Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL INATTENTION TO THE STOCK MARKET WITH INFORMATION COSTS AND TRANSACTIONS COSTS"

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APPENDIX A

PROOF OF LEMMA 1: Since $e^{-\rho\tau_i}A(t_i,\tau_i)=\kappa b(\tau_i)\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}\,dt$, we have

$$\lim_{\tau_i\to 0}\tau_i b(\tau_i)=\lim_{\tau_i\to 0}\frac{e^{-\rho\tau_i}A(t_i,\tau_i)}{\dfrac{\kappa}{\tau_i}\displaystyle\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}\,dt}. \tag{A.1}$$

Equation (9a) states that the numerator on the right hand side of (A.1) has a positive finite limit as $\tau_i\to 0$. The limit of the denominator is $\lim_{\tau_i\to 0}\frac{\kappa}{\tau_i}\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}\,dt=\kappa c_{t_i^+}^{1-\alpha}$, which is positive and finite since we are confining attention to cases with positive (and finite) consumption. Therefore, statement (ii) holds.[31] Statement (iii) follows from the fact that $e^{-\rho\tau_i}A(t_i,\tau_i)=\kappa b(\tau_i)\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}\,dt$ and (9b) along with the assumptions that $\kappa>0$ and $c_t>0$.

Equation (11) and $\kappa>0$ can be used to rewrite (9c) as

$$\begin{aligned} &b(\tau_i)\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}\,dt+e^{-\rho\tau_i}b(\tau_{i+1})\int_{t_{i+1}}^{t_{i+1}+\tau_{i+1}}c_t^{1-\alpha}e^{-\rho(t-t_{i+1})}\,dt\\ &\quad> b(\tau_i+\tau_{i+1})\int_{t_i}^{t_i+\tau_i+\tau_{i+1}}c_t^{1-\alpha}e^{-\rho(t-t_i)}\,dt.\end{aligned} \tag{A.2}$$

To see the implications of (A.2) for $b(\tau_i)$, we first state the following lemma.

LEMMA 3: *Suppose $q_1b(z_1)+q_2b(z_2)>(q_1+q_2)b(z_1+z_2)$ for all positive q_i and z_i, $i=1,2$, and that $b(z)>0$ for all $z>0$. Then $b(z)$ is nonincreasing.*

PROOF: The assumption that $q_1b(z_1)+q_2b(z_2)>(q_1+q_2)b(z_1+z_2)$ for all positive q_i and z_i, $i=1,2$, implies that $q_1[b(z_1)-b(z_1+z_2)]+q_2[b(z_2)-$

[31] Let $\gamma=\lim_{\tau\to 0}\tau b(\tau)=\lim_{\tau\to 0}\frac{\tau}{1/b(\tau)}$, which, by L'Hopital's rule (and assuming that the derivative of $1/b(\tau)$ exists and is non-zero in a neighborhood of $\tau=0$) implies $\gamma=\frac{1}{\lim_{\tau\to 0}-b'(\tau)/b[(\tau)]^2}$ or $\lim_{\tau\to 0}\frac{b'(\tau)}{b[(\tau)]^2}=-\gamma^{-1}$. Then $\lim_{\tau\to 0}\frac{\tau b'(\tau)}{b(\tau)}=\lim_{\tau\to 0}\frac{\tau b(\tau)b'(\tau)}{[b(\tau)]^2}=[\lim_{\tau\to 0}\tau b(\tau)][\lim_{\tau\to 0}\frac{b'(\tau)}{[b(\tau)]^2}]=\gamma(-\gamma^{-1})=-1$.

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$b(z_1+z_2)] > 0$ for all positive q_i and z_i, $i = 1, 2$. Suppose that, contrary to what is to be proved, for some positive z_1 and z_2, $b(z_1) < b(z_1 + z_2)$. Then for any $q_1 > -q_2 \frac{b(z_2)-b(z_1+z_2)}{b(z_1)-b(z_1+z_2)}$, $q_1[b(z_1) - b(z_1+z_2)] + q_2[b(z_2) - b(z_1+z_2)] < 0$, which is a contradiction. Therefore, $b(z_1) \geq b(z_1 + z_2)$ for any positive z_1 and z_2. *Q.E.D.*

Applying Lemma 3 to (A.2) while setting $z_1 = \tau_i$, $z_2 = \tau_{i+1}$, $q_1 = \int_{t_i}^{t_i+\tau_i} c_t^{1-\alpha} \times e^{-\rho(t-t_i)}\,dt$, and $q_2 = e^{-\rho\tau_i} \int_{t_{i+1}}^{t_{i+1}+\tau_{i+1}} c_t^{1-\alpha} e^{-\rho(t-t_i)}\,dt$, implies that $b(\tau)$ is nonincreasing, which is statement (i) in Lemma 1. *Q.E.D.*

PROOF OF PROPOSITION 1: We start by proving the following lemma.

LEMMA 4: *Optimal behavior requires $y^s y^b = 0$. If the optimal asset transfer increases x, then $y^s < 0$. If the optimal transfer decreases x, then $y^b > 0$.*

PROOF: To prove that $y^s y^b = 0$, suppose $y^s y^b \neq 0$, which implies that $y^s < 0$ and $y^b > 0$. Now consider reducing y^b by $\varepsilon > 0$ and increasing y^s by $\varepsilon > 0$, which will have no effect on the value of S relative to the original transfer, but will increase X by $(\psi_s + \psi_b)\varepsilon > 0$ relative to the original transfer by reducing the amount of proportional transactions cost incurred. Therefore, it could not have been optimal for $y^s y^b \neq 0$. Hence, $y^s y^b = 0$.

The value function $V(X, S)$ is strictly increasing in X and S, so an optimal transfer will never decrease both X and S. Therefore, if the optimal transfer increases $x \equiv \frac{X}{S}$, then the optimal transfer cannot decrease X and must decrease S, which implies that $y^b = 0$ and $y^s < 0$. Similarly, if the optimal transfer decreases $x \equiv \frac{X}{S}$, then the optimal transfer cannot decrease S and must decrease X, which implies that $y^s = 0$ and $y^b > 0$. *Q.E.D.*

Proof of statement (ii)(a). Suppose that $x < \omega_1$. The definition of ω_1 in (25) implies that $v(x) \neq \widetilde{v}(x)$. The optimal asset transfer will change the value of x to some value z for which $v(z) = \widetilde{v}(z)$. The definition of ω_1 implies that such a z cannot be less than ω_1, so the optimal transfer increases x. Lemma 4 implies that $y^s < 0$.

Proof of statement (ii)(b). Suppose that on an observation date normalized to be $t = 0$, $X_0 < \omega_1 S_0$. Statement (ii)(a) implies that $y^s < 0$. Let (X^*, S^*) be the value of (X_{0+}, S_{0+}) resulting from the optimal value of y^s. Define $P \equiv \{(X, S) : X = X^* + (1 - \psi_s)z \text{ and } S = S^* - z \text{ for } z \in (0, S^*)\}$. Because (X^*, S^*) is the result of an optimal transfer of assets from the investment portfolio to the transactions account (and the fixed costs $\theta_X X_0$ and $\theta_S S_0$ have already been paid to reach (X^*, S^*)), there is no $(X^{**}, S^{**}) \in P$ such that $V(x^{**}S^{**}, S^{**}) \geq V(x^*S^*, S^*)$ and $V(x^{**}S^{**}, S^{**}) > \widetilde{V}(x^*S^*, S^*)$. [If there were such a (X^{**}, S^{**}), then either (a) $V(x^{**}S^{**}, S^{**}) > V(x^*S^*, S^*)$ or (b) $V(x^{**}S^{**}, S^{**}) = V(x^*S^*, S^*)$. If (a) holds, then (X^*, S^*) is not optimal. If (b) holds, then $V(x^*S^*, S^*) > \widetilde{V}(x^*S^*, S^*)$ and hence it cannot be optimal to remain at (X^*, S^*).] Now suppose that $x^* < \pi_1$. Then consider $(X^{***}, S^{***}) \in P$ for which $x^{***} \equiv \frac{X^{***}}{S^{***}}$ is between x^* and π_1. The definition of π_1 implies that $V(x^{***}S^{***}, S^{***}) \geq V(x^*S^*, S^*)$ and $V(x^{***}S^{***}, S^{**}) > \widetilde{V}(x^*S^*, S^*)$, which contradicts the statement that there is no $(X^{**}, S^{**}) \in P$ such that $V(x^{**}S^{**}, S^{**}) \geq V(x^*S^*, S^*)$ and $V(x^{**}S^{**}, S^{**}) > \widetilde{V}(x^*S^*, S^*)$. Hence, $x^* < \pi_1$ is not optimal.

Proof of statement (ii)(c). Consider the point (X_0, S_0) with $x_0 \equiv \frac{X_0}{S_0} = \omega_1$ and define D as the set of (X, S) for which $x < \omega_1$ and from which the consumer can instantaneously move to (X_0, S_0) by transferring assets from the investment portfolio to the transactions account. Specifically,

$$\text{(A.3)} \qquad D \equiv \big\{(X, S) \text{ with } X < \omega_1 S: \\ \exists y^s < 0 \text{ for which } (1-\theta_X)X - (1-\psi_s)y^s = X_0 \text{ and} \\ (1-\theta_S)S + y^s = S_0\big\}.$$

Define F as the set of (X, S) for which $x \geq \omega_1$ and to which the consumer can instantaneously move from any point in D by transferring assets from the investment portfolio to the transactions account. Specifically,

$$\text{(A.4)} \qquad F \equiv \big\{(X, S) \text{ with } X \geq \omega_1 S: \\ \exists y^s < 0 \text{ for which } X = X_0 - (1-\psi_s)y^s \text{ and } S = S_0 + y^s \geq 0\big\}.$$

Consider two arbitrary points (X_1, S_1) and (X_2, S_2) in set D. Since $x_1 < \omega_1$ and $x_2 < \omega_1$, the optimal value of y^s will be strictly negative starting from either point. Moreover, y^s must be large enough in absolute value so that the post-transfer value of (X, S) satisfies $x \equiv \frac{X}{S} \geq \omega_1$, because it is always optimal to transfer assets from the investment portfolio to the transactions account from any point in set D. Therefore, the post-transfer value of (X, S) will be an element of set F. Thus, regardless of whether the consumer starts from point (X_1, S_1) or (X_2, S_2), the consumer's choice of asset transfer can be described as choosing $(X^+, S^+) \in F$ to maximize the value function. Therefore, $V(X_1, S_1) = V(X_2, S_2)$, so all of the points in set D lie on the same indifference curve of $V(X, S)$. The slope of this indifference curve is $\frac{dX}{dS} = \frac{dX}{dy^s}\frac{dy^s}{dS} = -(1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$, which proves statement (ii)(c).

Proof of statement (ii)(d). We have shown that if $x < \omega_1$, then $m(x) = (1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$. The expression for $V(X_{t_j}, S_{t_j})$ in (21) can be used to rewrite the marginal rate of substitution, $m(x_{t_j}) \equiv \frac{V_S(X_{t_j}, S_{t_j})}{V_X(X_{t_j}, S_{t_j})}$, as $m(x_{t_j}) = \frac{(1-\alpha)v(x_{t_j})}{v'(x_{t_j})} - x_{t_j}$, so that

$$\text{(A.5)} \qquad \frac{(1-\alpha)v(x)}{v'(x)} - x = (1-\psi_s)\frac{1-\theta_S}{1-\theta_X} \quad \text{for} \quad 0 \le x < \omega_1,$$

which implies

$$\text{(A.6)} \qquad v(x) = \left[\frac{(1-\theta_X)x + (1-\theta_S)(1-\psi_s)}{(1-\theta_X)\omega_1 + (1-\theta_S)(1-\psi_s)}\right]^{1-\alpha} v(\omega_1) \quad \text{for} \quad 0 \le x \le \omega_1.$$

Proof of statement (i). We start by proving the following lemma.

LEMMA 5: *For sufficiently small* $\overline{x} > 0$, $\frac{1}{1-\alpha}\widetilde{v}(x) < \frac{1}{1-\alpha}v(x)$ *for all* $x \in (0, \overline{x})$.

PROOF: Substitute the expression for $U(C(t_j, \tau_j))$ from (16) into the restricted value function in (23) to obtain

$$\text{(A.7)} \qquad \widetilde{V}(X_{t_j}, S_{t_j}) = \max_{C(t_j,\tau_j),\phi_j,\tau_j} \left[1-(1-\alpha)\kappa b(\tau_j)\right]\frac{1}{1-\alpha}\left[h(\tau_j)\right]^{\alpha}\left[C(t_j,\tau_j)\right]^{1-\alpha} + e^{-\rho\tau_j}E_{t_j}\left\{V\left(e^{r_L\tau_j}\left(X_{t_j} - C(t_j,\tau_j)\right), R(t_j,\tau_j)S_{t_j}\right)\right\}.$$

Equation (**) in footnote 18 states that $C(t_j, \tau_j) = h(\tau_j)c_{t_j^+}$, so that

$$\text{(A.8)} \qquad \left[1-(1-\alpha)\kappa b(\tau_j)\right]\frac{1}{1-\alpha}\left[h(\tau_j)\right]^{\alpha}\left[C(t_j,\tau_j)\right]^{1-\alpha} = \frac{1}{1-\alpha}\left[1-(1-\alpha)\kappa b(\tau_j)\right]h(\tau_j)c_{t_j^+}^{1-\alpha}.$$

Substitute (A.8) into (A.7) to obtain

$$\text{(A.9)} \qquad \widetilde{V}(X_{t_j}, S_{t_j}) = \max_{C(t_j,\tau_j),\phi_j,\tau_j} \frac{1}{1-\alpha}\left[1-(1-\alpha)\kappa b(\tau_j)\right]h(\tau_j)c_{t_j^+}^{1-\alpha} + e^{-\rho\tau_j}E_{t_j}\left\{V\left(e^{r_L\tau_j}\left(X_{t_j} - C(t_j,\tau_j)\right), R(t_j,\tau_j)S_{t_j}\right)\right\}.$$

Because the choice of $C(t_j, \tau_j)$ must satisfy the constraint $X_{t_j} - C(t_j, \tau_j) \ge 0$, the partial derivative with respect to $C(t_j, \tau_j)$ of the maximand on the right hand side of (A.7) must be nonnegative. Therefore, differentiation of this maximand with respect to $C(t_j, \tau_j)$ yields

$$\text{(A.10)} \qquad \left[1-(1-\alpha)\kappa b(\tau_j)\right]\left[h(\tau_j)\right]^{\alpha}\left[C(t_j,\tau_j)\right]^{-\alpha} - e^{-(\rho-r_L)\tau_j}E_{t_j}\left\{V_X\left(e^{r_L\tau_j}\left(X_{t_j} - C(t_j,\tau_j)\right), R(t_j,\tau_j)S_{t_j}\right)\right\} \ge 0.$$

Since $V_X(\,) > 0$, $[h(\tau_j)]^\alpha [C(t_j, \tau_j)]^{-\alpha} > 0$, and $e^{-(\rho - r_L)\tau_j} > 0$, (A.10) implies that

$$1 - (1-\alpha)\kappa b\big(\tau_j^*\big) > 0, \tag{A.11}$$

where τ_j^* is the value of τ_j that maximizes the restricted value function. Equation (A.11) implies that we can confine attention to values of τ_j that are greater than $\overline{\tau} \equiv \inf\{\tau > 0 : \kappa(1-\alpha)b(\tau) < 1\}$. If $\alpha > 1$, then $1 - \kappa(1-\alpha)b(\tau_j) > 0$ for any positive value of τ_j, so $\overline{\tau} = 0$. However, if $\alpha < 1$, Lemma 1 implies $\overline{\tau} > 0$.

Now we consider the cases in which $\alpha < 1$ and $\alpha > 1$ separately.

