z)=E[\bar{r}(z,U)|Z=z]$. Using B.2.$ii(a,b)$ to interchange the derivative and integral, we obtain

$$
\begin{aligned}
R^N_{Y.Z}(z)&=\frac{\partial}{\partial z}E[\bar{r}(z,U)|Z=z]=\frac{\partial}{\partial z}\int_{\mathcal{U}_z}\bar{r}(z,u)\ f_{U|Z}(u|z)du\\
&=\int_{\mathcal{U}_z}\frac{\partial}{\partial z}\bar{r}(z,u)\ f_{U|Z}(u|z)du+\int_{\mathcal{U}_z}\bar{r}(z,u)\ \frac{\partial}{\partial z}f_{U|Z}(u|z)\ du\equiv T_1+T_2,
\end{aligned}
$$

where the product rule derivatives exist by B.2.$ii(c,d,e)$. In particular, to examine T_1 note that

$$
\bar{r}(z,u)\equiv E[r(\mathbf{1}\{U_X\le\nu(z)\},u,U_Y)]=E[\alpha(u,U_Y)]+\int_{-\infty}^{\nu(z)}E[\beta(u,U_Y)|U_X=t]f_{U_X}(t)dt.
$$

B.2.$ii(c,d)$, the Lebesgue differentiation theorem, and the chain rule give

$$
T_1=f_{U_X}(\nu(z))\frac{\partial}{\partial z}\nu(z)\int_{\mathcal{U}_z}E[\beta(u,U_Y)|U_X=\nu(z)]f_{U|Z}(u|z)du=f_{U_X}(\nu(z))\frac{\partial}{\partial z}\nu(z)\ \bar{\beta}(0,1|\nu(z),z),
$$

where we make use of (9) and (10) in the last equality.

To examine T_2, B.2$ii(a,e)$ and B.2$(i.c)$ enable integration by parts which gives:

$$T_2 = \bar{r}(z,u)\frac{\partial}{\partial z}F_{U|Z}(u|z)\Big|_{\underline{u}}^{\bar{u}} - \int_{\mathcal{U}_z}\frac{\partial}{\partial u}\bar{r}(z,u)\frac{\partial}{\partial z}F_{U|Z}(u|z)du = -\int_{\mathcal{U}_z}\bar{\delta}_Y(u;z)\frac{\partial}{\partial z}F_{U|Z}(u|z)du,$$

where we use B.2$(i.d)$ in the last equality.
Dividing $R^N_{Y.Z}(z)$ by $R^N_{X.Z}(z)$ gives the result since, by B.2$(ii.c)$, we have

$$R^N_{X.Z}(z) \equiv \frac{\partial}{\partial z}E(X|Z=z) = \frac{\partial}{\partial z}\int_{-\infty}^{\nu(z)} f_{U_X}(t)dt = f_{U_X}(\nu(z))\frac{\partial}{\partial z}\nu(z) \neq 0.$$

$(ii.b)$ Similarly, condition (11), B.2.$i(f,g)$, B.2.$ii(a,c,e,f)$, and integration by parts give

$$R^{LIV}_{W.Z}(z) = \frac{1}{R^N_{X.Z}(z)}\int_{\mathcal{U}_z}\bar{q}(u)\frac{\partial}{\partial z}f_{U|Z}(u|z)du = -\frac{1}{R^N_{X.Z}(z)}\int_{\mathcal{U}_z}\bar{\delta}_W(u)\frac{\partial}{\partial z}F_{U|Z}(u|z)du.$$

Proof of Corollary 6.2: (i) Since $\bar{\delta}_Y(u;z) = d(u,z)\bar{\delta}_W(u)$ for a.e. $u \in \mathcal{U}_z$, we have:

$$B(\nu(z) < U_X \leq \nu(z^*), z^*) = -\frac{1}{R^N_{X.Z}(z,z^*)}\int_{\mathcal{U}_z} d(u,z)\bar{\delta}_W(u)[F_{U|Z}(u|z^*) - F_{U|Z}(u|z)]du.$$

