Supplementary Material

# Supplement to "A discrete choice model for partially ordered alternatives"

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Section S1 of this supplement provides derivation of formulae used to compute price elasticities in the empirical application in Section 6 of the main text Aristodemou and Rosen (2022). Section S2 provides estimates of features of the distribution of elasticities using simulated data from DGP2 as described in Appendix D of Aristodemou and Rosen (2022).

## S1. Derivation of elasticities

The elasticity of the quantity sold of brand $b$'s product of quality $y$ with respect to the price $p_{k\ell}$ of product $(k,\ell)$ is

$$\eta_{bykl} = \frac{\partial \wp_{by}}{\partial p_{k\ell}} \frac{p_{k\ell}}{\wp_{by}}.$$

We consider elasticities conditional on $Z = z$. Prices $p_{k\ell}$ can be plugged in directly and $\wp_{by} = \wp_{by}(z;\zeta)$ is given in (5.3) in the text. It is then additionally necessary to compute derivatives $\frac{\partial \wp_{by}(z;\zeta)}{\partial p_{k\ell}}$.

For this, we start with (5.3) with the function $\Delta$ the difference of bivariate normal CDF values defined in (5.4) and parameters $m_1^-$, $m_2^-$, $m_1^+$, $m_2^+$ defined in (5.6) and (5.5),

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respectively. Notation

$$g_{b,y} \equiv g_b(y; z, \theta), \qquad \tilde{g}_{b,y} \equiv \sigma_b^{-1} g_{b,y}, \qquad \tilde{z}^*_{by} \equiv \sigma_b^{-1} z^*_{by}$$

will also be used. There are three cases to consider, as follows, where $\Delta_j$ is used to denote the partial derivative of $\Delta$ with respect to its $j_{th}$ argument, $j = 1, \ldots, 4$. The expressions below additionally make use of the fact that for any $(k, \ell)$: $\frac{\partial m_2^-}{\partial p_{k\ell}} = \frac{\partial m_2^+}{\partial p_{k\ell}} = 0$:

1. $g_{b,y} < z^*_{by} < g_{b,y+1}$.

$$\wp_{by}(z, \zeta) = \Delta\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \Delta\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big).$$

Referring back to equations (5.1), (A.6), and (A.11) in the proof of Proposition 2, this can be equivalently written

$$\wp_{by}(z, \zeta) = \frac{1}{\sigma_b} \int_{g_b(y;z,\theta)}^{z^*_{by}} \Phi\left( \frac{h_b(y, z, v, \theta) - \rho \frac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b}\right) dv$$

$$+ \frac{1}{\sigma_b} \int_{z^*_{by}}^{g_b(y+1;z,\theta)} \Phi\left( \frac{h_b(y, z, v, \theta) - \rho \frac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b}\right) dv,$$

where it follows from steps in the proof of Proposition 2 that

$$\Delta\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) = \frac{1}{\sigma_b} \int_{g_b(y;z,\theta)}^{z^*_{by}} \Phi\left( \frac{h_b(y, z, v, \theta) - \rho \frac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b}\right) dv,$$

and

$$\Delta\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) = \frac{1}{\sigma_b} \int_{z^*_{by}}^{g_b(y+1;z,\theta)} \Phi\left( \frac{h_b(y, z, v, \theta) - \rho \frac{\sigma_d}{\sigma_b} v}{\sigma_d \sqrt{1-\rho^2}} \right) \phi\left(\frac{v}{\sigma_b}\right) dv.$$

Consequently, we see that

$$\frac{\partial \wp_{by}(z, \zeta)}{\partial z^*_{by}} = \Delta_2\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \Delta_1\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) = 0$$

and, therefore,

$$\frac{\partial \wp_{by}(z, \zeta)}{\partial p_{k\ell}} = \left\{ \begin{array}{c} \sigma_b^{-1}\left( \frac{\partial g_{b,y}}{\partial p_{k\ell}} \Delta_1\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \frac{\partial z^*_{by}}{\partial p_{k\ell}} \Delta_2\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) \right) \\ + \sigma_b^{-1}\left( \frac{\partial z^*_{by}}{\partial p_{k\ell}} \Delta_1\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \right. \\ \left. + \frac{\partial g_{b,y+1}}{\partial p_{k\ell}} \Delta_2\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \right) \\ + \frac{\partial m_1^-}{\partial p_{k\ell}} \Delta_3\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \frac{\partial m_1^+}{\partial p_{k\ell}} \Delta_3\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \end{array} \right\}$$