Case I: $\alpha < 1$. When $\alpha < 1$, $\tau^* > \overline{\tau} > 0$. Since $C(t_j, \tau_j) = h(\tau_j)c_{t_j^+}$, then

$$c_{t_j^+} = \frac{C(t_j, \tau_j^*)}{h(\tau_j^*)} < \frac{X_{t_j}}{h(\overline{\tau})}, \tag{A.12}$$

where the inequality follows from the constraint $C(t_j, \tau_j^*) \le X_{t_j}$ and the facts that $h(\tau_j)$ is strictly increasing in τ_j and $\tau_j^* > \overline{\tau}$. Equation (A.12) implies $\lim_{X_{t_j} \to 0} c_{t_j^+} = 0$. Therefore, taking the limits of both sides of (A.9) as $X_{t_j} \to 0$, and using the facts that $0 \le C(t_j, \tau_j^*) \le X_{t_j}$ and $\tau_j^* > \overline{\tau} > 0$ implies

$$\begin{aligned} \lim_{X_{t_j} \to 0} \widetilde{V}(X_{t_j}, S_{t_j}) &= \lim_{X_{t_j} \to 0} e^{-\rho \tau_j^*} E_{t_j}\big\{V\big(0, R\big(t_j, \tau_j^*\big) S_{t_j}\big)\big\} \\ &= \lim_{X_{t_j} \to 0} e^{-\rho \tau_j^*} E_{t_j}\big\{\big[R\big(t_j, \tau_j^*\big)\big]^{1-\alpha}\big\} \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(0). \end{aligned} \tag{A.13}$$

Use (B.9) and the fact that $\tau^* > \overline{\tau}$ to obtain

$$\lim_{X_{t_j} \to 0} \widetilde{V}(X_{t_j}, S_{t_j}) < \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(0) = V(0, S_{t_j}). \tag{A.14}$$

Case II: $\alpha > 1$. We start by showing that optimal $y^s(t_j) < 0$, when $x_{t_j} = 0$. Suppose, contrary to what is to be proved, that it is optimal to set $y^s(t_j) = 0$ when $x_{t_j} = 0$, which implies that $c_t = 0$ for all $t \in [t_j, t_{j+1}]$ and $x_{t_{j+1}} = 0$. In turn, $x_{t_{j+1}} = 0$ implies $c_t = 0$ for all $t \in [t_{j+1}, t_{j+2}]$ and so on ad infinitum. Accordingly, $\frac{1}{1-\alpha} v(0)$ is $-\infty$ when $\alpha > 1$. Clearly, $\frac{1}{1-\alpha} v(0)$ is smaller than the value associated with the policy of setting $y^s(t_j) = -(1-\theta_S)S_{t_j}$, so that $X_{t_j^+} = (1-\psi_s)(1-\theta_S)S_{t_j}$ and then consuming optimally from the transactions account over the infinite future, never incurring any information costs or transactions costs. As we show in (A.26), the value of such a policy is given by $\frac{1}{1-\alpha}\chi^{-\alpha} X_{t_j^+}^{1-\alpha}$, which is finite. Accordingly, the policy of setting $y^s(t_j) = 0$ whenever $x_{t_j} = 0$ cannot be optimal.

We show next that $\lim_{x_{t_j}\to 0} \frac{1}{1-\alpha} v(x_{t_j}) \geq \frac{1}{1-\alpha} v(0)$. Let $x^*_{t_j^+}$ denote the optimal value of $x_{t_j^+}$ associated with the optimal transfer $y^s(t_j)$ when $x_{t_j} = 0$. Value matching implies that $\frac{1}{1-\alpha} v(0) S_{t_j}^{1-\alpha} = \frac{1}{1-\alpha} v(x^*_{t_j^+}) S_{t_j^+}^{1-\alpha}$. Now we will compute the size of the transfer y^s that changes x_t from arbitrary x_{t_j} at time t_j to $x_{t_j^+}$ at time t_j^+. When $y^b = 0$, (4) and (5) imply that

$$x^*_{t_j^+} = \frac{(1-\theta_X)x_{t_j} - (1-\psi_s)\dfrac{y^s}{S_{t_j}}}{(1-\theta_S) + \dfrac{y^s}{S_{t_j}}}.$$

Solving for $\frac{y^s}{S_{t_j}}$ gives

$$\frac{y^s}{S_{t_j}} = \frac{(1-\theta_X)x_{t_j} - (1-\theta_S)x^*_{t_j^+}}{x^*_{t_j^+} + 1 - \psi_s}.$$

Furthermore, when $x_{t_j} = 0$, then $\frac{S_{t_j^+}}{S_{t_j}} = (1-\theta_S) + \frac{y^s}{S_{t_j}} = (1-\theta_s) - \frac{(1-\theta_S)x^*_{t_j^+}}{x^*_{t_j^+}+1-\psi_s} = (1-\theta_s)\frac{1-\psi_s}{x^*_{t_j^+}+1-\psi_s}$ and, accordingly,

$$\text{(A.15)} \qquad \frac{v(0)}{v(x^*_{t_j^+})} = \left((1-\theta_s)\frac{1-\psi_s}{x^*_{t_j^+}+1-\psi_s}\right)^{1-\alpha}.$$

Now take $\varepsilon > 0$ and suppose that $x_{t_j} = \varepsilon$. For sufficiently small $\varepsilon > 0$, set $\frac{y^s}{S_{t_j}} = \frac{(1-\theta_X)\varepsilon - (1-\theta_S)x^*_{t_j^+}}{x^*_{t_j^+}+1-\psi_s}$, which will be negative as ε approaches 0. By construction, this feasible transfer implies that $x_{t_j^+} = x^*_{t_j^+}$. Moreover, $\frac{S_{t_j^+}}{S_{t_j}} = (1-\theta_s) + \frac{(1-\theta_X)\varepsilon - (1-\theta_S)x^*_{t_j^+}}{x^*_{t_j^+}+1-\psi_s} = (1-\theta_s)\frac{1-\psi_s}{x^*_{t_j^+}+1-\psi_s} + (1-\theta_X)\frac{\varepsilon}{x^*_{t_j^+}+1-\psi_s}$. Accordingly,

$$\text{(A.16)} \qquad \frac{1}{1-\alpha} v\big(x^*_{t_j^+}\big)\left[(1-\theta_s)\frac{1-\psi_s}{x^*_{t_j^+}+1-\psi_s} + (1-\theta_X)\frac{\varepsilon}{x^*_{t_j^+}+1-\psi_s}\right]^{1-\alpha} \leq \frac{1}{1-\alpha} v(\varepsilon).$$

Using (A.15) to solve for $v(x^*_{t_j^+})$, substituting the resulting expression inside (A.16), and taking limits on both sides of (A.16) as $\varepsilon = x_{t_j} \to 0$ implies $\lim_{x_{t_j}\to 0} \frac{1}{1-\alpha} v(x_{t_j}) \geq \frac{1}{1-\alpha} v(0)$.

Next we show that $\lim_{x_{t_j}\to 0} v(x_{t_j}) = v(0)$. The proof proceeds by contradiction. Indeed, suppose that $\lim_{x_{t_j}\to 0} \frac{1}{1-\alpha} v(x_{t_j}) > \frac{1}{1-\alpha} v(0)$. Then for any t_j, it cannot be optimal to set $C(t_j, \tau_j) = X_{t_j}$, so that $X_{t_j+1} = 0$. [To see why, suppose otherwise. If it were optimal to set $X_{t_j+1} = 0$, then consider the following deviation: Reduce $C(t_j, \tau_j)$ by an arbitrarily small $\varepsilon > 0$, so that $X_{t_j+1} = e^{r_L \tau_j}\varepsilon$. This deviation is feasible for sufficiently small $\varepsilon > 0$, because $C(t_j, \tau_j) = 0$ can never be optimal when $\alpha > 1$. The deviation changes the value of the program by $\Lambda(\varepsilon) \equiv [1 - (1-\alpha)\kappa b(\tau_j)] \times [U(C(t_j, \tau_j) - \varepsilon) - U(C(t_j, \tau_j))] + e^{-\rho\tau_j} E_{t_j}\{[V(e^{r_L\tau_j}\varepsilon, S_{t_{j+1}}) - V(0, S_{t_{j+1}})]\}$. For given X_{t_j} and τ_j, $\lim_{\varepsilon\to 0}[1 - (1-\alpha)\kappa b(\tau_j)] \times [U(C(t_j, \tau_j) - \varepsilon) - U(C(t_j, \tau_j))] = 0$, so that $\lim_{\varepsilon\to 0} \Lambda(\varepsilon) = e^{-\rho\tau_j} \frac{1}{1-\alpha} \lim_{\varepsilon\to 0} E_{t_j}\{S_{t_{j+1}}^{1-\alpha}[v(\frac{e^{r_L\tau_j}\varepsilon}{S_{t_{j+1}}}) - v(0)]\}$. Since the function $\frac{1}{1-\alpha} v(x_t)$ is increasing in x_t, and $\alpha > 1$, it follows that $v(\frac{e^{r_L\tau_j}\varepsilon}{S_{t_{j+1}}})$ is increasing as ε decreases to 0. Therefore, the monotone convergence theorem, along with the supposition that $\lim_{x_{t_j}\to 0} \frac{1}{1-\alpha} v(x_{t_j}) > \frac{1}{1-\alpha} v(0)$, implies that $\lim_{\varepsilon\to 0} \Lambda(\varepsilon) = e^{-\rho\tau_j} \frac{1}{1-\alpha} E_{t_j}(S_{t_{j+1}}^{1-\alpha}[\lim_{\varepsilon\to 0} v(\frac{e^{r_L\tau_j}\varepsilon}{S_{t_{j+1}}}) - v(0)]) > 0$. Accordingly, there always exists small enough $\varepsilon > 0$, so that the deviation dominates the supposed optimal path, a contradiction.]

Next we show that for any $\delta > 0$, there exists a $z \in (0, \delta)$ such that if $x_{t_j} = z$ on observation date t_j, then $y^s(t_j) < 0$. The proof proceeds by contradiction. Suppose otherwise, that is, suppose that there exists a $\delta > 0$, such that it is optimal to set $y^s = 0$ for all $x_{t_j} \in (0, \delta)$. Now fix $T > 0$, and take $x_{t_j} < \delta$. Let $\bar{t}_{j+1}$ denote the last observation date before $t_j + T$. We will show next that under this (counterfactual) supposition, the discounted sum of the observation costs $\sum_{t_k\in[t_j,\bar{t}_{j+1}]} e^{-\rho(t_k - t_j)}(1-\alpha)\kappa b(\tau_k)U(C(t_k, \tau_k))$ approaches infinity with probability approaching 1 as $x_{t_j} \to 0$.

To start, we note that because $\alpha > 1$, it must be the case that $c_{t_j^+} > 0$. (Otherwise utility would be negatively infinite between t_j^+ and $t_j^+ + \tau_j$, and that would make the value function unboundedly negative.) Since $C(X_{t_j}) = c_{t_j^+} h(\tau_j) < X_{t_j}$, this implies that $\lim_{x_{t_j}\to 0} h(\tau_j) = 0$ or, equivalently, $\lim_{x_{t_j}\to 0} \tau_j = 0$. Now note that $x_{t_{j+1}} < x_{t_j} \frac{e^{r_L\tau_j}}{R(t_j,\tau_j)}$, so that $\lim_{x_{t_j}\to 0} \Pr(x_{t_{j+1}} > \delta) = 0$.

More generally, for any $\varepsilon \in (0, \delta)$, as long as (i) $x_{t_j} < \varepsilon$ and (ii) $x_{t_j} \times \max_{t_k\in[t_j,\bar{t}_j]} \prod_{t_i\in[t_j,t_k]} \frac{e^{r_L\tau_i}}{R(t_i,\tau_i)} < \varepsilon$, it follows that $\max_{t_k\in[t_j,\bar{t}_{j+1}]} x_{t_k} < x_{t_j} \times \max_{t_k\in[t_j,\bar{t}_j]} \prod_{t_i\in[t_j,t_k]} \frac{e^{r_L\tau_i}}{R(t_i,\tau_i)} < \varepsilon$. Next we show that the probability that $\max_{t_k\in[t_j,\bar{t}_{j+1}]} x_{t_k} \leq \varepsilon$ approaches 1 as x_{t_j} approaches 0. Indeed, since $x_{t_j} \times$

$\max_{t_k\in[t_j,\bar{t}_j]}\prod_{t_i\in[t_j,t_k]}\frac{e^{r_L\tau_i}}{R(t_i,\tau_i)}<\varepsilon$ implies that $\max_{t_k\in[t_j,\bar{t}_{j+1}]}x_{t_k}<\varepsilon$, we obtain

$$\begin{aligned}\text{(A.17)}\quad &\Pr\Big(\max_{t_k\in[t_j,\bar{t}_{j+1}]}x_{t_k}>\varepsilon\Big)\\ &<\Pr\Big(x_{t_j}\max_{t_k\in[t_j,\bar{t}_j]}\prod_{t_i\in[t_j,t_k]}\frac{e^{r_L\tau_i}}{R(t_i,\tau_i)}>\varepsilon\Big)\\ &=\Pr\Big(\max_{t_k\in[t_j,\bar{t}_j]}\sum_{t_i\in[t_j,t_k]}\big(r_L\tau_i-\log R(t_i,\tau_i)\big)>\log\varepsilon-\log x_{t_j}\Big).\end{aligned}$$

Before proceeding, we make a few observations. We start by noting that $R(t_i,\tau_i)=\phi_i\frac{P_{t_i+\tau_i}}{P_{t_i}}+(1-\phi_i)e^{r_f\tau_i}=\phi_i e^{(\mu-\sigma^2/2)\tau_i+\sigma\Delta B_{t_{i+1}}}+(1-\phi_i)e^{r_f\tau_i}$, where $\Delta B_{t_{i+1}}\equiv B_{t_i+\tau_i}-B_{t_i}$ denotes the increments of the Brownian motion B_t between $t_i+\tau_i$ and t_i. Since $\mu-\frac{\sigma^2}{2}>r_f$, it follows that $R(t_i,\tau_i)>\phi_i e^{r_f\tau_i+\sigma\Delta B_{t_{i+1}}}+(1-\phi_i)e^{r_f\tau_i}=e^{r_f\tau_i}[\phi_i e^{\sigma\Delta B_{t_{i+1}}}+(1-\phi_i)]$. Therefore, letting $g(y)\equiv\log[\phi_i e^y+(1-\phi_i)]$, we obtain $\log R(t_i,\tau_i)>r_f\tau_i+g(\sigma\Delta B_{t_{i+1}})$, so that $r_L\tau_i-\log R(t_i,\tau_i)<(r_L-r_f)\tau_i-g(\sigma\Delta B_{t_{i+1}})$. Letting $z_{t_{i+1}}\equiv(r_L-r_f)\tau_i-g(\sigma\Delta B_{t_{i+1}})$, it follows that

$$\begin{aligned}\text{(A.18)}\quad &\Pr\Big(\max_{t_k\in[t_j,\bar{t}_j]}\sum_{t_i\in[t_j,t_k]}\big(r_L\tau_i-\log R(t_i,\tau_i)\big)>\log\varepsilon-\log x_{t_j}\Big)\\ &<\Pr\Big(\max_{t_k\in[t_j,\bar{t}_j]}\sum_{t_i\in[t_j,t_k]}z_{t_{i+1}}>\log\varepsilon-\log x_{t_j}\Big).\end{aligned}$$

We next observe that $g(0)=0, g'(y)=\frac{\phi_i e^y}{\phi_i e^y+(1-\phi_i)}\le 1$, and $g''(y)=\frac{\phi_i e^y(1-\phi_i)}{[\phi_i e^y+(1-\phi_i)]^2}\ge 0$. Therefore, if $y>0$, then $g(y)=g(0)+\int_0^y g'(y)\,dy\le y$. By a similar logic, if $y<0$, then $g(y)\ge y$. Accordingly, $y^2\ge g^2(y)$ and also $E(y^2)\ge E(g^2(y))$. Finally, since $g''(y)\ge 0$, Jensen's inequality implies that $E(g(y))\ge g(E(y))$. Accordingly,

$$\begin{aligned}\text{(A.19)}\quad E(z_{t_{i+1}})&=(r_L-r_f)\tau_i-E\big(g(\sigma\Delta B_{t_{i+1}})\big)\le(r_L-r_f)\tau_i-g\big[E(\sigma\Delta B_{t_{i+1}})\big]\\ &=(r_L-r_f)\tau_i<0,\end{aligned}$$

where the last equality in (A.19) follows from $E(\Delta B_{t_{i+1}})=0$ and $g(0)=0$. Now let $Z_{t_{l+1}}\equiv\sum_{t_j\le t_i\le t_l}(z_{t_{i+1}}-E_{t_i}(z_{t_{i+1}}))$. By construction, $Z_{t_{l+1}}$ is a martingale, and Jensen's inequality implies that $|Z_{t_{l+1}}|$ is a nonnegative submartingale.[32] Equation (A.19) implies that $\max_{t_k\in[t_j,\bar{t}_j]}\sum_{t_i\in[t_j,t_k]}z_{t_{i+1}}<\max_{t_k\in[t_j,\bar{t}_j]}Z_{t_{k+1}}\le$

[32] $E_{t_l}|Z_{t_l+1}|=E_{t_l}|Z_{t_l}+z_{t_{l+1}}-E_{t_l}(z_{t_{l+1}})|>|Z_{t_l}+E_{t_l}(z_{t_{l+1}}-z_{t_{l+1}})|=|Z_{t_l}|$.