Since $\bar{\delta}_W(u)[F_{U|Z}(u|z^*) - F_{U|Z}(u|z)]$ does not change sign for a.e. $u \in \mathcal{U}_z$, and

$$R^{Wald}_{W.X|Z}(z,z^*) = -\frac{1}{R^N_{X.Z}(z,z^*)}\int_{\mathcal{U}_x}\bar{\delta}_W(u)[F_{U|Z}(u|z^*) - F_{U|Z}(u|z)]du,$$

we have that $d_L(z) \leq d(u,z) \leq d_H(z)$ gives

$$B(\nu(z) < U_X \leq \nu(z^*), z^*) \in \{R^{Wald}_{W.X|Z}(z,z^*)d : d \in \mathcal{D}(z)\},$$

The bounds then follow from $\bar{\beta}(\nu(z) < U_X \leq \nu(z^*), z^*) = R^{Wald}_{Y.X|Z}(z,z^*) - B(\nu(z) < U_X \leq \nu(z^*), z^*)$.

$\mathcal{B}(\mathcal{D}(z))$ is sharp since for a given $z, z^* \in \mathcal{Z}$ and each $\bar{b}(0,1|\nu(z) < V_X \leq \nu(z^*), z^*) \in \mathcal{B}(\mathcal{D}(z))$ there exist random variables (V, V_X, V_Y, V_W) and functions ν^*, α^*, β^*, r^*, and q^*, such that

$$Y = r^*(X,V,V_Y) = \alpha^*(V,V_Y) + \beta^*(V,V_Y)X, \quad W = q^*(V,V_W), \text{ and } X = \mathbf{1}\{V_X \leq \nu^*(Z)\},$$

that satisfy the conditions in Theorem 6.1 and Corollary 6.2. For this, let $V_X = U_X$, $\nu^* = \nu$, and

$$d(z) = \frac{1}{R^{Wald}_{W.X|Z}(z,z^*)}[R^{Wald}_{Y.X|Z}(z,z^*) - \bar{b}(0,1|\nu(z) < U_X \leq \nu(z^*), z^*)]$$

(recall that if $R^{Wald}_{W.X|Z}(z,z^*) = 0$ then $\mathcal{B}(\mathcal{D}(z))$ is a singleton) so that $d(z) \in \mathcal{D}(z)$. Further, let d_W^{-1} and $d_Y(z)$ be any constants such that $d(z) = d_W^{-1}d_Y(z)$ (e.g. $d_W = 1$ and $d_Y(z) = d(z)$).

Then it suffices to let $r_1^*(\cdot)$ be given by

$$\begin{aligned}
\left[\begin{array}{c} r_1^*(0) \\ r_1^*(1) \end{array}\right] &\equiv \left[\begin{array}{cc} 1-p(z) & p(z) \\ 1-p(z^*) & p(z^*) \end{array}\right]^{-1} \left[\begin{array}{c} E(Y-Wd(z)|Z=z) \\ E(Y-Wd(z)|Z=z^*) \end{array}\right] \\
&= \frac{1}{p(z^*)-p(z)} \left[\begin{array}{c} p(z^*)E(Y-Wd(z)|Z=z) - p(z)E(Y-Wd(z)|Z=z^*) \\ -(1-p(z^*))E(Y-Wd(z)|Z=z) + (1-p(z))E(Y-Wd(z)|Z=z^*) \end{array}\right] \\
&= \left[\begin{array}{c} E(Y-Wd(z)|Z=z) - p(z)[R_{Y.X|Z}^{Wald}(z,z^*) - R_{W.X|Z}^{Wald}(z,z^*)d(z)] \\ E(Y-Wd(z)|Z=z) + (1-p(z))[R_{Y.X|Z}^{Wald}(z,z^*) - R_{W.X|Z}^{Wald}(z,z^*)d(z)] \end{array}\right],
\end{aligned}$$

where $p(z) = E(X|Z=z)$ (the matrix inverse exists since $R_{X.Z}^N(z,z^*) > 0$), and to define V, V_Y, V_W, r^*, and q^* as follows

$$\begin{aligned}
Y &\equiv r_1^*(X) + [E(W|Z)d_W^{-1}]d_Y(z) + \{Y - r_1^*(X) - E(W|Z)d_W^{-1}d_Y(z)\} \\
&\equiv r_1^*(X) + Vd_Y(z) + V_Y \equiv r^*(X,V,V_Y), \text{ and} \\
W &\equiv [E(W|Z)d_W^{-1}]d_W + [W - E(W|Z)] \equiv Vd_W + V_W \equiv q^*(V,V_W).
\end{aligned}$$