$$= \begin{Bmatrix} \sigma_b^{-1}\left(\dfrac{\partial g_{b,y}}{\partial p_{k\ell}}\Delta_1\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \dfrac{\partial g_{b,y+1}}{\partial p_{k\ell}}\Delta_2\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big)\right) \\ +\dfrac{\partial m_1^-}{\partial p_{k\ell}}\Delta_3\big(\tilde{g}_{b,y}, \tilde{z}^*_{by}, m_1^-, m_2^-\big) + \dfrac{\partial m_1^+}{\partial p_{k\ell}}\Delta_3\big(\tilde{z}^*_{by}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \end{Bmatrix}$$

2. $g_{b,y} \le g_{b,y+1} \le z^*_{by}$. Using similar arguments as in case 1 above,

$$\wp_{by}(z, \zeta) = \Delta\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big)$$
$$\Longrightarrow$$
$$\frac{\partial \wp_{by}(z, \zeta)}{\partial p_{k\ell}} = \begin{Bmatrix} \sigma_b^{-1}\left(\dfrac{\partial g_{b,y}}{\partial p_{k\ell}}\Delta_1\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big)\right. \\ \left. + \dfrac{\partial g_{b,y+1}}{\partial p_{k\ell}}\Delta_2\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big)\right) \\ + \dfrac{\partial m_1^-}{\partial p_{k\ell}}\Delta_3\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^-, m_2^-\big) \end{Bmatrix}$$

3. $z^*_{by} \le g_{b,y} \le g_{b,y+1}$. Again, following similar steps as in case 1, it follows that:

$$\wp_{by}(z, \zeta) = \Delta\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big)$$
$$\Longrightarrow$$
$$\frac{\partial \wp_{by}(z, \zeta)}{\partial p_{k\ell}} = \begin{Bmatrix} \sigma_b^{-1}\left(\dfrac{\partial g_{b,y}}{\partial p_{k\ell}}\Delta_1\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big)\right. \\ \left. + \dfrac{\partial g_{b,y+1}}{\partial p_{k\ell}}\Delta_2\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big)\right) \\ + \dfrac{\partial m_1^+}{\partial p_{k\ell}}\Delta_3\big(\tilde{g}_{b,y}, \tilde{g}_{b,y+1}, m_1^+, m_2^+\big) \end{Bmatrix}$$

Thus to compute these derivatives at a given value of parameters and a given value of observed variables, the following expressions (arguments suppressed) are needed:

$$\frac{\partial g_{b,y}}{\partial p_{k\ell}}, \qquad \frac{\partial g_{b,y+1}}{\partial p_{k\ell}}, \qquad \frac{\partial m_1^-}{\partial p_{k\ell}}, \qquad \frac{\partial m_1^+}{\partial p_{k\ell}}, \qquad \Delta_1, \Delta_2, \Delta_3.$$

Since $g_{b,y} = g_b(y; z, \theta) = \lambda_{b,y} - x_b\beta_b$ from (4.8), we have that

$$\frac{\partial g_{b,y}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b,y}}{\partial p_{k\ell}}, \qquad \frac{\partial g_{b,y+1}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b,y+1}}{\partial p_{k\ell}}.$$

In our application, with the linear specification $\alpha_{by} = \delta_b + \gamma_b p_{by}$, we have that at the estimated parameter vector $\alpha_{b2} > 2\alpha_{b1}$ for all observations. Under this inequality,

$$\lambda_{b1} = \alpha_{b1}, \qquad \lambda_{b2} = \alpha_{b2} - \alpha_{b1},$$

and it follows that

$$\frac{\partial g_{b,1}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b1}}{\partial p_{k\ell}} = 1[b = k, \ell = 1] \cdot \gamma_b,$$

$$\frac{\partial g_{b,2}}{\partial p_{k\ell}} = \frac{\partial \lambda_{b2}}{\partial p_{k\ell}} = \big(1[b=k,\ \ell=2] - 1[b=k,\ \ell=1]\big)\cdot \gamma_b,$$