$\max_{t_k\in[t_j,\bar{t}_j]} |Z_{t_{k+1}}|$ and, therefore,

$$\text{(A.20)}\quad \Pr\Bigl(\max_{t_k\in[t_j,\bar{t}_j]} \sum_{t_i\in[t_j,t_k]} z_{t_{i+1}} > \log\varepsilon - \log x_{t_j}\Bigr) < \Pr\Bigl(\max_{t_k\in[t_j,\bar{t}_j]} |Z_{t_{k+1}}| > \log\varepsilon - \log x_{t_j}\Bigr) \le \frac{E_{t_j}[Z^2_{\bar{t}_{j+1}}]}{(\log\varepsilon - \log x_{t_j})^2},$$

where the last inequality follows from Doob's inequality for submartingales applied to the process $|Z_{t_{l+1}}|$. Since $Z_{t_{l+1}}$ is a martingale,

$$\begin{aligned}\text{(A.21)}\quad E_{t_j}\bigl[Z^2_{\bar{t}_{j+1}}\bigr] &= E_{t_j}\Bigl\{\sum_{t_i\in[t_j,\bar{t}_j]} \bigl(z_{t_{i+1}} - E_{t_i}(z_{t_{i+1}})\bigr)^2\Bigr\} \\ &= E_{t_j}\Bigl\{\sum_{t_i\in[t_j,\bar{t}_j]} E_{t_i}\bigl\{g(\sigma\Delta B_{t_{i+1}}) - E_{t_i}\bigl[g(\sigma\Delta B_{t_{i+1}})\bigr]\bigr\}^2\Bigr\} \\ &= E_{t_j}\Bigl\{\sum_{t_i\in[t_j,\bar{t}_j]} E_{t_i}\bigl[g(\sigma\Delta B_{t_{i+1}})\bigr]^2 - \sum_{t_i\in[t_j,\bar{t}_j]} \bigl[E_{t_i} g(\sigma\Delta B_{t_{i+1}})\bigr]^2\Bigr\} \\ &\le E_{t_j}\Bigl\{\sum_{t_i\in[t_j,\bar{t}_j]} E_{t_i}\bigl[g(\sigma\Delta B_{t_{i+1}})\bigr]^2\Bigr\} \\ &\le E_{t_j}\Bigl\{\sum_{t_i\in[t_j,\bar{t}_j]} (\sigma\Delta B_{t_{i+1}})^2\Bigr\} \\ &\le \sigma^2 T,\end{aligned}$$

where the next to last inequality follows from $g^2(y) \le y^2$ for any y. Equations (A.17), (A.18), (A.20), and (A.21) imply $\lim_{x_{t_j}\to 0} \Pr(\max_{t_k\in[t_j,\bar{t}_{j+1}]} x_{t_k} > \varepsilon) = 0$. Since ε is an arbitrary number in $(0,\delta)$, it can be chosen arbitrarily close to 0. In turn, this implies that for any $t_k \in [t_j, \bar{t}_{j+1}]$, x_{t_k} approaches 0 with probability 1 as x_{t_j} becomes arbitrarily small. Accordingly, the lengths τ_k of all the inattention intervals between t_j and $\bar{t}_{j+1}$ approach 0 with probability approaching 1. Using this result together with (8) and assumption (9a) implies that the discounted sum of the observation costs $\sum_{t_k\in[t_j,\bar{t}_{j+1}]} e^{-\rho(t_k-t_j)}(1 -$

$\alpha)\kappa b(\tau_k)U(C(t_k, \tau_k))$ approaches infinity with probability approaching 1.[33] Accordingly, there cannot exist a $\delta > 0$, such that $y^s = 0$ for all $x_{t_j} < \delta$.

This finding implies that for any $\delta > 0$ (however small), there exists a $z \in (0, \delta)$ such that if $x_{t_j} = z$ on observation date t_j, then optimal $y^s(t_j) < 0$. Accordingly, it is possible to find a set of *positive* values $\mathcal{X} = [x^{(1)}, x^{(2)}, \dots]$ with the properties that (i) $\inf_{x\in\mathcal{X}} \mathcal{X} = 0$ and (ii) if $x_{t_j} \in \mathcal{X}$ on observation date t_j, then $y^s(t_j) < 0$. Now take some $x^{(n)} \in \mathcal{X}$. By definition, if on observation date t_j, $x_{t_j} = x^{(n)}$, then it is optimal to transfer funds from the investment portfolio to the transactions account by setting $y^s(t_j) < 0$. Let $x^{(n^*)}$ denote the associated post-transfer value of $x_{t_j^+}$. Since $\frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(x_{t_j}) = \frac{1}{1-\alpha} S_{t_j^+}^{1-\alpha} v(x_{t_j^+})$, $\frac{S_{t_j^+}}{S_{t_j}} = (1-\theta_S) + \frac{(1-\theta_X)x_{t_j} - (1-\theta_S)x^*_{t_j^+}}{x^*_{t_j^+} + 1 - \psi_s}$, $x_{t_j} = x^{(n)}$, and $x_{t_j^+} = x^{(n^*)}$, we have that

$$\text{(A.22)} \qquad \frac{v(x^{(n)})}{v(x^{(n^*)})} = \left(1 - \theta_S + \frac{(1-\theta_X)x^{(n)} - (1-\theta_S)x^{(n^*)}}{x^{(n^*)} + 1 - \psi_s}\right)^{1-\alpha}.$$

As we established at the beginning of the proof, it is always optimal to set $y^s < 0$ whenever $x_{t_j} = 0$ on an observation date. Let x_0^* denote the optimal post-transfer value of $x_{t_j^+}$ when $x_{t_j} = 0$. Since the consumer can choose any $y^s < 0$, optimality of $x_{t_j^+}$ requires that

$$\text{(A.23)} \qquad \begin{aligned} \frac{1}{1-\alpha} v(0) &= \frac{1}{1-\alpha} v(x_0^*)\left((1-\theta_S)\frac{1-\psi_s}{x_0^* + 1 - \psi_s}\right)^{1-\alpha} \\ &\geq \frac{1}{1-\alpha} v(x)\left((1-\theta_S)\frac{1-\psi_s}{x + 1 - \psi_s}\right)^{1-\alpha} \end{aligned}$$

for any $x > 0$. However, dividing (A.15) by (A.22) implies that

$$\text{(A.24)} \qquad \begin{aligned} &\frac{\frac{1}{1-\alpha} v(0)}{\frac{1}{1-\alpha} v(x^{(n)})} \\ &= \frac{\frac{1}{1-\alpha} v(x_0^*)}{\frac{1}{1-\alpha} v(x^{(n^*)})} \frac{\left((1-\theta_s)\frac{1-\psi_s}{x_0^* + 1 - \psi_s}\right)^{1-\alpha}}{\left(1 - \theta_S + \frac{x^{(n)}(1-\theta_X) - (1-\theta_S)x^{(n^*)}}{x^{(n^*)} + 1 - \psi_s}\right)^{1-\alpha}}. \end{aligned}$$

[33] We note that it would be impossible to set c_t arbitrarily close to infinity for almost all values between t_j and $\bar{t}_{j+1}$, since this would violate the constraint $X_{t_j} > \int_{t_j}^{\bar{t}_{j+1}} e^{-r_L(s-t_j)} c_s \, ds$.

Since $\inf_{x\in\mathcal{X}}\mathcal{X}=0$, it is possible to take the limit as $x^{(n)}\to 0$ on both sides of (A.24). Using the supposition that $\lim_{x^{(n)}\to 0}\frac{1}{1-\alpha}v(x^{(n)})>\frac{1}{1-\alpha}v(0)$ and noting that $\alpha>1$ gives

$$1<\lim_{x^{(n)}\to 0}\frac{\frac{1}{1-\alpha}v(0)}{\frac{1}{1-\alpha}v(x^{(n)})}=\frac{\frac{1}{1-\alpha}v(x_0^*)}{\frac{1}{1-\alpha}v(x^{(n^*)})}\frac{\left((1-\theta_s)\frac{1-\psi_s}{x_0^*+1-\psi_s}\right)^{1-\alpha}}{\left((1-\theta_s)\frac{1-\psi_s}{x^{(n^*)}+1-\psi_s}\right)^{1-\alpha}}. \tag{A.25}$$

The fact that $\alpha>1$ along with (A.25) implies that $\frac{1}{1-\alpha}v(x_0^*)((1-\theta_S)\times\frac{1-\psi_s}{x_0^*+1-\psi_s})^{1-\alpha}<\frac{1}{1-\alpha}v(x^{(n^*)})((1-\theta_s)\frac{1-\psi_s}{x^{(n^*)}+1-\psi_s})^{1-\alpha}$, which contradicts (A.23). Accordingly, $\lim_{x_n\longrightarrow 0}\frac{1}{1-\alpha}v(x_n)=\frac{1}{1-\alpha}v(0)$.

The continuity of the function v in a positive neighborhood of zero, together with the theorem of the maximum, implies the continuity of $\widetilde{v}$ in a positive neighborhood of zero. Moreover, noting that $y^s<0$ when $x_{t_j}=0$ implies that $\frac{1}{1-\alpha}\widetilde{v}(0)\equiv\lim_{x_{t_j}\to 0}\frac{1}{1-\alpha}\widetilde{v}(x_{t_j})<\frac{1}{1-\alpha}v(0)$. *Q.E.D.*

Proof of $\omega_1>0$. Since Lemma 5 implies that $\lim_{x_{t_j}\to 0}\frac{1}{1-\alpha}\widetilde{v}(x_{t_j})<\frac{1}{1-\alpha}v(0)$, there exists $\overline{x}>0$ such that $\frac{1}{1-\alpha}\widetilde{v}(x)<\frac{1}{1-\alpha}v(0)\le\frac{1}{1-\alpha}v(x)\ \forall x\in[0,\overline{x}]$. Therefore, $\omega_1\ge\overline{x}>0$.

Proof of $\pi_2\ge\pi_1$. To prove that $\pi_2\ge\pi_1$, suppose the contrary, that is, that $\pi_1>\pi_2$, and consider three points (X_A,S_A), (X_B,S_B), and (X_C,S_C), where $X_A=\pi_1 S_A$, $(X_B,S_B)=(\pi_1 S_A-(1-\psi_s)z^*,S_A+z^*)$ where $z^*\equiv\frac{\pi_1-\pi_2}{\pi_2+1-\psi_s}S_A$, which implies $X_B=\pi_2 S_B$, and $(X_C,S_C)=(\pi_2 S_B+(1+\psi_b)z^{**},S_B-z^{**})$ where $z^{**}\equiv\frac{\pi_1-\pi_2}{\pi_1+1+\psi_b}S_B$, which implies $X_C=\pi_1 S_C$. The definition of π_1 implies that $V(X_A,S_A)\ge V(X_B,S_B)$ and the definition of π_2 implies that $V(X_B,S_B)\ge V(X_C,S_C)$ so that $V(X_A,S_A)\ge V(X_C,S_C)$. But $S_C=S_B-z^{**}=S_B-\frac{\pi_1-\pi_2}{\pi_1+1+\psi_b}S_B=\frac{\pi_2+1+\psi_b}{\pi_1+1+\psi_b}S_B=\frac{\pi_2+1+\psi_b}{\pi_1+1+\psi_b}\frac{\pi_1+1-\psi_s}{\pi_2+1-\psi_s}S_A=(\frac{(\pi_1-\pi_2)(\psi_s+\psi_b)}{(\pi_1+1+\psi_b)(\pi_2+1-\psi_s)}+1)S_A>S_A$, since $\psi_s+\psi_b>0$. Therefore, since $X_C=\pi_1 S_C$ and $X_A=\pi_1 S_A$, we have $X_C>X_A$. Hence, since $V(X,S)$ is strictly increasing in X and S, we have $V(X_C,S_C)>V(X_A,S_A)$, which contradicts the earlier statement that $V(X_A,S_A)\ge V(X_C,S_C)$.

Proof of $\omega_1\le\pi_1$. We prove this statement using a geometric argument to show that $\omega_1>\pi_1$ leads to a contradiction. We consider three cases: $\theta_S<\theta_X$, $\theta_S>\theta_X$, and $\theta_S=\theta_X$.

Suppose that $\omega_1>\pi_1$ and consider the case in which $\theta_S<\theta_X$, so that in Figure 2(a), the line through points B, C, and E, which has slope $-(1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$, is steeper than the line through points C and D, which has slope $-(1-\psi_s)$. Statement (ii)(c) of Proposition 1 implies that for values of $x\equiv\frac{X}{S}$ less than ω_1, indifference curves of the value function are straight lines with slope $-(1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$. Therefore, $V(B)=V(C)=V(E)$, where the notation $V(j)$ indicates the value

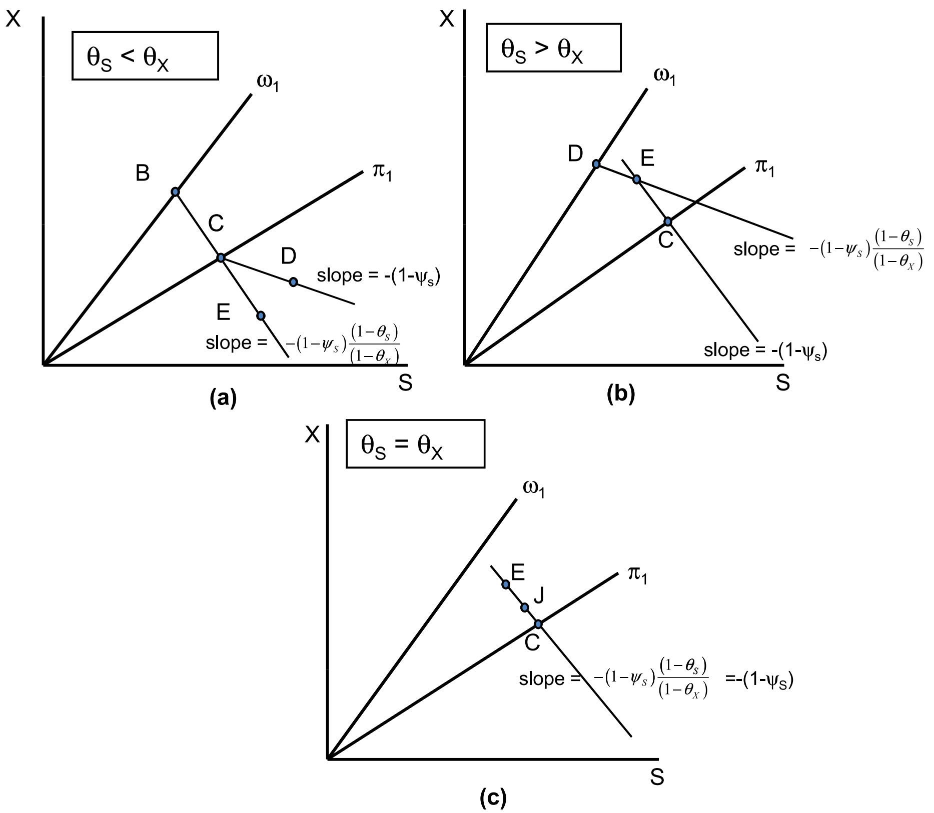

FIGURE 2.—Proof of $\omega_1 \leq \pi_1$.

of the value function evaluated at point j. The definition of π_1 implies that $V(C) \geq V(D)$. Therefore, $V(E) \geq V(D)$, which contradicts strict monotonicity of the value function since both X and S are larger at point D than at point E. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S < \theta_X$.

Suppose that $\omega_1 > \pi_1$ and consider the case in which $\theta_S > \theta_X$, so that in Figure 2(b) the line through points D and E, which has slope $-(1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$, is less steep than the line through points C and E, which has slope $-(1-\psi_s)$. Statement (ii)(c) of Proposition 1 implies that the line from point D through point E is an indifference curve and all points on this indifference curve are preferred to all points below and to the left of the indifference curve for which $x < \omega_1$. In particular, point E is preferred to all points below point E along the line through points E and C. Since the value of x at point E is higher than π_1, the fact that the value function evaluated at point E is greater than the value function, and hence greater than the restricted value function, evaluated at all points below point E with slope $-(1-\psi_s)$ contradicts the definition of π_1. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S > \theta_X$.

Suppose that $\omega_1 > \pi_1$ and consider the case in which $\theta_S = \theta_X$, so that in Figure 2(c), the slope of the line through points C and E is $-(1-\psi_s)\frac{1-\theta_S}{1-\theta_X} = -(1-\psi_s)$. Statement (ii)(c) of Proposition 1 implies that for values of $x \equiv \frac{X}{S} < \omega_1$, indifference curves of the value function are straight lines with slope $-(1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$ so points E and C are on the same indifference curve. Indeed, point E yields the same value of the value function as all points below point E on the line through points E and C. That is, for any point J below point E along the line through points E and C with $X \geq 0$, $V(E) = V(J)$. Since $x < \omega_1$ at point J, the definition of ω_1 implies that $V(J) > \widetilde{V}(J)$. Therefore, $V(E) = V(J) > \widetilde{V}(J)$. Since $x > \pi_1$ at point E, the facts that for arbitrary point J we have $V(E) = V(J)$ and $V(E) > \widetilde{V}(J)$ contradict the definition of π_1. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S = \theta_X$.

Putting together the cases in which $\theta_S < \theta_X$, $\theta_S > \theta_X$, and $\theta_S = \theta_X$, we have proved that $\omega_1 \leq \pi_1$.