In particular, for $\ddot{z} = z, z^*$ and $v \in \mathcal{V}_{\ddot{z}}$ we have

$$\begin{aligned}
E[V_Y|V=v, Z=\ddot{z}] &= E[Y - Wd(z) - r_1^*(X)|Z=\ddot{z}] \\
&= E[(1-p(\ddot{z}))r_1^*(0) + p(\ddot{z})r_1^*(1) - r_1^*(X)|Z=\ddot{z}] \\
&= E[r_1^*(0) + X(r_1^*(1) - r_1^*(0)) - r_1^*(X)|Z=\ddot{z}] = 0,
\end{aligned}$$

and $E(V_W|V=v, Z=\ddot{z}) = 0$. It follows that, for $\ddot{z} = z, z^*$ and all $v \in \mathcal{V}_{\ddot{z}}$, we have

$$\begin{aligned}
E[\alpha^*(v,V_Y)|V=v, Z=\ddot{z}] &= E[r^*(0,v,V_Y)|Z=\ddot{z}] = r_1^*(0) + vd_Y(z) = E[\alpha^*(v,V_Y)], \\
E[\beta^*(v,V_Y)|V_X, V=v, Z=\ddot{z}] &= E[r^*(1,v,V_Y) - r^*(0,v,V_Y)|V_X, Z=\ddot{z}] \\
&= r_1^*(1) - r_1^*(0) = E[\beta^*(v,V_Y)|V_X], \qquad \text{and} \\
E[q^*(v,V_W)|V=v, Z=\ddot{z}] &= vd_W = E[q^*(v,V_W)],
\end{aligned}$$

and, thus, the analogues of equations (10) and (11) hold. Further, we have

$$\begin{aligned}
&E[r^*(1,V,V_Y) - r^*(0,V,V_Y)|\nu^*(z) < V_X \leq \nu^*(z^*), z^*)] = r_1^*(1) - r_1^*(0) \\
&\qquad = R_{Y.X|Z}^{Wald}(z,z^*) - R_{W.X|Z}^{Wald}(z,z^*)d(z) = \bar{b}(0,1|\nu(z) < U_X \leq \nu(z^*), z^*).
\end{aligned}$$

Last, $E[\frac{\partial}{\partial v}q^*(v,V_W)] = d_W$ does not change sign and $F_{V|Z}(v|\ddot{z})$ is degenerate:

$$F_{V|Z}(v|\ddot{z}) = \Pr[E(W|Z=\ddot{z})d_W^{-1} \leq v|Z=\ddot{z}] = H(v - E(W|Z=\ddot{z})d_W^{-1}),$$

where $H(t) \left\{\begin{array}{ll} 1 & \text{if } 0 \leq t \\ 0 & \text{if } t < 0 \end{array}\right.$ is the Heaviside step function, so that $F_{V|Z}(v|z) - F_{V|Z}(v|z^*)$ is either nonnegative or nonpositive for all v. Here, we view $F_{V|Z}(v|\ddot{z})$ as a limiting case as $\tau \to 0$ for a sequence $F^{\tau}_{V|Z}(v|\ddot{z})$ that, along with the additively separable functions r^* and q^*, satisfy the regularity conditions in B.2 to interchange the order of well defined derivative and integral (see e.g., Bracewell, 1986).

(ii) The bounds obtain using similar arguments to (i). The sharpness proof constructs V, V_X, V_Y, V_W, ν^*, r^*, and q^* analogously to (i). In particular, for a given $z \in \mathcal{Z}$ and each $\bar{b}(0,1|\nu(z),z) \in B(D(z))$, set $d_W^{-1} d_Y(z) = d(z) = \frac{1}{R^{LIV}_{W.X|Z}(z)}[R^{LIV}_{Y.X|Z}(z) - \bar{b}(0,1|\nu(z),z)]$ and $z^* = z + e$. By letting $e \to 0$ and redefining $r_1^*(0)$ and $r_1^*(1)$ as the limits, we obtain

$$\begin{aligned}\bar{b}(0,1|\nu^*(z),z) &= r_1^*(1) - r_1^*(0)\\ &= \lim_{e\to 0} R^{Wald}_{Y.X|Z}(z,z^*) - R^{Wald}_{W.X|Z}(z,z^*)d(z) = R^{LIV}_{Y.X|Z}(z) - R^{LIV}_{W.X|Z}(z)d(z).\end{aligned}$$

Here, $\frac{\partial}{\partial z}F_{V|Z}(v|z) = -R^N_{W.Z}(z)d_W^{-1}\delta(v - E(W|Z=z)d_W^{-1})$ where δ is the Dirac delta function with an impulse concentrated at $E(W|Z=z)d_W^{-1}$.