If there were observations for which instead $\alpha_{b2} \le 2\alpha_{b1}$, then for these observations we would have

$$\lambda_{b1} = \lambda_{b2} = \alpha_{b2}/2,$$

and given the linear specification $\alpha_{by} = \delta_b + \gamma_b p_{by}$, for each $y \in \{1, 2\}$,

$$\frac{\partial g_{b,y}}{\partial p_{k\ell}} = \frac{\partial \lambda_{by}}{\partial p_{k\ell}} = 1[b=k,\ \ell=2]\cdot \frac{\gamma_b}{2}.$$

Now consider $\frac{\partial z^*_{by}}{\partial p_{k\ell}}$. Recall that $z^*_{by} \equiv \frac{\alpha_{d2}+\alpha_{by}-2\alpha_{d1}}{y} - x_b\beta_b$ and, therefore, for $y \in \{1, 2\}$ and $\tilde{y} = 3 - y$:

$$\frac{\partial z^*_{by}}{\partial p_{by}} = \frac{\gamma_b}{y}, \qquad \frac{\partial z^*_{by}}{\partial p_{d1}} = \frac{-2\gamma_d}{y}, \qquad \frac{\partial z^*_{by}}{\partial p_{d2}} = \frac{\gamma_d}{y}, \qquad \frac{\partial z^*_{by}}{\partial p_{b\tilde{y}}} = 0.$$

The variables $m_1^-$ and $m_1^+$ are defined in (5.6) and (5.5). Applying the linear specification $\alpha_{by} = \delta_b + \gamma_b p_{by}$ and making explicit dependence on $(b, y)$ yields

$$m_1^-(b, y) \equiv \frac{y x_b\beta_b + \delta_d + \gamma_d p_{d1} - \delta_b - \gamma_b p_{by} - x_d\beta_d}{\sqrt{\sigma_b^2 y^2 - 2\rho\sigma_b\sigma_d y + \sigma_d^2}}, \tag{S1}$$

$$m_1^+(b, y) \equiv \frac{y x_b\beta_b + \delta_d + \gamma_d p_{d2} - \delta_b - \gamma_b p_{by} - 2x_d\beta_d}{\sqrt{\sigma_b^2 y^2 - 4\rho\sigma_b\sigma_d y + 4\sigma_d^2}}. \tag{S2}$$

Thus, for $\tilde{y} \ne y$,

$$\frac{\partial m_1^-(b,y)}{\partial p_{by}} = \frac{-\gamma_b}{\sqrt{\sigma_b^2 y^2 - 2\rho\sigma_b\sigma_d y + \sigma_d^2}}, \quad \frac{\partial m_1^-(b,y)}{\partial p_{b\tilde{y}}} = 0,$$

$$\frac{\partial m_1^-(b,y)}{\partial p_{d1}} = \frac{\gamma_d}{\sqrt{\sigma_b^2 y^2 - 2\rho\sigma_b\sigma_d y + \sigma_d^2}}, \quad \frac{\partial m_1^-(b,y)}{\partial p_{d2}} = 0,$$

$$\frac{\partial m_1^+(b,y)}{\partial p_{by}} = \frac{-\gamma_b}{\sqrt{\sigma_b^2 y^2 - 4\rho\sigma_b\sigma_d y + 4\sigma_d^2}}, \quad \frac{\partial m_1^+(b,y)}{\partial p_{b\tilde{y}}} = 0,$$

$$\frac{\partial m_1^+(b,y)}{\partial p_{d2}} = \frac{\gamma_d}{\sqrt{\sigma_b^2 y^2 - 4\rho\sigma_b\sigma_d y + 4\sigma_d^2}}, \quad \frac{\partial m_1^+(b,y)}{\partial p_{d1}} = 0.$$