To prove $\omega_2 \geq \pi_2$, use a set of arguments similar to the proof that $\omega_1 \leq \pi_1$.

Proof of $\omega_2 < \infty$. We prove that ω_2 is finite by showing that if the investment portfolio has zero value on an observation date, the consumer will use some of the liquid assets in the transactions account to buy assets for the investment portfolio. We use proof by contradiction. That is, suppose that time 0 is an observation date, and that at this observation date, the transactions account has a balance $X_0 > 0$ and the investment portfolio has a zero balance so that $S_0 = 0$ and x_0 is infinite. Suppose that whenever the investment portfolio has zero value on an observation date, the consumer does not transfer any assets to the investment portfolio. Then the consumer will simply consume from the transactions account over the infinite future, never incurring any information costs or transactions costs. In this case, with the values of the variables denoted with asterisks, $c^*_{0^+} = \frac{X_0}{h(\infty)} = \chi X_0$ and $c^*_t = \exp(-\frac{\rho - r_L}{\alpha}t)c^*_{0^+} = \chi X^*_t$, so $X^*_t = \exp(-\frac{\rho - r_L}{\alpha}t)X_0$. Equation (16) implies that lifetime utility is

$$U^* = \frac{1}{1-\alpha}\big[h(\infty)\big]^{\alpha} X_0^{1-\alpha} = \frac{1}{1-\alpha}\chi^{-\alpha} X_0^{1-\alpha}. \quad \text{(A.26)}$$

Now consider an alternative feasible path that sets $c_t = c^*_t$ for $0 < t \leq T$ and at time 0^+ transfers to the investment portfolio any liquid assets in the transactions account that will not be needed to finance consumption until time T. Under this alternative policy, the present value of consumption up to date T is $h(T)c^*_{0^+} = h(T)\chi X_0$, so

$$X_{0^+} = h(T)\chi X_0. \quad \text{(A.27)}$$

The consumer uses $(1-\theta_X-\chi h(T))X_0$ liquid assets to purchase assets in the investment portfolio. After paying the transactions cost,

$$S_{0^+}=\frac{1-\theta_X-\chi h(T)}{1+\psi_b}X_0. \tag{A.28}$$

Suppose that the consumer invests the investment portfolio entirely in the riskless bond. At time T, the transactions account has a zero balance, and the investment portfolio is worth $S_T=\exp(r_f T)\frac{1-\theta_X-\chi h(T)}{1+\psi_b}X_0$. The consumer transfers the entire investment portfolio to the transactions account, so that after paying the transactions costs, the balance in the transactions account is

$$X_{T^+}=(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\exp(r_f T)\big[1-\theta_X-\chi h(T)\big]X_0. \tag{A.29}$$

Define $P\equiv\frac{X_{T+}}{X_T^*}$ as the ratio of the transactions account balance at time T^+ under this alternative policy to the transactions account balance under the initial policy. Use (A.29) and $X_T^*=\exp(-\frac{\rho-r_L}{\alpha}T)X_0$, along with $\chi\equiv\frac{\rho-(1-\alpha)r_L}{\alpha}$, to obtain

$$P\equiv\frac{X_{T^+}}{X_T^*}=(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}F(T), \tag{A.30}$$

where

$$F(T)\equiv\exp\big[(r_f-r_L)T\big]\big[1-\theta_X\exp(\chi T)\big]. \tag{A.31}$$

Equation (A.30) and $X_T^*=\exp(-\frac{\rho-r_L}{\alpha}T)X_0$ imply

$$X_{T^+}=(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}F(T)\exp\left(-\frac{\rho-r_L}{\alpha}T\right)X_0. \tag{A.32}$$

Now choose T to maximize $F(T)$. Differentiate $F(T)$ and set the derivative equal to zero to obtain

$$\exp(-\chi\widehat{T})=\left(1+\frac{\chi}{r_f-r_L}\right)\theta_X<1, \tag{A.33}$$

where $\widehat{T}$ is the optimal value of T, and the inequality follows from the assumption that $\theta_X < \overline{\theta_X}$ and the fact that $\frac{\chi}{r_f - r_L} > 0$.[34] Use (A.33) to evaluate $F(\widehat{T})$ to obtain

$$F(\widehat{T}) = \left(1 + \frac{\chi}{r_f - r_L}\right)^{-1-(r_f - r_L)/\chi} \frac{\chi}{r_f - r_L} \theta_X^{-(r_f - r_L)/\chi}. \tag{A.34}$$

Use (A.33) and the definition of $h(T)$ to obtain

$$\chi h(\widehat{T}) = 1 - \left(1 + \frac{\chi}{r_f - r_L}\right)\theta_X. \tag{A.35}$$

The present value of lifetime utility under the alternative plan is

$$\begin{aligned} U = {} & \left[1 - (1-\alpha)\kappa b(\widehat{T})\right] \frac{1}{1-\alpha} \left[h(\widehat{T})\right]^{\alpha} [X_{0+}]^{1-\alpha} \\ & + \exp(-\rho \widehat{T}) \frac{1}{1-\alpha} \left[h(\infty)\right]^{\alpha} [X_{\widehat{T}+}]^{1-\alpha}. \end{aligned} \tag{A.36}$$

Substitute (A.27) and (A.32) into (A.36), and use the fact that $h(\infty) = \frac{1}{\chi}$ to obtain

$$\begin{aligned} U = {} & \left[1 - (1-\alpha)\kappa b(\widehat{T})\right] \frac{1}{1-\alpha} h(\widehat{T}) [\chi X_0]^{1-\alpha} \\ & + \exp(-\rho \widehat{T}) \frac{1}{1-\alpha} \chi^{-\alpha} \\ & \times \left[(1-\theta_S) \frac{1-\psi_s}{1+\psi_b} F(\widehat{T}) \exp\left(-\frac{\rho - r_L}{\alpha} \widehat{T}\right) X_0\right]^{1-\alpha}. \end{aligned} \tag{A.37}$$

Now divide the utility under the alternative plan in (A.37) by the utility under the initial plan in (A.26), and use the definition of χ and the fact that $\chi h(T) = 1 - \exp(-\chi T)$ to obtain

$$\begin{aligned} \frac{U}{U^*} = {} & \left[1 - (1-\alpha)\kappa b(\widehat{T})\right]\left[1 - \exp(-\chi \widehat{T})\right] \\ & + \exp(-\chi \widehat{T}) \left[(1-\theta_S) \frac{1-\psi_s}{1+\psi_b} F(\widehat{T})\right]^{1-\alpha}. \end{aligned} \tag{A.38}$$

[34]From (27), $\overline{\theta_X} \equiv [(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\frac{\chi}{r_f - r_L + \chi}]^{\chi/(r_f - r_L)} \frac{r_f - r_L}{r_f - r_L + \chi}$, which implies $(1 + \frac{\chi}{r_f - r_L})\overline{\theta_X} = [(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\frac{\chi}{r_f - r_L + \chi}]^{\chi/(r_f - r_L)} < 1$ because $(1-\theta_S)\frac{1-\psi_s}{1+\psi_b} < 1$, $\frac{\chi}{r_f - r_L} > 0$, and hence $\frac{\chi}{r_f - r_L + \chi} < 1$.

Then rearrange to obtain

$$\frac{U}{U^*} = 1 + \left(\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}F(\widehat{T})\right]^{1-\alpha} - \left[1+(1-\alpha)\kappa b(\widehat{T})\big(\exp(\chi\widehat{T})-1\big)\right]\right)\exp(-\chi\widehat{T}). \tag{A.39}$$

If $\alpha < 1$, utility under the alternative plan, U, will exceed U^* if $\frac{U}{U^*} > 1$; if $\alpha > 1$, utility under the alternative plan, U, will exceed U^* if $\frac{U}{U^*} < 1$. A sufficient condition for U to exceed U^*, regardless of whether α is less than or greater than 1, is[35]

$$\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\right]F(\widehat{T}) > \left[1+(1-\alpha)\kappa b(\widehat{T})\big(\exp(\chi\widehat{T})-1\big)\right]^{1/(1-\alpha)}. \tag{A.40}$$

Multiply both sides of (A.34) by $(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}$ to obtain

$$\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\right]F(\widehat{T}) = \left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\frac{\chi}{r_f-r_L+\chi}\right]\left(\frac{r_f-r_L}{r_f-r_L+\chi}\right)^{(r_f-r_L)/\chi}\theta_X^{-(r_f-r_L)/\chi}. \tag{A.41}$$

Use the definition of $\overline{\theta_X}$ in (27) and the assumption that $\theta_X < \overline{\theta_X}$ to write (A.41) as

$$\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\right]F(\widehat{T}) = \left(\frac{\theta_X}{\overline{\theta_X}}\right)^{-(r_f-r_L)/\chi} > 1. \tag{A.42}$$

Substitute (A.42) into (A.40) to obtain the sufficient condition for U to exceed U^*:

$$\left(\frac{\theta_X}{\overline{\theta_X}}\right)^{-(r_f-r_L)/\chi} > \left[1+(1-\alpha)\kappa b(\widehat{T})\big(\exp(\chi\widehat{T})-1\big)\right]^{1/(1-\alpha)}. \tag{A.43}$$

[35]If $\alpha > 1$, then κ must be less than $\widehat{\kappa} \equiv \frac{1}{\alpha-1}\frac{1}{b(\widehat{T})(\exp(\chi\widehat{T})-1)}$ so that the right hand side of (A.40) is defined. Since we assume that $\kappa < \overline{\kappa}$ in (28) and $\widehat{\kappa} = [1-(\frac{\theta_X}{\overline{\theta_X}})^{-((r_f-r_L)/\chi)(1-\alpha)}]^{-1}\overline{\kappa} > \overline{\kappa}$, we have $\kappa < \widehat{\kappa}$.

Regardless of whether α is larger or smaller than 1, the condition in (A.43) is satisfied if $\theta_X < \overline{\theta_X}$ and $\kappa < \overline{\kappa}$, where

$$\overline{\kappa} \equiv \frac{\left(\dfrac{\theta_X}{\overline{\overline{\theta_X}}}\right)^{-((r_f - r_L)/\chi)(1-\alpha)} - 1}{(1-\alpha) b(\widehat{T})(\exp(\chi \widehat{T}) - 1)}. \tag{A.44}$$

Since $\theta_X < \overline{\theta_X}$ and $\kappa < \overline{\kappa}$, the original plan, in which the consumer does not buy any assets in the investment portfolio, is not optimal.

The proof of statement (i) is now complete.

Proof of statement (iii). The proof of statement (iii) follows the proof of statement (ii).

The proof of Proposition 1 is now complete. *Q.E.D.*

To prepare for the proof of Proposition 2, we state and prove the following lemma.

LEMMA 6: *If $C(t_j, \tau_j) \leq X_{t_j}$, then, for sufficiently small $\theta_S \geq 0$, $y^s(t_j) = 0$.*

PROOF: Consider some path for c_t, X_t, S_t, $y^s(t)$, and $y^b(t)$, $t \in [t_j, t_{j+1}]$, and let c_t^0, X_t^0, S_t^0, $y^{s,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that $C(t_j, \tau_t) \leq X_{t_j}^0$ and (contrary to what is to be proved) that $y^{s,0}(t_j) < 0$, so that Lemma 4 implies that $y^{b,0}(t_j) = 0$. Consider a deviation from $y^{s,0}(t_j) < 0$ that reduces $-y^s(t_j)$ to zero so that $X_{t_j^+}$ changes by $y^{s,0}(t_j)(1-\psi_s) + \theta_X X_{t_j}^0$ and $S_{t_j^+}$ increases by $-y^{s,0}(t_j) + \theta_S S_{t_j}^0$. Since under the deviation, $X_{t_j^+} = X_{t_j} = X_{t_j}^0 \geq C(t_j, \tau_j)$, it is feasible to maintain $c_t = c_t^0$ for $t_j \leq t \leq t_{j+1}$, and we suppose that the consumer does so. Also suppose that the consumer invests the additional assets in the investment portfolio in the riskless bond, which pays a rate of return r_f. Thus, at the next observation date t_{j+1}, the transactions account will have changed by $\Delta^X \equiv [y^{s,0}(t_j)(1-\psi_s) + \theta_X X_{t_j}^0]e^{r_L \tau_j}$ and the investment portfolio will have increased by $\Delta^S \equiv [-y^{s,0}(t_j) + \theta_S S_{t_j}^0]e^{r_f \tau_j} > 0$, relative to the original path. The deviation at time t_{j+1} depends on the direction of the transfer along the original path at time t_{j+1}.

(i) If $y^{s,0}(t_{j+1}) < 0$, increase $-y^s(t_{j+1})$ by $(1-\theta_S)\Delta^S$, which makes the value of the investment portfolio under the deviation equal to the value under the original path. Compared to the original path, the transactions account at time t_{j+1}^+ changes by $\xi \equiv (1-\theta_X)\Delta^X + (1-\psi_s)(1-\theta_S)\Delta^S$. Using the definitions of Δ^S and Δ^X implies

$$\begin{aligned}\xi &= [-y^{s,0}(t_j)](1-\psi_s)[(1-\theta_S)e^{r_f \tau_j} - (1-\theta_X)e^{r_L \tau_j}] \\ &\quad + (1-\theta_X)\theta_X X_{t_j}^0 e^{r_L \tau_j} + (1-\psi_s)(1-\theta_S)\theta_S S_{t_j}^0 e^{r_f \tau_j},\end{aligned}$$

which in turn implies that $\lim_{\theta_S\to 0}\xi = [-y^{s,0}(t_j)](1-\psi_s)[e^{r_f\tau_j} - (1-\theta_X)e^{r_L\tau_j}] + (1-\theta_X)\theta_X X^0_{t_j} e^{r_L\tau_j} > 0$.

(ii) If the consumer would not have transferred assets in either direction between the investment portfolio and the transactions account at time t_{j+1}, then $\omega_1 \le x^0_{t_{j+1}} \le \omega_2$. We begin by showing that $\frac{S^0_{t_{j+1}}}{S^0_{t_j}} = \frac{S^0_{t_{j+1}}}{X^0_{t_{j+1}}}(\frac{X^0_{t_{j+1}}}{X^0_{t_j}})(\frac{X^0_{t_j}}{S^0_{t_j}}) = \frac{1}{x^0_{t_{j+1}}}(\frac{X^0_{t_{j+1}}}{X^0_{t_j}})x^0_{t_j}$ is bounded above by a quantity that is finite and F_{t_j}-measurable. First, the fact that $\omega_1 \le x^0_{t_{j+1}} \le \omega_2$ implies that $\frac{1}{x^0_{t_{j+1}}} \le \frac{1}{\omega_1}$, which is finite since $\omega_1 > 0$. Second, $X^0_{t_{j+1}} = [(1-\theta_X)X^0_{t_j} - (1-\psi_s)y^{s,0}(t_j)]e^{r_L\tau_j} - C(t_j,\tau_j)e^{r_L\tau_j}$ so that $\frac{X^0_{t_{j+1}}}{X^0_{t_j}} = [(1-\theta_X) - (1-\psi_s)\frac{y^{s,0}(t_j)}{X^0_{t_j}}]e^{r_L\tau_j} - \frac{C(t_j,\tau_j)}{X^0_{t_j}}e^{r_L\tau_j}$, which is finite and F_{t_j}-measurable. Third, since $-y^{s,0}(t_j) > 0$, we know that $S^0_{t_j} \ge \frac{1}{1-\theta_s}[-y^{s,0}(t_j)] > 0$, which implies that $x^0_{t_j} \equiv \frac{X^0_{t_j}}{S^0_{t_j}}$ is finite; it is also F_{t_j}-measurable. Therefore, $\frac{S^0_{t_{j+1}}}{S^0_{t_j}} = \frac{1}{x^0_{t_{j+1}}}(\frac{X^0_{t_{j+1}}}{X^0_{t_j}})x^0_{t_j}$ is bounded above by $\frac{1}{\omega_1}(\frac{X^0_{t_{j+1}}}{X^0_{t_j}})x^0_{t_j}$, which is the product of three quantities that are finite and F_{t_j}-measurable.