Comment on the Sharpness in Corollary 6.2 under Global Mean Independence: Consider strengthening the local conditions (10) and (11) in Corollary 6.2 to require the global conditions:

$$\begin{aligned}E[\alpha(u,U_Y)|U,Z] &= E[\alpha(u,U_Y)],\\ E[\beta(u,U_Y)|U_X,U,Z] &= E[\beta(u,U_Y)|U_X], \quad \text{and}\\ E[q(u,U_W)|U,Z] &= E[q(u,V_W)] \qquad \text{for all } u \in \mathcal{U}.\end{aligned} \tag{14}$$

Then, provided the regularity conditions in B.2 can be suitably adjusted so that $\nu(\cdot)$ is differentiable a.e. and $E(X|Z=s) = \int_{-\infty}^{\nu(s)} f_{U_X}(t)dt$ and $E(Y - Wd(z)|Z=s) = \int_{\mathcal{U}_s}[\bar{r}(s,u) - \bar{q}(u)d(z)]\, f_{U|Z}(u|s)du$ are absolutely continuous on $\mathcal{Z}$ (see e.g. Talvila (2001) for sufficient regularity conditions), the bounds $\mathcal{B}(\mathcal{D}(z))$ in Corollary 6.2 remain sharp. In particular, let V_X, ν^*, d_W^{-1}, $d_Y(z)$, V, V_W, and q^* be defined as in the proof of Corollary 6.2. For the Y equation, let $V_Y \equiv (Y, V_X, Z)$ and define α^* and β^* such that

$$\beta^*(V,V_Y) = \tilde{\beta}(V_X) \text{ where } \tilde{\beta}(\nu^*(t)) = \begin{cases} \frac{\frac{\partial}{\partial z}E(Y-Wd(z)|Z=t)}{\frac{\partial}{\partial z}E(X|Z=t)} & \text{if } \frac{\partial}{\partial z}E(X|Z=t) \neq 0 \\ 0 & \text{if } \frac{\partial}{\partial z}E(X|Z=t) = 0 \end{cases}, \text{ and}$$

$$\begin{aligned}Y &\equiv \tilde{\beta}(V_X)X + [E(W|Z)d_W^{-1}]d_Y(z) + \{Y - \tilde{\beta}(V_X)X - E(W|Z)d(z)\}\\ &\equiv \beta^*(V,V_Y)X + [Vd_Y(z) + \alpha_1^*(V_Y)] \equiv \beta^*(V,V_Y)X + \alpha^*(V,V_Y)\end{aligned}$$

Note that $E[\alpha_1^*(V_Y)|Z] = 0$ since for all $s \in \mathcal{Z}$ we have:

$$\begin{aligned}E[\tilde{\beta}(V_X)X|Z=s] &= \int_{-\infty}^{\nu^*(s)} \tilde{\beta}(v)f_{V_X}(v)dv = \int_{-\infty}^{s} \tilde{\beta}(\nu^*(t))f_{V_X}(\nu^*(t))\frac{\partial \nu^*(t)}{\partial t}dt\\ &= \int_{-\infty}^{s} \frac{\frac{\partial}{\partial z}E(Y - Wd(z)|Z=t)}{\frac{\partial}{\partial z}E(X|Z=t)}\frac{\partial}{\partial z}E(X|Z=t)dt = E(Y - Wd(z)|Z=s).\end{aligned}$$

It follows that $\alpha^*(V, V_Y)$, $\beta^*(V, V_Y)$, and $q^*(v, V_W)$ satisfy the analogue of condition (14) since:

$$\begin{aligned} E[\alpha^*(v, V_Y)|V, Z] &= E[vd_Y(z) + \alpha_1^*(V_Y)|Z] = vd_Y(z) = E[\alpha^*(v, V_Y)], \\ E[\beta^*(v, V_Y)|V_X, V, Z] &= E[\beta^*(v, V_Y)|V_X] = \tilde{\beta}(V_X) = E[\beta^*(v, V_Y)|V_X], \quad \text{and} \\ E[q^*(v, V_W)|V, Z] &= E[vd_W + V_W|Z] = vd_W = E[q^*(v, V_W)]. \end{aligned}$$