Finally, there is

$$\Delta(h, k, m_1, m_2) \equiv \Phi_2(k, m_1, m_2) - \Phi_2(h, m_1, m_2),$$

where $\Phi_2(x, y, \rho)$ denotes the CDF of a bivariate normal vector with standard normal marginals and correlation coefficient $\rho$ evaluated at $(x, y)$. Thus the partial derivatives of $\Delta(h, k, m_1, m_2)$ are obtained as

$$\Delta_1(h, k, m_1, m_2) = -\frac{\partial \Phi_2}{\partial h}(h, m_1, m_2),$$
$$\Delta_2(h, k, m_1, m_2) = \frac{\partial \Phi_2}{\partial k}(k, m_1, m_2),$$
$$\Delta_3(h, k, m_1, m_2) = \frac{\partial \Phi_2}{\partial m_1}(k, m_1, m_2) - \frac{\partial \Phi_2}{\partial m_1}(h, m_1, m_2).$$

Expressions for the relevant partial derivatives of the bivariate normal CDF are

$$\frac{\partial \Phi_2}{\partial x}(x, y, \rho) = \phi(x)\Phi\left(\frac{y - \rho x}{\sqrt{1-\rho^2}}\right),$$
$$\frac{\partial \Phi_2}{\partial y}(x, y, \rho) = \phi(y)\Phi\left(\frac{x - \rho y}{\sqrt{1-\rho^2}}\right),$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density and cumulative distribution function, respectively.

## S2. Estimates of elasticities from simulated data

The data used in Section 6.3 of Aristodemou and Rosen (2022) is not publicly available, but code used to estimate features of the distribution of household elasticities is included in our replication code. In order to demonstrate usage, the file Compute_POR_Elasticities.R was used to compute quantities reported in Table 6 of the main text instead using data simulated from DGP2, which is described in Appendix D.[S1] The resulting estimates using 2000 observations generated from DGP2 are reported here in Table S1.

[S1] In DGP2 prices have bounded support on $\mathbb{R}_+$ and higher-quality products are priced higher than lower-quality products.

Table S1. Estimated means and 0.2, 0.5, and 0.8 quantiles of household elasticities using a sample of 2000 observations from DGP2.

| | Price elasticities | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\partial \log \wp_{by}}{\partial \log p_{k\ell}}$ | | $p_{11}$ | | | $p_{12}$ | | | $p_{21}$ | | | $p_{22}$ | |
| $\wp_{11}$ | | | | | | | | | | | | |
| Mean | | −4.764 | | | 2.984 | | | 0.782 | | | 0.423 | |
| Quantiles | −6.966 | −3.768 | −2.298 | 1.355 | 2.428 | 4.389 | 0.338 | 0.631 | 1.171 | 0.080 | 0.374 | 0.728 |
| $\wp_{12}$ | | | | | | | | | | | | |
| Mean | | 1.888 | | | −4.122 | | | 0.014 | | | 0.489 | |
| Quantiles | 0.991 | 1.780 | 2.523 | −5.300 | −3.790 | −2.630 | 0.000 | 0.000 | 0.000 | 0.256 | 0.462 | 0.707 |
| $\wp_{21}$ | | | | | | | | | | | | |
| Mean | | 1.263 | | | 0.024 | | | −7.106 | | | 5.017 | |
| Quantiles | 0.865 | 1.252 | 1.632 | 0.000 | 0.000 | 0.000 | −10.167 | −5.761 | −3.773 | 2.620 | 4.302 | 7.256 |
| $\wp_{22}$ | | | | | | | | | | | | |
| Mean | | 0.236 | | | 0.402 | | | 1.464 | | | −3.161 | |
| Quantiles | 0.051 | 0.216 | 0.395 | 0.204 | 0.379 | 0.589 | 0.674 | 1.331 | 2.098 | −4.284 | −2.845 | −1.849 |
| $\wp_{0}$ | | | | | | | | | | | | |
| Mean | | 1.086 | | | 0.000 | | | 1.329 | | | 0.000 | |
| Quantiles | 0.707 | 1.067 | 1.466 | 0.000 | 0.000 | 0.000 | 0.807 | 1.377 | 1.848 | 0.000 | 0.000 | 0.000 |

## References

Aristodemou, Eleni and Adam M. Rosen (2022), "A discrete choice model for partially ordered alternatives." *Quantitative Economics*, 13 (3), 863–906. [1, 5]

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