For sufficiently small $\theta_S \ge 0$, the alternative path sets $y^s(t_{j+1})$ equal to $-(1-\theta_S)\Delta^S + \theta_S S^0_{t_{j+1}} = -S^0_{t_j}\{(1-\theta_S)[-\frac{y^{s,0}(t_j)}{S^0_{t_j}}]e^{r_f\tau_j} + \theta_S[(1-\theta_S)e^{r_f\tau_j} - \frac{S^0_{t_{j+1}}}{S^0_{t_j}}]\}$, which is negative because $-\frac{y^{s,0}(t_j)}{S^0_{t_j}} > 0$ and $\frac{S^0_{t_{j+1}}}{S^0_{t_j}}$ is bounded above by an F_{t_j}-measurable quantity. With $y^s(t_{j+1}) = -(1-\theta_S)\Delta^S + \theta_S S^0_{t_{j+1}}$, the value of the investment portfolio on the alternative path equals the value on the hypothesized optimal path. Compared to the hypothesized optimal path, the transactions account at time t^+_{j+1} changes by $\xi_2 \equiv (1-\theta_X)\Delta^X - \theta_X X^0_{t_{j+1}} - (1-\psi_s)[-(1-\theta_S)\Delta^S + \theta_S S^0_{t_{j+1}}]$. Use the definitions of Δ^X and Δ^S to obtain $\xi_2 = (1-\psi_s)[-y^{s,0}(t_j)][(1-\theta_S)e^{r_f\tau_j} - (1-\theta_X)e^{r_L\tau_j}] + \theta_X[(1-\theta_X)X^0_{t_j}e^{r_L\tau_j} - X^0_{t_{j+1}}] + (1-\psi_s)(1-\theta_S)\theta_S S^0_{t_j}e^{r_f\tau_j} - (1-\psi_s)\theta_S S^0_{t_{j+1}}$.

Now use the fact that $X^0_{t_{j+1}} = [(1-\theta_X)X^0_{t_j} - (1-\psi_s)y^{s,0}(t_j)]e^{r_L\tau_j} - C(t_j,\tau_j)\times e^{r_L\tau_j}$ to obtain $(1-\theta_X)X^0_{t_j}e^{r_L\tau_j} - X^0_{t_{j+1}} = (1-\psi_s)y^{s,0}(t_j)e^{r_L\tau_j} + C(t_j,\tau_j)e^{r_L\tau_j}$, substitute this expression into the expression for ξ_2, and factor out $S^0_{t_j}$ to obtain

$$\xi_2 = S^0_{t_j}\Bigg\{(1-\psi_s)\left[\frac{-y^{s,0}(t_j)}{S^0_{t_j}}\right]\left[(1-\theta_S)e^{r_f\tau_j} - e^{r_L\tau_j}\right] + \theta_X\frac{C(t_j,\tau_j)}{S^0_{t_j}}e^{r_L\tau_j} + (1-\psi_s)(1-\theta_S)\theta_S e^{r_f\tau_j} - (1-\psi_s)\theta_S\frac{S^0_{t_{j+1}}}{S^0_{t_j}}\Bigg\}.$$

Since $\frac{S^0_{t_{j+1}}}{S^0_{t_j}}$ is bounded above by a quantity that is F_{t_j}-measurable and finite, $\lim_{\theta_S\to 0}\xi_2 = S^0_{t_j}\{(1-\psi_s)[\frac{-y^{s,0}(t_j)}{S^0_{t_j}}][e^{r_f\tau_j}-e^{r_L\tau_j}]+\theta_X\frac{C(t_j,\tau_j)}{S^0_{t_j}}e^{r_L\tau_j}\}>0$.

(iii) If $y^{b,0}(t_{j+1})>0$, the deviation depends on whether $(1-\theta_S)\Delta^S$ is larger or smaller than $y^{b,0}(t_{j+1})$. (a) If $(1-\theta_S)\Delta^S > y^{b,0}(t_{j+1})$, set $y^s(t_{j+1}) = -(1-\theta_S)\Delta^S + y^{b,0}(t_{j+1}) < 0$ and set $y^b(t_{j+1}) = 0$ so that the value of the investment portfolio at time t^+_{j+1} is the same for the deviation and for the original path. Compared to the original path, the transactions account at time t^+_{j+1} changes by $\xi_3 \equiv (1-\theta_X)\Delta^X + (1-\psi_s)[(1-\theta_S)\Delta^S - y^{b,0}(t_{j+1})] + (1+\psi_b)y^{b,0}(t_{j+1}) = (1-\theta_X)\Delta^X + (1-\psi_s)(1-\theta_S)\Delta^S + (\psi_s+\psi_b)y^{b,0}(t_{j+1})$. Using the definitions of Δ^X and Δ^S, rewrite ξ_3 as $\xi_3 = (1-\psi_s)[-y^{s,0}(t_j)][(1-\theta_S)e^{r_f\tau_j} - (1-\theta_X)e^{r_L\tau_j}] + (1-\theta_X)\theta_X X^0_{t_j}e^{r_L\tau_j} + (1-\psi_s)(1-\theta_S)\theta_S S^0_{t_j}e^{r_f\tau_j} + (\psi_s+\psi_b)y^{b,0}(t_{j+1})$. Therefore,

$$\lim_{\theta_S\to 0}\xi_3 = (1-\psi_s)\big[-y^{s,0}(t_j)\big]\big[e^{r_f\tau_j} - (1-\theta_X)e^{r_L\tau_j}\big] + (1-\theta_X)\theta_X X^0_{t_j}e^{r_L\tau_j} + (\psi_s+\psi_b)y^{b,0}(t_{j+1}) > 0.$$

(b) If $(1-\theta_S)\Delta^S < y^{b,0}(t_{j+1})$, set $y^b(t_{j+1}) = y^{b,0}(t_{j+1}) - (1-\theta_S)\Delta^S > 0$ and set $y^s(t_{j+1}) = 0$ so that the value of the investment portfolio at time t^+_{j+1} is the same for the deviation and for the original path. Compared to the original path, the transactions account at time t^+_{j+1} changes by $\xi_4 \equiv (1-\theta_X)\Delta^X + (1+\psi_b)(1-\theta_S)\Delta^S$. Using the definitions of Δ^X and Δ^S, rewrite ξ_4 as $\xi_4 = [-y^{s,0}(t_j)][(1+\psi_b)(1-\theta_S)e^{r_f\tau_j} - (1-\theta_X)(1-\psi_s)e^{r_L\tau_j}] + (1-\theta_X)\theta_X X^0_{t_j}e^{r_L\tau_j} + (1+\psi_b)(1-\theta_S)\theta_S S^0_{t_j}e^{r_f\tau_j}$. Therefore, $\lim_{\theta_S\to 0}\xi_4 = [-y^{s,0}(t_j)][(1+\psi_b)e^{r_f\tau_j} - (1-\theta_X)(1-\psi_s)e^{r_L\tau_j}] + (1-\theta_X)\theta_X X^0_{t_j}e^{r_L\tau_j} > 0$.

(c) If $(1-\theta_S)\Delta^S = y^{b,0}(t_{j+1})$, set $y^b(t_{j+1}) = y^s(t_{j+1}) = 0$. Compared to the original path, the deviation increases $S_{t^+_{j+1}}$ by $\Delta^S + \theta_S S^0_{t_{j+1}} - y^{b,0}(t_{j+1}) = \theta_S S^0_{t_{j+1}} + \theta_S\Delta^S = \theta_S S_{t_{j+1}} > 0$. Compared to the original path, the transactions account at time t^+_{j+1} changes by $\xi_5 \equiv \Delta^X + \theta_X X^0_{t_{j+1}} + (1+\psi_b)y^{b,0}(t_{j+1}) = \Delta^X + \theta_X X^0_{t_{j+1}} + (1+\psi_b)(1-\theta_S)\Delta^S$. Using the definitions of Δ^X and Δ^S, rewrite ξ_5 as $\xi_5 = [-y^{s,0}(t_j)][(1+\psi_b)(1-\theta_S)e^{r_f\tau_j} - (1-\psi_s)e^{r_L\tau_j}] + \theta_X X^0_{t_j}e^{r_L\tau_j} + \theta_X X^0_{t_{j+1}} + (1+\psi_b)(1-\theta_S)\theta_S S^0_{t_j}e^{r_f\tau_j}$. Therefore, $\lim_{\theta_S\to 0}\xi_5 = [-y^{s,0}(t_j)][(1+\psi_b)e^{r_f\tau_j} - (1-\psi_s)e^{r_L\tau_j}] + \theta_X X^0_{t_j}e^{r_L\tau_j} + \theta_X X^0_{t_{j+1}} > 0$.

To summarize, we have shown that along all possible branches, the deviation leads to an unchanged or increased value of $S_{t^+_{j+1}}$ and an increased value of $X_{t^+_{j+1}}$ (because ξ_i, $i = 1, 2, 3, 4, 5$, have positive limits for θ_S approaching 0) for sufficiently small $\theta_S \geq 0$. Therefore, the hypothesized optimal path could not have been optimal. Therefore, the optimal value of $y^s(t_j) = 0$. *Q.E.D.*

PROOF OF PROPOSITION 2: Consider some path for c_t, X_t, S_t, $y^s(t)$, and $y^b(t)$, $t \in [t_j, t_{j+1}]$, and let c_t^0, X_t^0, S_t^0, $y^{s,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that $x_{t_j} < \omega_1$ and (contrary to what is to be proved) $X_{t_{j+1}}^0 > 0$. Since $\kappa > 0$, the consumer will not continuously observe the value of the investment portfolio. That is, $\tau_j > 0$. If $x_{t_j} < \omega_1$ on an observation date t_j, then Proposition 1 implies that optimal $y^s(t_j) < 0$. Since $X_{t_j^+}^0 = X_{t_j}^0 - (1-\psi_s)y^{s,0}(t_j) - \theta_X X_{t_j}^0$, we have $-y^{s,0}(t_j) = \frac{1}{1-\psi_s}[X_{t_j^+}^0 - X_{t_j}^0 + \theta_X X_{t_j}^0] = \frac{1}{1-\psi_s}[X_{t_j^+}^0 - C(t_j, \tau_j) + C(t_j, \tau_j) - X_{t_j}^0 + \theta_X X_{t_j}^0]$. Then use the fact that $e^{-r_L \tau_j} X_{t_{j+1}}^0 = X_{t_j^+}^0 - C(t_j, \tau_j)$ and Lemma 6 (which implies that since $y^{s,0}(t_j) < 0$, $C(t_j, \tau_t) > X_{t_j}^0$) to deduce that $-y^{s,0}(t_j) = \frac{1}{1-\psi_s}[e^{-r_L \tau_j} X_{t_{j+1}}^0 + (C(t_j, \tau_j) - X_{t_j}^0) + \theta_X X_{t_j}^0] > \frac{1}{1-\psi_s} e^{-r_L \tau_j} X_{t_{j+1}}^0 > 0$. We will show that there exists a deviation from this choice that increases the consumer's expected lifetime utility and, hence, $X_{t_{j+1}}^0 > 0$ cannot be optimal.

Consider a deviation in which the consumer reduces $-y^s(t_j)$ by $\frac{X_{t_j^+}^0 - C(t_j, \tau_j)}{1-\psi_s} = \frac{e^{-r_L \tau_j} X_{t_{j+1}}^0}{1-\psi_s}$ and invests this amount in the riskless bond in the investment portfolio. With this deviation, the value of the investment portfolio at time t_{j+1} will exceed its value under the original policy by $\frac{X_{t_{j+1}}^0}{1-\psi_s} e^{(r_f - r_L)\tau_j}$ and the transactions account will have a zero balance at time t_{j+1}.

The deviation from the original path at time t_{j+1} depends on whether, and in which direction, the consumer would transfer assets between the transactions account and the investment portfolio under the original path at that time. First, consider the case in which $y^{s,0}(t_{j+1}) < 0$ so that the consumer transfers assets from the investment portfolio to the transactions account at time t_{j+1}. In this case, the consumer can increase $-y^s(t_{j+1})$ by $(1-\theta_S)\frac{X_{t_{j+1}}^0}{1-\psi_s} e^{(r_f - r_L)\tau_j}$, which leaves the value of the investment portfolio at time t_{j+1}^+ equal to its value on the original path. Compared to the original path, this deviation will change the balance in the transactions account at time t_{j+1}^+ by $-(1-\theta_X)X_{t_{j+1}}^0 + (1-\theta_S)X_{t_{j+1}}^0 e^{(r_f - r_L)\tau_j} = [(1-\theta_S)e^{(r_f - r_L)\tau_j} - (1-\theta_X)]X_{t_{j+1}}^0$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when $\theta_S \geq 0$ is sufficiently small.

Second, consider the case in which the consumer would not make any transfers between the investment portfolio and the transactions account at time t_{j+1} under the original policy. Since the consumer does not make any transfers at time t_{j+1}, if the original path were optimal, Proposition 1 implies that $0 < \omega_1 \leq \frac{X_{t_{j+1}}^0}{S_{t_{j+1}}^0} \leq \omega_2$, which implies that $S_{t_{j+1}}^0 \leq \frac{X_{t_{j+1}}^0}{\omega_1}$. In this case, under the deviation, the consumer sets $-y^s(t_{j+1}) = (1-\theta_S)\frac{X_{t_{j+1}}^0}{1-\psi_s} e^{(r_f - r_L)\tau_j} - \theta_S S_{t_{j+1}}^0$. Therefore, $-y^s(t_{j+1}) \geq [\frac{1-\theta_S}{1-\psi_s} e^{(r_f - r_L)\tau_j} - \frac{\theta_S}{\omega_1}] X^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. (Proposition 1 states that $\omega_1 > 0$ for all admissible values of $\theta_S \geq 0$, including $\theta_S = 0$, so that $\lim_{\theta_S \to 0} \frac{\theta_S}{\omega_1} = 0$.) With this transfer, the value of assets in the investment portfolio at time t^+_{j+1} will be the same under the deviation as under the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-X^0_{t_{j+1}} - (1-\psi_s) y^s(t_{j+1}) = -X^0_{t_{j+1}} + (1-\theta_S) X^0_{t_{j+1}} e^{(r_f - r_L)\tau_j} - (1-\psi_s)\theta_S S^0_{t_{j+1}} = [(1-\theta_S) e^{(r_f - r_L)\tau_j} - 1] X^0_{t_{j+1}} - (1-\psi_s)\theta_S S^0_{t_{j+1}} \geq ((1-\theta_S) e^{(r_f - r_L)\tau_j} - 1 - (1-\psi_s)\frac{\theta_S}{\omega_1}) X^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when $\theta_S \geq 0$ is sufficiently small.

Third, consider the case in which $y^{b,0}(t_{j+1}) > 0$ so that the consumer transfers assets from the transactions account to the investment portfolio at time t_{j+1}. If $y^{b,0}(t_{j+1}) > (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j}$, the deviation reduces $y^b(t_{j+1})$ by $(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j}$ and sets $y^s(t_{j+1}) = 0$, which will leave the value of the investment portfolio at time t^+_{j+1} under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X) X^0_{t_{j+1}} + (1+\psi_b)(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} = [(1-\theta_S)\frac{1+\psi_b}{1-\psi_s} e^{(r_f - r_L)\tau_j} - (1-\theta_X)] X^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when $\theta_S \geq 0$ is sufficiently small. If $y^{b,0}(t_{j+1}) < (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j}$, the deviation sets $y^b(t_{j+1}) = 0$ and sets $-y^s(t_{j+1}) = (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} - y^{b,0}(t_{j+1}) > 0$, which will leave the value of the investment portfolio at time t^+_{j+1} under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X) X^0_{t_{j+1}} + (1+\psi_b) y^{b,0}(t_{j+1}) + (1-\psi_s)[(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} - y^{b,0}(t_{j+1})] = [(1-\theta_S) e^{(r_f - r_L)\tau_j} - (1-\theta_X)] X^0_{t_{j+1}} + (\psi_b + \psi_s) y^{b,0}(t_{j+1})$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small. Finally, if $y^{b,0}(t_{j+1}) = (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j}$, the deviation sets $y^s(t_{j+1}) = y^b(t_{j+1}) = 0$. Compared to the original path, the deviation changes $S_{t^+_{j+1}}$ by $\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} + \theta_S S^0_{t_{j+1}} - y^{b,0}(t_{j+1}) = \frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} + \theta_S S^0_{t_{j+1}} - (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} = \theta_S S^0_{t_{j+1}} + \theta_S \frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} > 0$. Compared to the original path, the deviation changes $X_{t_{j+1}^+}$ by $-X^0_{t_{j+1}} + \theta_X X^0_{t_{j+1}} + (1+\psi_b) y^{b,0}(t_{j+1}) = -(1-\theta_X) X^0_{t_{j+1}} + (1+\psi_b)(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f - r_L)\tau_j} = [(1-\theta_S)\frac{1+\psi_b}{1-\psi_s} e^{(r_f-r_L)\tau_j} - (1-\theta_X)] X^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$.