Further, we have that

$$E[\beta^*(V, V_Y)|\nu^*(z) < V_X \leq \nu^*(z^*), Z = z^*] = \frac{\int_{\nu^*(z)}^{\nu^*(z^*)} \tilde{\beta}(t) f_{V_X}(t)dt}{\Pr[\nu^*(z) < V_X \leq \nu^*(z^*)]} = R^{Wald}_{Y.X|Z}(z, z^*) - R^{Wald}_{W.X|Z}(z, z^*)d(z)$$

and

$$E[\beta^*(V, V_Y)|V_X = \nu^*(z), Z = z] = \tilde{\beta}(\nu^*(z)) = \frac{\frac{\partial}{\partial z}E(Y - Wd(z)|Z = z)}{\frac{\partial}{\partial z}E(X|Z = z)} = R^{LIV}_{Y.X|Z}(z) - R^{LIV}_{W.X|Z}(z)d(z).$$

We leave a detailed study of the sharpness of $\mathcal{B}(\mathcal{D}(z))$ under stronger (mean) independence conditions to other work.

Proof of Theorem 7.1 Let $\hat{Q} \equiv diag(\frac{1}{n}\sum_{i=1}^n \tilde{H}_i\tilde{G}_i', \frac{1}{n}\sum_{i=1}^n \tilde{H}_i\tilde{G}_i')$ and $\hat{M} \equiv \frac{1}{n}\sum_{i=1}^n (\tilde{H}_i'\epsilon_{Y.G|H,i}, \tilde{H}_i'\epsilon_{W.G|H,i})'$. By (i) and since $E(\tilde{H}\tilde{G}')$, and thus Q, is finite and nonsingular, $\hat{Q}^{-1}$ exists in probability for all n sufficiently large. The result then obtains from

$$\sqrt{n}((\hat{R}'_{Y.G|H}, \hat{R}'_{W.G|H})' - (R'_{Y.G|H}, R'_{W.G|H})') = \hat{Q}^{-1}\sqrt{n}\hat{M} = (\hat{Q}^{-1} - Q^{-1})\sqrt{n}\hat{M} + Q^{-1}\sqrt{n}\hat{M},$$

since (i) gives $\hat{Q}^{-1} - Q^{-1} = o_p(1)$ and (ii) gives $\sqrt{n}\hat{M} \xrightarrow{d} N(0, \Xi)$, with Ξ finite and positive definite.

Proof of Theorem A.1 (i) By $(i.b)$ we have

$$Cov(Z, Y|S = s) = Cov(Z, X|S = s)\bar{\beta}(s) + Cov(Z, U|S = s)\bar{\delta}_Y(s),$$

and thus, by $(i.a)$,

$$R_{Y.X|Z}(s) = \bar{\beta}(s) + R_{U.X|Z}(s)\bar{\delta}_Y(s).$$

(ii) By $(ii.b)$ we have

$$Cov(Z, W|S = s) = Cov(Z, U|S = s)\bar{\delta}_W(s),$$

and thus, by $(i.a)$ and $(ii.a)$,

$$R_{W.X|Z}(s)\bar{\delta}(s) = R_{U.X|Z}(s)\bar{\delta}_Y(s).$$

Proof of Theorem A.2 (i) By $(i.b)$, we have

$$Cov(Z, Y) = Cov(Z, X)\bar{\beta} + Cov(Z, U)\bar{\delta}_Y.$$

By $(i.c)$ and $(i.a)$, we have

$$R_{Y.X|Z} = \bar{\beta} + Cov(Z,X)^{-1}Cov(Z,U)\bar{\delta}_Y = \bar{\beta} + Cov(Z,X)^{-1}\begin{bmatrix} 0 \\ Cov(Z_2,U) \end{bmatrix}\bar{\delta}_Y.$$

(ii) By $(ii.b)$, we have

$$Cov(Z_2,X_1) = Cov(Z_2,(\bar{\delta}'_{X_1}U + \eta_{X_1})] = Cov(Z_2,U)\bar{\delta}_{X_1}.$$

By $(ii.a)$, it follows that

$$B = Cov(Z,X)^{-1}\begin{bmatrix} 0 \\ Cov(Z_2,X_1) \end{bmatrix}\bar{\delta}_{X_1}^{-1}\bar{\delta}_Y.$$