We have shown that the deviation path dominates the original path; hence it cannot be optimal for $X_{t_{j+1}}$ to be positive. Since the optimal value of $X_{t_{j+1}} = 0$, we have $x_{t_{j+1}} = 0 < \omega_1$, which implies $x_{t_{j+2}} = 0$ and so on, ad infinitum. *Q.E.D.*

PROOF OF LEMMA 2: Lemma 11 states that the optimal value of ϕ_j is positive. Since $\tau_j > 0$ as a consequence of the information cost, there exists some $\delta > 0$ such that between any two consecutive observation dates, t_j and $t_{j+1} = t_j + \tau_j$, $\Pr\{e^{-r_L \tau_j} R(t_j, \tau_j) > \frac{\omega_2}{\omega_1}\} \geq \delta$. Therefore, since $x_{t_{j+1}} \equiv \frac{X_{t_{j+1}}}{S_{t_{j+1}}} = \frac{e^{r_L \tau_j}}{R(t_j,\tau_j)} \frac{X_{t_j^+} - C(t_j,\tau_j)}{S_{t_j^+}} < \frac{e^{r_L \tau_j}}{R(t_j,\tau_j)} \frac{X_{t_j^+}}{S_{t_j^+}} = \frac{x_{t_j^+}}{e^{-r_L \tau_j} R(t_j,\tau_j)} \leq \frac{\omega_2}{e^{-r_L \tau_j} R(t_j,\tau_j)}$ (where the final inequality follows from Corollary 1), then $\Pr\{x_{t_{j+1}} < \omega_1\} \geq \delta$. Let $t_k \geq t_j$ be the first observation date at which $x_{t_k} < \omega_1$. Then by Williams (1991, p. 233), $\Pr\{t_k < \infty\} = 1$ and $E\{t_k\} < \infty$. *Q.E.D.*

PROOF OF PROPOSITION 3: Lemma 2 states that eventually, with probability 1, $x_{t_j} < \omega_1$ on an observation date. Proposition 2 implies that when this event occurs, $x_{t_{j+1}} = 0$ on the next observation date and on all subsequent observation dates, provided that $\theta_S \geq 0$ is sufficiently small. Since the optimal value of τ_j is simply a function of x_{t_j}, τ_j will be constant when x_{t_j} becomes constant. *Q.E.D.*

PROPOSITION 5: *Let $T^s(t_j, t) \equiv \int_{t_j}^t dY^s(t) \leq 0$ denote the cumulative transfer process from the investment portfolio to the transactions account from time t_j to time $t \in [t_j, t_{j+1}]$, and let $T^b(t_j, t) \equiv \int_{t_j}^t dY^b(t) \geq 0$ denote the cumulative transfer process from the transactions account to the investment portfolio from time t_j to time $t \in [t_j, t_{j+1}]$. We define automatic transfers as F_{t_j}-measurable functions $T^s(t_j, t)$ and $T^b(t_j, t)$ that satisfy three requirements*: (i) *$T^s(t_j, t)$ is nonincreasing in t*, (ii) *$T^b(t_j, t)$ is nondecreasing in t, and* (iii) *given $T^s(t_j, t)$ and $T^b(t_j, t)$, along with the F_{t_j}-measurable path of consumption from t_j to t_{j+1}, $X_t \geq 0$ and $S_t \geq 0$ for any path of P_t. If the consumer can utilize automatic transfers and $\theta_X = \theta_S = 0$, then the stochastic process for x_{t_j} is eventually, with probability* 1, *absorbed at zero and the time between consecutive observations is constant.*

To prepare for the proof of Proposition 5, we first introduce some notation and then prove three ancillary lemmas.

Define $F^s(t, z; r)$ to be the (negative of the) future value, as of time z, of transfers from the investment portfolio to the transactions account from time t until, but not including, time z. The future value is computed using the discount rate r. Formally, $F^s(t, z; r) \equiv \lim_{x \nearrow z} \int_t^x e^{r(x-v)}\, dY^s(v)$, where

$dY^s(v) \leq 0$ denotes the increments of the cumulative transfer from the investment portfolio to the transactions account (so that $F^s(t, z; 0) = T^s(t, z)$). We use the notation $F^s(t, t^+; r)$ to capture potential lump-sum transfers at time t ($F^s(t, t^+; r) = \lim_{z \searrow t} F^s(t, z; r)$, which equals $y^s(t)$ using the notation in the baseline version of the model with transfers confined to observation dates). Similarly, $F^b(t, z, r)$ is the future value, as of time z, of transfers from the transactions account to the investment portfolio from time t until, but not including, time z (so that $F^b(t, z; 0) = T^b(t, z)$). The notation $F^b(t, t^+; r)$ captures lump-sum transfers from the transactions account to the investment account at time t. Finally, $\text{FVC}(t, z) \equiv \int_t^z c_v e^{r_L(z-v)}\, dv$ is the future value, as of time z, of consumption from time t to z, compounded at the rate r_L.

We next prove the three ancillary lemmas.

LEMMA 7: *Along an optimal path that includes the possibility of automatic transfers, if $\theta_X = \theta_S = 0$ and if $X_t > 0$ for all $t \in [t_j, t_{j+1}]$, then $F^s(t_j, t_{j+1}, r_L) = 0$.*

PROOF: Assume otherwise, that is, suppose that for an optimal path, $X_t^0 > 0$ for all $t \in [t_j, t_{j+1}]$ and yet $F^{s,0}(t_j, t_{j+1}, r_L) < 0$. Now consider the following deviation: Do not transfer any assets from the investment portfolio to the transactions account until the next observation time, t_{j+1}, or until the transactions account under this deviation reaches a nonpositive balance, whichever comes first. Formally, denote this time as $t^* \equiv \min\{t_{j+1}, \inf\{t : \widetilde{X}_t \leq 0\}\}$, where $\widetilde{X}_t$ is the balance in the transactions account under this deviation. We next argue that $t^* \neq t_j$ and hence that $t^* > t_j$. We proceed by contradiction. Suppose, contrary to what is to be proved, that $t^* = t_j$, so that $0 \geq \widetilde{X}_{t_j^+}$. Since (i) $\widetilde{X}_{t_j^+} = X_{t_j} - (1 - \psi_s)\widetilde{F}^s(t_j, t_j^+; r_L) - (1 + \psi_b)F^{b,0}(t_j, t_j^+; r_L)$, (ii) $X_{t_j} > 0$, and (iii) $\widetilde{F}^s(t_j, t_j^+; r_L)$ cannot be positive under any circumstance, then $\widetilde{X}_{t_j^+}$ can be nonpositive only if $F^{b,0}(t_j, t_j^+; r_L) > 0$. But if the original path is optimal, then $F^{b,0}(t_j, t_j^+; r_L) > 0$ and Lemma 4 imply that $F^{s,0}(t_j, t_j^+; r_L) = 0$. Since $X^0_{t_j^+} = X_{t_j} - (1 - \psi_s)F^{s,0}(t_j, t_j^+; r_L) - (1 + \psi_b)F^{b,0}(t_j, t_j^+; r_L)$, the fact that $F^{s,0}(t_j, t_j^+; r_L) = 0$ implies that $0 < X^0_{t_j^+} = X_{t_j} - (1 + \psi_b)F^{b,0}(t_j, t_j^+; r_L) \leq \widetilde{X}_{t_j^+}$, which contradicts $0 \geq \widetilde{X}_{t_j^+}$ above. Therefore, $t^* > t_j$.

Also, by construction, $t^* \leq t_{j+1}$ and $F^{s,0}(t_j, t^*, r_L) < 0$.[36] To complete the construction of the deviation, suppose that between t_j and t^* the consumer

[36]To show that $F^{s,0}(t_j, t^*, r_L) < 0$, we proceed in steps: First, we show that $F^{s,0}(t_j, t^{*+}, r_L) < 0$ by distinguishing two cases: (i) if $t^* = t_{j+1}$, then $F^{s,0}(t_j, t^*, r_L) < 0$ by assumption, and (ii) if $t^* < t_{j+1}$, then $\widetilde{X}_{t^{*+}} \leq 0$. Note that if $F^{s,0}(t_j, t^{*+}, r_L)$ were zero, and hence equal to $\widetilde{F}^s(t_j, t^{*+}, r_L)$ under the deviation, then $X_t^0 = \widetilde{X}_t$ for all $t \in [t_j, t^{*+}]$. But $X_t^0 > 0$ for all $t \in [t_j, t_{j+1}]$, which is inconsistent with $\widetilde{X}_{t^{*+}} \leq 0$. Having established that $F^{s,0}(t_j, t^{*+}, r_L) < 0$, we next show that $F^{s,0}(t_j, t^*, r_L) < 0$. Suppose otherwise, that is, suppose that $F^{s,0}(t_j, t^*, r_L) = 0$ so that $F^{s,0}(t_j, t^{*+}, r_L) = F^{s,0}(t^*, t^{*+}, r_L)$. Since $F^{s,0}(t_j, t^{*+}, r_L) < 0$, it follows that $F^{s,0}(t^*, t^{*+}, r_L) < 0$.

invests the funds she would have transferred into the transactions account in riskless bonds in the investment portfolio. At time t^*, the consumer sets $\widetilde{F}^s(t^*, t^{*+}, r_L) = F^{s,0}(t^*, t^{*+}, r_L) + F^{s,0}(t_j, t^*, r_f) < 0$. From t^{*+} to t_{j+1}, the consumer simply follows the same transfer and consumption policies she would have followed under the original path.

Under this deviation, the consumption process does not change between t_j and t^* or between t^{*+} and t_{j+1}, so that consumption is unchanged in $[t_j, t_{j+1}]$. Moreover, at time t^{*+}, the investment portfolio has the same value as under the original path, and since the consumer follows the same transfer policies from t^{*+} onward, the investment portfolio at t_{j+1} is the same under the deviation as under the original path. The transactions account changes by $(1-\psi_s)[F^{s,0}(t_j, t^*, r_L) - F^{s,0}(t_j, t^*, r_f)] > 0$ at t^{*+}. Since the consumer follows the same transfer policies from t^{*+} onward, the deviation increases the transactions account at time t_{j+1} relative to the original path by $(1-\psi_s)e^{r_L(t_{j+1}-t^*)} \times [F^{s,0}(t_j, t^*, r_L) - F^{s,0}(t_j, t^*, r_f)] > 0$. Hence, the original path could not have been optimal. *Q.E.D.*

LEMMA 8: *Along an optimal path that includes the possibility of automatic transfers, let* $\bar{t} = \inf\{t \geq t_j : X_t = 0\}$. *If* $\theta_X = \theta_S = 0$, *then* $X_t = 0$ *for all* $t \geq \bar{t}$.

PROOF: Suppose that there are no transactions costs ($\psi_s = \psi_b = 0$). In that case, the consumer can move freely and instantaneously between the investment portfolio and the transactions account. The allocation between the investment portfolio and the transactions account is part of an asset allocation problem with three assets: risky equity, riskless bonds paying r_f, and riskless liquid assets paying $r_L < r_f$. In the absence of the requirement $X_t \geq 0$, there would be an arbitrage opportunity that would send the holding of riskless bonds in the investment portfolio to infinity and the holding of the liquid assets in the transactions account to minus infinity. Given the requirement $X_t \geq 0$ and the ability to undertake costless transfers between X_t and S_t, the consumer would immediately set $X_t = 0$, and then would keep X_t at zero forever by setting $F^b(t, \infty) = 0$ and $\int_t^z dT^s = -\int_t^z c_s\, ds$ so that $F^s(t, z, r_L) = -\mathrm{FVC}(t, z)$ for any $z \geq t$; in words, the consumer would transfer infinitesimal amounts from S_t to X_t as needed to finance instantaneous consumption. Any allocation to riskless bonds would take place exclusively inside the investment portfolio and on observation dates, the consumer would simply adjust the consumption rate.

Now introduce transactions costs so that $\psi_s + \psi_b > 0$. We will prove that, also in this case, it is optimal to keep $X_t = 0$ for $t \geq \bar{t}$. Let c_t^{**}, X_t^{**}, and S_t^{**} denote values of c_t, X_t, and S_t along an optimal path for $\psi_s + \psi_b > 0$ and $t \geq \bar{t}$. Now consider the case with $\psi_s = \psi_b = 0$, and let c_t^*, $\mathrm{FVC}^*(\,)$, $F^{s*}(\,)$, and

But then $F^{b,0}(t^*, t^{*+}, r_L) = 0$ so $\widetilde{X}_{t^{*+}} = \widetilde{X}_{t^*} - (1-\psi_s)\widetilde{F}^s(t^*, t^{*+}; r_L) - (1+\psi_b)\widetilde{F}^b(t^*, t^{*+}; r_L) = \widetilde{X}_{t^*} - (1-\psi_s)\widetilde{F}^s(t^*, t^{*+}; r_L) - (1+\psi_b)F^{b,0}(t^*, t^{*+}; r_L) \geq \widetilde{X}_{t^*} = X_t^0 > 0$. So under the deviation, X_t is positive both at time t^* and at time t^{*+}, which contradicts the definition of t^*.

$F^{b*}(\,)$ denote the values of c_t, FVC( ), $F^s(\,)$, and $F^b(\,)$ in this case. In this case, setting $c_t^* = \frac{1}{1-\psi_s}c_t^{**}$ is feasible. To see this, simply set $c_t^* = \frac{1}{1-\psi_s}c_t^{**}$, and keep the observation dates, the allocations within the investment portfolio, and the transfers between the investment portfolio and the transactions account unchanged. Clearly the path of S_t does not change, so to show feasibility, it suffices to show that the path of X_t^* is nonnegative. To that end, note that for arbitrary ψ_s and ψ_b, and any feasible consumption and transfer policies, the dynamics of X_t for $t \geq \bar{t}$ are characterized by

$$X_t = -\text{FVC}(\bar{t}, t) - (1-\psi_s)F^s(\bar{t}, t; r_L) - (1+\psi_b)F^b(\bar{t}, t; r_L). \tag{A.45}$$

For the optimal path associated with $\psi_s + \psi_b > 0$, we have

$$X_t^{**} = -\text{FVC}^{**}(\bar{t}, t) - (1-\psi_s)F^{s**}(\bar{t}, t; r_L) - (1+\psi_b)F^{b**}(\bar{t}, t; r_L). \tag{A.46}$$

For the alternative path, which has $\psi_s = \psi_b = 0$, we have $\text{FVC}^*(\bar{t}, t) = \frac{1}{1-\psi_s}\text{FVC}^{**}(\bar{t}, t)$, $F^{s*}(\bar{t}, t; r_L) = F^{s**}(\bar{t}, t; r_L)$, and $F^{b*}(\bar{t}, t; r_L) = F^{b**}(\bar{t}, t; r_L)$, which implies

$$X_t^* = -\frac{1}{1-\psi_s}\text{FVC}^{**}(\bar{t}, t) - F^{s**}(\bar{t}, t; r_L) - F^{b**}(\bar{t}, t; r_L). \tag{A.47}$$

Dividing (A.46) by $1-\psi_s$, recognizing that $\frac{1+\psi_b}{1-\psi_s} > 1$ when $\psi_s + \psi_b > 0$, and then using (A.47) yields

$$\frac{1}{1-\psi_s}X_t^{**} = -\frac{1}{1-\psi_s}\text{FVC}^{**}(\bar{t}, t) - F^{s**}(\bar{t}, t; r_L) - \frac{1+\psi_b}{1-\psi_s}F^{b**}(\bar{t}, t; r_L) \tag{A.48}$$

$$\leq -\frac{1}{1-\psi_s}\text{FVC}^{**}(\bar{t}, t) - F^{s**}(\bar{t}, t; r_L) - F^{b**}(\bar{t}, t; r_L) \tag{A.49}$$

$$= X_t^*. \tag{A.50}$$

Since the original path was feasible with $X_t^{**} \geq 0$, (A.48) implies that $X_t^* \geq \frac{1}{1-\psi_s}X_t^{**} \geq 0$ for all t. Therefore, it is feasible to set $c_t^* = \frac{1}{1-\psi_s}c_t^{**}$ when $\psi_s = \psi_b = 0$. Accordingly, letting $V_{\bar{t}}^{(\psi_s,\psi_b)}$ denote the time-$\bar{t}$ value function of the consumer when the transactions costs parameters are ψ_s and ψ_b, we obtain $\frac{1}{(1-\psi_s)^{1-\alpha}}V_{\bar{t}}^{(\psi_s,\psi_b)} \leq V_{\bar{t}}^{(0,0)}$ or, equivalently, $V_{\bar{t}}^{(\psi_s,\psi_b)} \leq (1-\psi_s)^{1-\alpha}V_{\bar{t}}^{(0,0)}$. In words, $(1-\psi_s)^{1-\alpha}V_{\bar{t}}^{(0,0)}$ provides an upper bound to $V_{\bar{t}}^{(\psi_s,\psi_b)}$. Next observe that when $\psi_s + \psi_b > 0$, the policy that sets $c_t^{**} = (1-\psi_s)c_t^*$, $F^{b**}(\bar{t}, t; r_L) = 0$, and $F^{s*}(\bar{t}, t; r_L) = F^{s**}(\bar{t}, t; r_L) = -\text{FVC}^{**}(t, t_1)$ for all $t \geq \bar{t}$ keeps $X_t = 0$ for all $t \geq \bar{t}$, is feasible, and delivers welfare equal to $(1-\psi_s)^{1-\alpha}V_{\bar{t}}^{(0,0)}$. That is, for

$\psi_s + \psi_b > 0$, this policy attains the upper bound $(1-\psi_s)^{1-\alpha}V_{\bar{t}}^{(0,0)}$ and hence is optimal. *Q.E.D.*

LEMMA 9: *Along an optimal path that includes the possibility of automatic transfers, if $\theta_X = \theta_S = 0$ and if $F^s(t_j, t_{j+1}; r_L) < 0$, then optimal $X_{t_{j+1}} = 0$.*

PROOF: Lemma 7 implies that if $F^s(t_j, t_{j+1}; r_L) < 0$, then $\bar{t} \equiv \inf\{t \geq t_j : X_t = 0\} < t_{j+1}$. Then Lemma 8 implies that $X_t = 0$ for all $t \geq \bar{t}$, so that in particular, $X_{t_j+1} = 0$. *Q.E.D.*

PROOF OF PROPOSITION 5: The arguments of Lemma 2, appropriately adjusted for automatic transfers, imply that if, along an optimal path, x_{t_j} becomes smaller than some number $\Omega_1 > 0$ on some observation date t_j, then $C(t_j, t_{j+1}) > X_{t_j}$, which requires $F^s(t_j, t_{j+1}; r_L) < 0$. Accordingly Lemma 9 implies $X_{t_{j+1}} = 0$, which implies $x_t = 0$ for all $t \geq t_{j+1}$ (by Lemma 8) so that, in particular, $x_{t_{j+k}} = 0$ for all $k \geq 1$.