Proof of Theorem A.3 (i) By $(i.b)$, we have

$$Cov(Z,Y) = Cov(Z,X)\bar{\beta} + Cov(Z,U)\bar{\delta}_Y$$

and $(i.a)$ gives

$$R_{Y.X|Z} = \bar{\beta} + R_{U.X|Z}\bar{\delta}_Y.$$

(ii) By $(ii.b)$, we have

$$Cov(Z_2,X_1) = Cov(Z_2,(\bar{\delta}'_{X_1}U + \eta_{X_1})) = Cov(Z_2,U)\bar{\delta}_{X_1},$$

and $Z_1 = X_1$, $(ii.b)$, and $(ii.c)$ give

$$\begin{aligned} Cov(Z_1,X_2) &= \bar{\delta}'_{X_1}Cov(U,X_2) = \bar{\delta}'_{X_1}Cov(U,(\bar{\delta}'_{X_2}U + \eta_{X_2})) = \bar{\delta}'_{X_1}Var(U)\bar{\delta}_{X_2}, \text{ and} \\ Cov(Z_1,U) &= Cov((\bar{\delta}'_{X_1}U + \eta_{X_1}),U) = \bar{\delta}'_{X_1}Var(U). \end{aligned}$$

By $(i.a)$ and $(ii.a)$, we have

$$B = Cov(Z,X)^{-1}\begin{bmatrix} Cov(Z_1,X_2)\bar{\delta}_{X_2}^{-1}\bar{\delta}_Y \\ Cov(Z_2,X_1)\bar{\delta}_{X_1}^{-1}\bar{\delta}_Y \end{bmatrix}.$$

Proof of Corollary A.4 (i) The result follows from the expression for $\bar{\beta}$ in Theorem A.3. (ii) The result obtains from substituting for $U' = (X'_1 - \eta'_{X_1})\bar{\delta}_{X_1}^{-1}$ in the Y equation. (iii) Recall that $Cov(Z,X)^{-1}$ is given by (e.g. Baltagi, 1999, p. 185):

$$Cov(Z,X)^{-1} = \begin{bmatrix} Cov(Z_1,X_1), & Cov(Z_1,X_{2,3}) \\ Cov(Z_2,X_1), & Cov(Z_2,X_{2,3}) \end{bmatrix}^{-1} = \begin{bmatrix} P_1^{-1}, & -R_{X_{2,3}.X_1|Z_1}P_2^{-1} \\ -R_{X_1.X_{2,3}|Z_2}P_1^{-1}, & P_2^{-1} \end{bmatrix},$$

where

$$\begin{aligned} P_1 &\equiv Cov(Z_1,X_1) - Cov(Z_1,X_{2,3})Cov(Z_2,X_{2,3})^{-1}Cov(Z_2,X_1) = Cov(\epsilon_{Z_1.Z_2|X_{2,3}},X_1) \\ P_2 &\equiv Cov(Z_2,X_{2,3}) - Cov(Z_2,X_1)Cov(Z_1,X_1)^{-1}Cov(Z_1X_{2,3}) = Cov(\epsilon_{Z_2.Z_1|X_1},X_{2,3}). \end{aligned}$$

The result then follows from

$$B = \left[\begin{array}{c} P_1^{-1}Cov(Z_1, X_2)\bar{\delta}_2 - R_{X_{2,3}.X_1|Z_1}P_2^{-1}Cov(Z_2, X_1)\bar{\delta}_1 \\ -R_{X_1.X_{2,3}|Z_2}P_1^{-1}Cov(Z_1, X_2)\bar{\delta}_2 + P_2^{-1}Cov(Z_2, X_1)\bar{\delta}_1 \end{array} \right].$$

Proof of Theorem A.5 $(i.a)$ By conditions B.1$(i.a)$, (6a), and the proof of Theorem 5.1, we have that:

$$\bar{r}(x, U) = E[\ddot{r}(x, U_Y)] + U'\bar{\delta}_Y$$

and

$$B(x, x^*) = E[\bar{r}(x, U)|X = x^*] - E[\bar{r}(x, U)|X = x] = R^N_{U.X}(x, x^*)\bar{\delta}_Y.$$

$(i.b)$ Similarly, by conditions B.1$(i.e)$ and (7), $\bar{q}(U) = \bar{\alpha}'_W + U'\bar{\delta}_W$ and

$$R^N_{W.X}(x, x^*) = E[\bar{q}(U)|X = x^*] - E[\bar{q}(U)|X = x] = R^N_{U.X}(x, x^*)\bar{\delta}_W,$$

and thus, by $(i.b.1)$,

$$R^N_{W.X}(x, x^*)\bar{\delta} = R^N_{U.X}(x, x^*)\bar{\delta}_Y.$$

$(ii.a)$ By $(ii.a.2)$, $\frac{\partial}{\partial x}E[\ddot{r}(x, U_Y)] = E[\frac{\partial}{\partial x}\ddot{r}(x, U_Y)]$ (see e.g. Corbae, Stinchcombe, and Zeman (2009, Theorem 7.5.17) or Bartle (1966, corollary 5.9)). Then $(i.a)$ and $(ii.a.1)$ yield

$$R^N_{Y.X}(x) = \frac{\partial}{\partial x}E[\bar{r}(x, U)|X = x] = \bar{\beta}(x) + R^N_{U.X}(x)\bar{\delta}_Y.$$

$(ii.b)$ By $(i.b)$ and $(ii.a.1)$, we have

$$R^N_{W.X}(x)\bar{\delta} = R^N_{U.X}(x)\bar{\delta}_Y.$$

Proof of Theorem A.6: $(i.a)$ By conditions B.2$(i.a)$, (9,10), and the proof of Theorem 6.1, we have that

$$\bar{r}(z, U) = E[\ddot{r}(0, U_Y)] + U'\bar{\delta}_Y + E[\mathbf{1}\{U_X \le \nu(z)\}[\ddot{r}(1, U_Y) - \ddot{r}(0, U_Y)]].$$

and, since $R^N_{X.Z}(z, z^*) = \Pr[\nu(z) < U_X \le \nu(z^*)] > 0$,

$$B(\nu(z) < U_X \le \nu(z^*)) = \frac{B_\gamma(z, z^*|z^*)}{R^N_{X.Z}(z, z^*)} = \frac{E[\bar{r}(z, U)|Z = z^*] - E[\bar{r}(z, U)|Z = z]}{R^N_{X.Z}(z, z^*)} = R^{Wald}_{U.X|Z}(z, z^*)\bar{\delta}_Y.$$

$(i.b)$ Similarly, by conditions B.2$(i.e)$ and (11), we have $\bar{q}(U) = \bar{\alpha}'_W + U'\bar{\delta}_W$ and

$$R^{Wald}_{W.X|Z}(z, z^*) = \frac{R^N_{W.Z}(z, z^*)}{R^N_{X.Z}(z, z^*)} = \frac{E[\bar{q}(U)|Z = z^*] - E[\bar{q}(U)|Z = z]}{R^N_{X.Z}(z, z^*)} = R^{Wald}_{U.X|Z}(z, z^*)\bar{\delta}_W,$$

and thus, by $(i.b.1)$,

$$R^{Wald}_{W.X|Z}(z, z^*)\bar{\delta} = R^{Wald}_{U.X|Z}(z, z^*)\bar{\delta}_Y.$$

$(ii.a)$ To characterize $\bar{\beta}(\nu(z))$, note that by $(ii.a.1)$

$$R^N_{Y.Z}(z) = \frac{\partial}{\partial z} E[\bar{r}(z,U)|Z=z] = \frac{\partial}{\partial z} E[\mathbf{1}\{U_X \leq \nu(z)\}[\ddot{r}(1,U_Y) - \ddot{r}(0,U_Y)]] + R^N_{U.Z}(z)\bar{\delta}_Y$$

The result obtains, after division by $R^N_{X.Z}(z) > 0$, since $(ii.a.2)$ and arguments similar to the proof of Theorem 6.1$(ii.a)$ give

$$\frac{\partial}{\partial z} E[\mathbf{1}\{U_X \leq \nu(z)\}[\ddot{r}(1,U_Y) - \ddot{r}(0,U_Y)]] = \bar{\beta}(\nu(z)) \times R^N_{X.Z}(z).$$