Next we argue that eventually, with probability 1, there will exist some $k \geq 1$, such that $x_{t_{j+k}} \leq \Omega_1$. We start by observing that in the presence of automatic transfers, $X_{t_{j+1}}$ is F_{t_j}-measurable.[37] Lemmas 7 and 8 imply that as long as $X_{t_{j+1}} > 0$, it follows that $F^s(t_j, t_{j+1}; r_L) = 0$, which, together with the fact that consumption and transfers from the transactions account to the investment account are both nonnegative, implies that $X_{t_{j+1}} \leq e^{r_L \tau_j} X_{t_j}$ and $S_{t_{j+1}} \geq S_{t_j} R(t_j, \tau_j)$. Accordingly, $x_{t_{j+1}} = \frac{X_{t_{j+1}}}{S_{t_{j+1}}} \leq \frac{e^{r_L\tau_j} X_{t_j}}{S_{t_j} R(t_j,\tau_j)} = x_{t_j} \frac{e^{r_L\tau_j}}{R(t_j,\tau_j)}$. Taking logs gives $\log x_{t_{j+1}} \leq \log x_{t_j} + r_L\tau_j - \log R(t_j, \tau_j)$. Taking expectations as of time t_j gives $E_{t_j} \log x_{t_{j+1}} \leq \log x_{t_j} + r_L\tau_j - E_{t_j} \log R(t_j, \tau_j)$. We next observe that $-E_{t_j} \log R(t_j, \tau_j) \leq \max_{\phi_j \in [0,1]}\{-E_{t_j} \log R(t_j, \tau_j)\} = -r_f\tau_j$.[38] Accordingly, $\log x_{t_j}$ is bounded above by a random walk with drift $r_L - r_f$, which is strictly negative. Since a random walk with negative drift eventually, with probability 1, becomes smaller than any finite number (and in particular $\log \Omega_1$) with probability 1, there will exist a k, such that $x_{t_{j+k}} \leq \Omega_1$. Therefore, as discussed above, $x_{t_{j+k+n}} = 0$ for all $n \geq 1$.

Since the optimal value of τ_j is simply a function of x_{t_j} and since x_{t_j} eventually, with probability 1, becomes constant (namely, zero), the inattention intervals τ_j will eventually become constant with probability 1. *Q.E.D.*

[37] Since any transfers from the investment portfolio must be F_{t_j}-measurable, and feasible, these transfers will not be financed from the risky holdings in the investment portfolio.

[38] Note that $-E_{t_j} \log R(t_j, \tau_j)$ is a convex function of ϕ_j, since $\frac{\partial^2[-E_{t_j} \log R(t_j,\tau_j)]}{(\partial\phi_j)^2} = E_{t_j}\{\frac{1}{R^2(t_j,\tau_j)}[\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f\tau_j}]^2\} > 0$. Hence the maximum value of $-E_{t_j} \log R(t_j, \tau_j)$ for $\phi_j \in [0, 1]$ is attained either when $\phi_j = 0$, or when $\phi_j = 1$. When $\phi_j = 0$, $-E_{t_j} \log R(t_j, \tau_j) = -r_f\tau_j$, whereas when $\phi_j = 1$, $-E_{t_j} \log R(t_j, \tau_j) = -(\mu - \frac{\sigma^2}{2})\tau_j$. Given the maintained assumption $(\mu - \frac{\sigma^2}{2}) > r_f$, it follows that $\max_{\phi_j \in [0,1]}\{-E_{t_j} \log R(t_j, \tau_j)\} = -r_f\tau_j$.

The following lemma proves that although x_t is eventually absorbed at zero with probability 1, this absorption need not occur immediately.

LEMMA 10: *Suppose that we allow automatic transfers*, $\theta_X = \theta_S = 0$, *and* x_{t_j} *is sufficiently large*. *Then optimal* $X_{t_j^+} > 0$ *so that* x_t *is not immediately absorbed at zero*.

PROOF: Let X_t^0 be the value of X_t along the hypothesized optimal path, and suppose, contrary to what is to be proved, that $X^0_{t_j^+} = 0$, which implies that $F^{b,0}(t_j, t_j^+, r_L) = \frac{X^0_{t_j}}{1+\psi_b}$ and $F^{s,0}(t_j, t, r_L) = -\frac{\text{FVC}(t_j,t)}{1-\psi_s}$ for $t > t_j$. Define τ^* such that $\frac{1-\psi_s}{1+\psi_b} e^{(r_f - r_L)\tau^*} = 1$ and note that for $0 \le \tau^{**} < \tau^*$, any dollar transferred from the transactions account to the investment portfolio at time t_j, invested in the riskless bond, and then transferred back to the transactions account at time $t_j + \tau^{**}$ will be worth less at time $t_j + \tau^*$ than a dollar simply left in the transactions account from t_j to $t_j + \tau^{**}$. Now let τ^{***} be a positive number less than $\min\{t_{j+1} - t_j, \tau^*\}$ that is small enough that $e^{-r_L\tau^{***}}\,\text{FVC}(t_j, t_j + \tau^{***}) < X^0_{t_j}$. Consider an alternative path that sets $F^b(t_j, t_j^+, r_L) = \frac{X^0_{t_j} - e^{-r_L\tau^{***}}\,\text{FVC}(t_j, t_j+\tau^{***})}{1+\psi_b} > 0$ and does not change any other transfers from the transactions account to the investment portfolio so that $F^b(t_j, t, 0) = F^{b,0}(t_j, t, 0) - \frac{e^{-r_L\tau^{***}}\,\text{FVC}(t_j, t_j+\tau^{***})}{1+\psi_b}$ for $t > t_j^+$. In addition, the alternative path sets $F^s(t_j, t_j + \tau^{***}, r_L) = 0$ and then maintains $F^s(t_j + \tau^{***}, t, r_L) = F^{s,0}(t_j + \tau^{***}, t, r_L)$ for all $t \in (t_j + \tau^{***}, t_{j+1})$. Suppose that any changes in the size of the investment portfolio affect only the amount invested in riskless bonds. Relative to the originally hypothesized optimal path, the alternative path changes $S_{t_j+\tau^{***}}$ by $\Delta^S \equiv -e^{r_f\tau^{***}} e^{-r_L\tau^{***}} \frac{\text{FVC}(t_j, t_j+\tau^{***})}{1+\psi_b} - F^{s,0}(t_j, t_j + \tau^{***}, r_f)$, where the first term reflects the reduction in $S_{t_j+\tau^{***}}$ arising from the reduced transfer into the investment portfolio at time t_j and the second term reflects the fact that the consumer does not need to transfer assets from the investment portfolio to the transactions account to finance the original path of consumption until $t_j + \tau^{***}$. Relative to the originally hypothesized optimal path, the alternative path changes $X_{t_j+\tau^{***}}$ by $\Delta^X \equiv (1 + \psi_b)[\frac{e^{-r_L\tau^{***}}\,\text{FVC}(t_j, t_j+\tau^{***})}{1+\psi_b}]e^{r_L\tau^{***}} + (1 - \psi_s)F^{s,0}(t_j, t_j + \tau^{***}, r_L)$, where the first term reflects the increase in $X_{t_j+\tau^{***}}$ that arises from the reduction in the transfer out of the transactions account at time t_j and the second term reflects the reduction in transfers into the transactions account between t_j and $t_j + \tau^{***}$. Use the fact that $-F^{s,0}(t_j, t_j + \tau^{***}, r_f) \ge -F^{s,0}(t_j, t_j + \tau^{***}, r_L) = \frac{\text{FVC}(t_j, t_j+\tau^{***})}{1-\psi_s}$ to obtain $\Delta^S \ge -e^{r_f\tau^{***}} e^{-r_L\tau^{***}} \frac{\text{FVC}(t_j, t_j+\tau^{***})}{1+\psi_b} + \frac{\text{FVC}(t_j, t_j+\tau^{***})}{1-\psi_s} = [-\frac{1-\psi_s}{1+\psi_b} e^{(r_f - r_L)\tau^{***}} + 1]\frac{\text{FVC}(t_j, t_j+\tau^{***})}{1-\psi_s} > 0$ since $\tau^{***} > \tau^*$. Observe that $\Delta^X \equiv \text{FVC}(t_j, t_j + \tau^{***}) + (1 - \psi_s)F^{s,0}(t_j, t_j + \tau^{***}, r_L) = 0$. Since $\Delta^S > 0$ and $\Delta^X = 0$, the original path could not be optimal. Therefore, optimal $X_{t_j^+} > 0$. *Q.E.D.*

PROPOSITION 6: *Define $V(0, S_{t_j}; \psi_s)$ as the value function for a given value of the transactions cost parameter ψ_s on observation date t_j when $(X_{t_j}, S_{t_j}) = (0, S_{t_j})$, and define $\pi_1(\psi_s)$ as the optimal return value of $x_{t_j^+}$ for $x_{t_j} < \omega_1$. Suppose that θ_S is sufficiently small that for any admissible value of ψ_s, if $x_{t_j} < \omega_1$ on observation date t_j, then on all subsequent observation dates $x_{t_{j+1}} = 0$. Then the following statements hold*:

(i) $V(0, S_{t_j}; \psi_s) = (1 - \psi_s)^{1-\alpha} V(0, S_{t_j}; 0)$.

(ii) *The optimal observation dates $t_k = t_j + (k - j)\tau^*$ for $k \geq j$ are invariant to ψ_s.*

(iii) $\pi_1(\psi_s) = (1 - \psi_s)\pi_1(0)$.

PROOF: Suppose that $\psi_s = 0$ and let $\{S_t^*\}_{t=t_j}^{t=\infty}$ be the path of S_t under the optimal policy starting from observation date t_j when the consumer observes $X_{t_j} = 0$ and $S_{t_j} = S_{t_j}^*$. Let τ^* be the constant optimal interval of time between consecutive observations so that observation date $t_k = t_j + (k - j)\tau^*$ for $k \geq j$. For any observation date $t_k \geq t_j$, the transactions account balance will be $X_{t_k} = 0$, and immediately after each observation date, the transactions account balance will be $X_{t_k^+} = X_{t_k^+}^* \equiv \pi_1(0) S_{t_k^+}^*$. Since $0 = X_{t_{k+1}}^* = e^{r_L \tau^*}(X_{t_k^+}^* - C(t_k, \tau^*))$, we have $C(t_k, \tau^*) = X_{t_k^+}^*$.

Now let ψ_s take an arbitrary admissible value and suppose that the consumer continues to observe the value of the investment portfolio on dates $t_k = t_j + (k - j)\tau^*$ for $k \geq j$ and maintains the same path of S_t, that is, that $S_t = S_t^*$ for $t \geq t_j$. Since the consumer will make the same transfers out of the investment portfolio as in the initial case with $\psi_s = 0$, a feasible path of the transaction account balance immediately after each observation date would be $X_{t_k^+} = (1 - \psi_s) X_{t_k^+}^*$, which supports a feasible path of consumption $C(t_k, \tau^*) = (1 - \psi_s) X_{t_k^+}^*$. Therefore, $V(0, S_{t_j}; \psi_s) \geq (1 - \psi_s)^{1-\alpha} V(0, S_{t_j}; 0)$.

A similar argument, starting with an arbitrary admissible value of ψ_s less than 1, implies $V(0, S_{t_j}; 0) \geq (\frac{1}{1-\psi_s})^{1-\alpha} V(0, S_{t_j}; \psi_s)$. Therefore, $V(0, S_{t_j}; \psi_s) \geq (1 - \psi_s)^{1-\alpha} V(0, S_{t_j}; 0) \geq V(0, S_{t_j}; \psi_s)$, which implies $V(0, S_{t_j}; \psi_s) = (1 - \psi_s)^{1-\alpha} V(0, S_{t_j}; 0)$ (statement (i)). We showed that by maintaining the same observation dates when ψ_s is positive as when $\psi_s = 0$ allows a path of consumption that achieves $V(0, S_{t_j}; \psi_s) \geq (1 - \psi_s)^{1-\alpha} V(0, S_{t_j}; 0) = V(0, S_{t_j}; \psi_s)$. Similarly, by maintaining the same observation dates when $\psi_s = 0$ as when ψ_s is positive allows a path of consumption that achieves $V(0, S_{t_j}; 0) \geq (\frac{1}{1-\psi_s})^{1-\alpha} V(0, S_{t_j}; \psi_s) = V(0, S_{t_j}; 0)$. Therefore, we have proven statement (ii). For any observation date $t_k \geq t_j$, $x_{t_k^+} = \pi_1(\psi_s)$. Therefore, $\pi_1(\psi_s) = \frac{X_{t_k^+}}{S_{t_k^+}} = \frac{(1-\psi_s) X_{t_k^+}^*}{S_{t_k^+}^*} = (1 - \psi_s)\pi_1(0)$, which proves statement (iii). *Q.E.D.*

PROOF OF PROPOSITION 4: At each observation date t_j the consumer chooses the share ϕ_j of the investment portfolio to allocate to equity to maximize $E_{t_j}\{V(X_{t_{j+1}}, S_{t_{j+1}})\}$ subject to the constraints $0 \leq \phi_j \leq 1$. Using (2) and (3), we can write the Lagrangian for this constrained maximization as

$$\text{(A.51)} \quad \mathcal{L}_j = E_{t_j}\left\{V\left(X_{t_{j+1}}, \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} S_{t_j^+} + (1-\phi_j)e^{r_f \tau_j} S_{t_j^+}\right)\right\} + \delta_j S_{t_j^+} \phi_j + \nu_j S_{t_j^+}(1-\phi_j),$$

where $\delta_j S_{t_j^+} \geq 0$ is the Lagrange multiplier on the constraint $\phi_j \geq 0$ and $\nu_j S_{t_j^+} \geq 0$ is the Lagrange multiplier on the constraint $\phi_j \leq 1$. Differentiating the Lagrangian in (A.51) with respect to ϕ_j, setting the derivative equal to zero, and then dividing both sides by $S_{t_j^+}$ yields

$$\text{(A.52)} \quad E_{t_j}\left\{V_S(X_{t_{j+1}}, S_{t_{j+1}})\left(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f \tau_j}\right)\right\} = \nu_j - \delta_j.$$

Next, we prove the following lemma.

LEMMA 11: *We have* $\phi_j > 0$ *and* $\delta_j = 0$.