$(ii.b)$ The result obtains, after division by $R^N_{X.Z}(z) \neq 0$, since $(i.b)$ and $(ii.a.1)$ give

$$R^N_{W.Z}(z)\bar{\delta} = R^N_{U.Z}(z)\bar{\delta}_Y.$$

Proof of Theorem A.7: $(i.a)$ From the proof of Theorem 5.1, condition (6a) gives

$$B(x, x^*|x^*, s) = E[\bar{r}(x,U)|X = x^*] - E[\bar{r}(x,U)|X = x].$$

The expression for $B(x, x^*|x^*, s)$ follows since for $\ddot{x} = x, x^*$:

$$\begin{aligned}
E[\bar{r}(x,U)|X = \ddot{x}] &= \sum_{h=0}^{L} \bar{r}(x,u_h) f_{U|X}(u_h|\ddot{x}) \\
&= \bar{r}(x,u_0)[1 - \sum_{h=1}^{L} f_{U|X}(u_h|\ddot{x})] + \sum_{h=1}^{L} \bar{r}(x,u_h) f_{U|X}(u_h|\ddot{x}) \\
&= \bar{r}(x,u_0) + \sum_{h=1}^{L} f_{U|X}(u_h|\ddot{x})[\bar{r}(x,u_h) - \bar{r}(x,u_0)] \\
&= \bar{r}(x,u_0) + \sum_{h=1}^{L} f_{U|X}(u_h|\ddot{x})[\sum_{g=1}^{h} \bar{r}(x,u_g) - \bar{r}(x,u_{g-1})] \\
&= \bar{r}(x,u_0) + \sum_{g=1}^{L} [\bar{r}(x,u_g) - \bar{r}(x,u_{g-1})] \sum_{h=g}^{L} f_{U|X}(u_h|\ddot{x}) \\
&= \bar{r}(x,u_0) + \sum_{g=1}^{L} [\bar{r}(x,u_g) - \bar{r}(x,u_{g-1})][1 - F_{U|X}(u_{g-1}|\ddot{x})].
\end{aligned}$$

$(i.b)$ A similar derivation gives the expression for $R^N_{W|X}(x, x^*|x^*)$.

$(ii.a)$ From the proof of Theorem 5.1, condition (6a) gives

$$R^N_{Y.X}(x) = \frac{\partial}{\partial x} E[r(x,U,U_Y)|X = x] = \frac{\partial}{\partial x} E[\bar{r}(x,U)|X = x].$$

Since $\frac{\partial}{\partial x}\bar{r}(x,u_g)$ (by B.1(ii.c)) and $\frac{\partial}{\partial x} f_{U|X}(u_g|x)$ exist and are finite for all $u_g \in \{u_0, u_1, ..., u_L\}$,

$$R^N_{Y.X}(x) = \sum_{g=0}^{L} [\frac{\partial}{\partial x}\bar{r}(x,u_g)] f_{U|X}(u_g|x) + \sum_{g=0}^{L} \bar{r}(x,u_g)[\frac{\partial}{\partial x} f_{U|X}(u_g|x)] = \bar{\beta}(x|x) + B(x|x),$$

where the last equality makes use of conditions B.1(ii.c) and (6b). Further,

$$
\begin{aligned}
B(x|x) &= \sum_{h=0}^{L} \bar{r}(x,u_h)\frac{\partial}{\partial x} f_{U|X}(u_h|x) \\
&= \bar{r}(x,u_0)\frac{\partial}{\partial x}[1-\sum_{h=1}^{L} f_{U|X}(u_h|x)] + \sum_{h=1}^{L} \bar{r}(x,u_h)\frac{\partial}{\partial x} f_{U|X}(u_h|x) \\
&= \sum_{h=1}^{L} \frac{\partial}{\partial x} f_{U|X}(u_h|x)[\bar{r}(x,u_h)-\bar{r}(x,u_0)] \\
&= \sum_{h=1}^{L} \frac{\partial}{\partial x} f_{U|X}(u_h|x)[\sum_{g=1}^{h} \bar{r}(x,u_g)-\bar{r}(x,u_{g-1})] \\
&= \sum_{g=1}^{L}[\bar{r}(x,u_g)-\bar{r}(x,u_{g-1})]\sum_{h=g}^{L} \frac{\partial}{\partial x} f_{U|X}(u_h|x) \\
&= -\sum_{g=1}^{L} \bar{\delta}_Y(u_{g-1},u_g;x)\frac{\partial}{\partial x} F_{U|X}(u_{g-1}|x).
\end{aligned}
$$

$(ii.b)$ A similar derivation gives the expression for $R^N_{W|X}(x|x)$.

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