PROOF: We proceed by contradiction. Suppose that $\phi_j = 0$, which implies that $\nu_j = 0$ and that $S_{t_{j+1}}$ is known at time t_j. Therefore, (A.52) can be written as $V_S(X_{t_{j+1}}, S_{t_{j+1}})E_{t_j}\{(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f \tau_j})\} = -\delta_j \leq 0$, which is a contradiction because $V_S(X_{t_{j+1}}, S_{t_{j+1}}) > 0$ and, by assumption, the expected equity premium, $E_{t_j}\{(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f \tau_j})\}$, is positive. Therefore, ϕ_j must be positive, which implies $\delta_j = 0$. *Q.E.D.*

To replace the marginal valuation of the investment portfolio $V_S(X_{t_{j+1}}, S_{t_{j+1}})$ by a function of the marginal utility of consumption, first use the definition of the marginal rate of substitution $m(x_{t_{j+1}})$ to obtain

$$\text{(A.53)} \quad V_S(X_{t_{j+1}}, S_{t_{j+1}}) = m(x_{t_{j+1}})V_X(X_{t_{j+1}}, S_{t_{j+1}}).$$

Then use the envelope theorem to obtain

$$\text{(A.54)} \quad V_X(X_{t_{j+1}}, S_{t_{j+1}}) = \left[1 - (\mathbf{1}_{\{y^b(t_{j+1})>0\}} + \mathbf{1}_{\{y^s(t_{j+1})<0\}})\theta_X\right] \times \left(1 - (1-\alpha)\kappa b(\tau_{j+1})\right)U'\left(C(t_{j+1}, \tau_{j+1})\right),$$

which implies that $V_X(X_{t_{j+1}}, S_{t_{j+1}})$, the increase in expected lifetime utility made possible by a \$1 increase in $X_{t_{j+1}}$, equals the increase in utility that would accompany an increase of $1-(\mathbf{1}_{\{y^b(t_{j+1})>0\}}+\mathbf{1}_{\{y^s(t_{j+1})<0\}})\theta_X$ dollars in $C(t_{j+1},\tau_{j+1})$. That is, if the consumer transfers assets between the investment portfolio and the transactions account at time t_{j+1}, a \$1 increase in $X_{t_{j+1}}$ would allow $C(t_{j+1},\tau_{j+1})$ to increase by $1-\theta_X$ dollars; otherwise, $C(t_{j+1},\tau_{j+1})$ can increase by \$1. Differentiate (16) with respect to $C(t_j,\tau_j)$ and use (**) in footnote 18 to obtain

$$U'\big(C(t_j,\tau_j)\big)=c_{t_j^+}^{-\alpha}. \tag{A.55}$$

Substitute (A.54) into (A.53) and use (A.55) to obtain

$$\begin{aligned} V_S(X_{t_{j+1}},S_{t_{j+1}}) &= m(x_{t_{j+1}})\big[1-(\mathbf{1}_{\{y^b(t_{j+1})>0\}}+\mathbf{1}_{\{y^s(t_{j+1})<0\}})\theta_X\big] \\ &\quad\times\big(1-(1-\alpha)\kappa b(\tau_{j+1})\big)c_{t_{j+1}^+}^{-\alpha}. \end{aligned} \tag{A.56}$$

Substituting the right hand side of (A.56) for $V_S(X_{t_{j+1}},S_{t_{j+1}})$ in (A.52) and using Lemma 11 to set $\delta_j=0$ yields

$$\begin{aligned} E_{t_j}\bigg\{ & m(x_{t_{j+1}})\big[1-(\mathbf{1}_{\{y^b(t_{j+1})>0\}}+\mathbf{1}_{\{y^s(t_{j+1})<0\}})\theta_X\big] \\ &\times\big(1-(1-\alpha)\kappa b(\tau_{j+1})\big)c_{t_{j+1}^+}^{-\alpha}\bigg(\frac{P_{t_{j+1}}}{P_{t_j}}-e^{r_f\tau_j}\bigg)\bigg\}=\nu_j. \end{aligned} \tag{A.57}$$

In standard models without information costs and transfer costs, and without the constraints $0\le\phi_j\le 1$, the corresponding Euler equation, which is widely used in financial economics, is

$$E_t\bigg\{c_s^{-\alpha}\bigg(\frac{P_s}{P_t}-e^{r_f(s-t)}\bigg)\bigg\}=0 \quad \text{for} \quad s>t. \tag{A.58}$$

In general, the Euler equation in the presence of information costs and transactions costs in (A.57) differs from the standard Euler equation in (A.58) in five ways: (i) the Euler equation in (A.57) contains the Lagrange multiplier on the constraint $\phi_j\le 1$, but this Lagrange multiplier does not appear in the standard Euler equation; (ii) the Euler equation in (A.57) contains the marginal rate of substitution $m(x_{t_{j+1}})$, which is a random variable, but this marginal rate of substitution is absent (or implicitly equal to a constant) in the standard Euler equation[39]; (iii) the Euler equation in (A.57) contains the term $1-(\mathbf{1}_{\{y^b(t_{j+1})>0\}}+\mathbf{1}_{\{y^s(t_{j+1})<0\}})\theta_X$, which reflects the additional fixed transfer cost associated with having an additional dollar in the transactions account; (iv) the

[39] If assets could be transferred without any resource costs (i.e., if $\theta_X=\theta_S=\psi_s=\psi_b=0$), then $m(x_{t_j})=1$ at all observation dates and, hence, can be eliminated from (A.57).

Euler equation in (A.57) contains the term $1-(1-\alpha)\kappa b(\tau_{j+1})$, which reflects the utility cost of the next observation; and (v) in the presence of information costs, the Euler equation holds only for rates of return between observation dates, whereas the Euler equation in the standard case holds for rates of return between any arbitrary pair of dates because all dates are observation dates in the standard case. We show that in the long run, in an interesting special case, the first four of these differences disappear. Before showing this result, we prove the following lemma.

LEMMA 12: *Suppose that θ_S is sufficiently small in the sense described in the proof of Proposition* 2. *If $x_{t_j} \leq \omega_1$, then* (i) $\phi_j < 1$ *if* $\alpha > \frac{\mu - r_f}{\sigma^2}$ *and* (ii) $\phi_j = 1$ *if* $\alpha \leq \frac{\mu - r_f}{\sigma^2}$.

PROOF: Proposition 2 implies that if $x_{t_j} \leq \pi_1$, then $x_{t_{j+1}} = 0$. The optimal value of ϕ_j, $0 \leq \phi_j \leq 1$, maximizes $E_{t_j}\{V(X_{t_{j+1}}, S_{t_{j+1}})\} = \frac{1}{1-\alpha} E_{t_j}\{S_{t_{j+1}}^{1-\alpha} v(0)\}$, which is equivalent to maximizing $\varphi(\phi_j;\alpha) \equiv \frac{1}{1-\alpha} E_{t_j}\{[\phi_j \frac{P_{t_j+\tau_j}}{P_{t_j}} + (1-\phi_j)\times e^{r_f\tau_j}]^{1-\alpha}\}$. Define α^* such that $\arg\max_{\phi_j} \varphi(\phi_j;\alpha^*) = 1$ and note that $\varphi'(1;\alpha^*) = 0$.

Differentiating the definition of $\varphi(\phi_j;\alpha)$ with respect to ϕ_j and setting $\phi_j = 1$ yields

$$\varphi'(1;\alpha) = E_{t_j}\left\{\left(\frac{P_{t_j+\tau_j}}{P_{t_j}}\right)^{1-\alpha}\right\} - e^{r_f\tau_j} E_{t_j}\left\{\left(\frac{P_{t_j+\tau_j}}{P_{t_j}}\right)^{-\alpha}\right\}.$$

Use the fact that $\frac{P_{t_j+\tau_j}}{P_{t_j}}$ is log normal to obtain

$$\begin{aligned}\varphi'(1;\alpha) = {} & \exp\left[(1-\alpha)\left(\mu - \frac{1}{2}\alpha\sigma^2\right)\tau_j\right] \\ & - e^{r_f\tau_j}\exp\left[-\alpha\left(\mu + \frac{1}{2}(-\alpha-1)\sigma^2\right)\tau_j\right].\end{aligned}$$

Further rearrangement yields

$$\begin{aligned}\varphi'(1;\alpha) = {} & \exp\left[\left(-\alpha\mu + r_f - \frac{1}{2}\alpha(1-\alpha)\sigma^2\right)\tau_j\right] \\ & \times \left[\exp\big((\mu - r_f)\tau_j\big) - \exp\big(\alpha\sigma^2\tau_j\big)\right],\end{aligned}$$

which implies that

$$\varphi'(1;\alpha) \lesseqgtr 0 \quad \text{as} \quad \alpha \gtreqless \alpha^* \equiv (\mu - r_f)/\sigma^2.$$

Differentiate $\varphi(\phi_j;\alpha)$ twice with respect to ϕ_j to obtain

$$\varphi''(\phi_j;\alpha) = -\alpha E_{t_j}\left\{\left(\phi_j \frac{P_{t_j+\tau_j}}{P_{t_j}} + (1-\phi_j)e^{r_f\tau_j}\right)^{-\alpha-1}\left(\frac{P_{t_j+\tau_j}}{P_{t_j}} - e^{r_f\tau_j}\right)^2\right\}$$
$$< 0,$$

which implies that $\varphi(\phi_j;\alpha)$ is concave. If $\alpha > \alpha^*$, then $\varphi'(1;\alpha) < 0$, so the concavity of $\varphi(\phi_j;\alpha)$ implies that the optimal value of ϕ_j is less than 1 and the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\nu_j = 0$. If $\alpha \leq \alpha^*$, then $\varphi'(1;\alpha) \geq 0$, so the concavity of $\varphi(\phi_j;\alpha)$ implies that the optimal value of ϕ_j equals 1. If $\alpha < \alpha^*$, the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\nu_j > 0$. *Q.E.D.*

Suppose that θ_S is sufficiently small so that in the long run, the stochastic process for x_{t_j} is absorbed at zero. Lemma 12 implies that if the coefficient of relative risk aversion α exceeds $\frac{\mu-r_f}{\sigma^2}$, then in the long run, the constraint $\phi_j \leq 1$ does not bind and, hence, $\nu_j = 0$. In this case, the first of the five differences between the Euler equation in (A.57) and the standard Euler equation disappears. In addition, in the long run, $x_{t_j} = 0$ on each observation date t_j, so (i) $m(x_{t_j}) = (1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$ on each observation date, (ii) the consumer sells assets from the investment portfolio on each observation date so $1 - (\mathbf{1}_{\{y^b_{t_{j+1}}>0\}} + \mathbf{1}_{\{y^s_{t_{j+1}}<0\}})\theta_X = 1 - \theta_X$ on each observation date, and (iii) the time between consecutive observations is constant so $1-(1-\alpha)\kappa b(\tau_{j+1})$ is constant. Using the fact that $\nu_j = 0$ and dividing both sides of (A.57) by $(1-\psi_s)(1-\theta_S)(1-(1-\alpha)\kappa b(\tau_{j+1}))$ proves Proposition 4. *Q.E.D.*

APPENDIX B: BASIC PROPERTIES OF THE OPTIMIZATION PROBLEM AND THE VALUE FUNCTION

The goal of this section is to establish some basic properties of the optimization problem that we consider in the paper. Specifically, we show that the value function $V(X_{t_j}, S_{t_j})$ is finite, homogeneous of degree $1-\alpha$, continuous, and satisfies the Bellman equation (20). Moreover, we show that there exist policies that attain the supremum on the right hand side of (20) and that these policies are optimal. The main result is formulated in Proposition 7. In preparation for Proposition 7, we state and prove four lemmas.

LEMMA 13: *Let $a_j \equiv \{C(t_j,\tau_j), y^b(t_j), y^s(t_j), \phi_j, \tau_j\}$ denote a strategy that is feasible given X_{t_j} and S_{t_j}, and let*

$$\mathcal{U}(a_{j=1,\dots,\infty}) \equiv E_{t_j}\left\{\int_{t_j}^{\infty} \frac{1}{1-\alpha} c_t^{1-\alpha} e^{-\rho(t-t_j)}\,dt - \sum_{i=j}^{\infty} A(t_i,\tau_i)e^{-\rho(t_i+\tau_i-t_j)}\right\}$$

denote the expected payoff from following the policy $a_{j=1,\dots,\infty}$. *Furthermore, let* $V(X_{t_j}, S_{t_j}) = \sup_{\alpha_{j=1,\dots,\infty}} \mathcal{U}(\alpha_{j=1,\dots,\infty})$ *denote the value function of the problem. Then* $V(X_{t_j}, S_{t_j})$ *satisfies* (20).

PROOF: First, we observe that $-\infty < V(X_{t_j}, S_{t_j}) < \infty$. To see this, we note first that there exist policies that are feasible and lead to a finite $\mathcal{U}(\alpha_{j=1,\dots,\infty})$. For instance, setting $\tau_1 = \infty$, $y_s(t_j) = -[1-\theta_S]S_{t_j}$, and $C(t_j, \infty) = X_{t_j^+} = (1-\theta_X)X_{t_j} + (1-\psi_s)(1-\theta_S)S_{t_j}$ implies the discounted utility $\frac{1}{1-\alpha}\chi^{-\alpha}X_{t_j^+}^{1-\alpha}$, where $\chi \equiv \frac{\rho-(1-\alpha)r_L}{\alpha} > 0$. Accordingly, the value function is bounded below. Furthermore, the value function is bounded above by the value function that can be attained if we remove all transactions and observations costs ($\kappa = \theta_S = \theta_X = \psi_s = \psi_b = 0$). But by removing all these frictions, the problem becomes identical to the standard, continuous-time Merton (1971) problem with portfolio weights restricted to $\phi \in [0, 1]$. Since that problem has a finite value function (as long as (7) holds), we conclude that the value function is bounded above and, hence, is finite.[40]

From this point on, the proof mimics closely the proof given in Stokey and Lucas (1989, Theorem 4.2). We observe that the definition of $V(X_{t_j}, S_{t_j})$ implies that

$$\text{(B.1)} \qquad V(X_{t_j}, S_{t_j}) \geq \mathcal{U}(a_{j=1,\dots,\infty}) \quad \text{for all } a_{j=1,\dots,\infty}$$

and that for (arbitrarily small) $\delta > 0$, there exists some strategy $a_{j=1,\dots,\infty}$ such that

$$\text{(B.2)} \qquad \mathcal{U}(a_{j=1,\dots,\infty}) \geq V(X_{t_j}, S_{t_j}) - \delta.$$

(If (B.2) were not true, then we must have

$$V(X_{t_j}, S_{t_j}) = \sup_{\alpha_{j=1,\dots,\infty}} \mathcal{U}(a_{j=1,\dots,\infty}) < V(X_{t_j}, S_{t_j}) - \delta,$$

which is absurd.) Now, take $\varepsilon > 0$. To show that $V(X_{t_j}, S_{t_j})$ satisfies (20), we will show that

$$\text{(B.3)} \qquad V(X_{t_j}, S_{t_j}) \geq \left[1-(1-\alpha)\kappa b(\tau_j)\right] \times U\big(C(t_j, \tau_j)\big) + e^{-\rho\tau_j} E_{t_j}\left\{V\left(e^{r_L\tau_j}\big(X_{t_j^+} - C(t_j, \tau_j)\big), R(t_j, \tau_j)S_{t_j^+}\right)\right\}$$

[40]For a detailed analysis of the infinite-horizon version of Merton's problem and the condition for its value function to be finite, see, for example, the monograph Karatzas and Shreve (1998, p. 149).

for all a_j and that there exists a_j such that for any (arbitrarily small) $\varepsilon > 0$,

$$V(X_{t_j}, S_{t_j}) \leq \big[1-(1-\alpha)\kappa b(\tau_j)\big] \times U\big(C(t_j, \tau_j)\big) + e^{-\rho\tau_j} E_{t_j}\big\{V\big(e^{r_L\tau_j}\big(X_{t_j^+} - C(t_j, \tau_j)\big), R(t_j, \tau_j)S_{t_j^+}\big)\big\} + \varepsilon. \tag{B.4}$$

To show (B.3), note that by (B.2) there exists a policy sequence $a' = a'_{t_j, t'_{j+1}, \ldots}$ such that $\mathcal{U}(\alpha'_{t'_{j+1}, t'_{j+2}, \ldots}) \geq V(X_{t'_{j+1}}, S_{t'_{j+1}}) - \frac{\varepsilon}{2}$. Moreover, using the definition of $U(C(t_j, \tau_j))$ implies the existence of a policy $c'_{t\in(t_j, t_j+\tau'_j)}$ such that $\frac{1}{1-\alpha}[1-(1-\alpha)\kappa b(\tau'_j)] \times \int_{t_j}^{t_j+\tau'_j} (c'_t)^{1-\alpha} e^{-\rho(t-t_j)}\, dt \geq [1-(1-\alpha)\kappa b(\tau'_j)] \times U(C(t_j, \tau'_j)) - \frac{\varepsilon}{2}$. Accordingly,

$$\begin{aligned}
V(X_{t_j}, S_{t_j}) &\geq \mathcal{U}\big(a'_{t'_j, t'_{j+1}, \ldots}\big) \\
&= \frac{1}{1-\alpha}\bigg\{\big[1-(1-\alpha)\kappa b\big(\tau'_j\big)\big] \int_{t_j}^{t_j+\tau'_j} \big(c'_t\big)^{1-\alpha} e^{-\rho(t-t_j)}\, dt\bigg\} \\
&\quad + e^{-\rho\tau'_j} E_{t_j} \mathcal{U}\big(\alpha'_{t'_{j+1}, t'_{j+2}, \ldots}\big) \\
&\geq \big[1-(1-\alpha)\kappa b\big(\tau'_j\big)\big] \times U\big(C\big(t_j, \tau'_j\big)\big) - \frac{\varepsilon}{2} \\
&\quad + e^{-\rho\tau'_j} E_{t_j} V\big(e^{r_L\tau_j}\big(X'_{t_j^+} - C\big(t_j, \tau'_j\big)\big), R\big(t_j, \tau'_j\big)S'_{t_j^+}\big) - e^{-\rho\tau'_j}\frac{\varepsilon}{2} \\
&\geq \big[1-(1-\alpha)\kappa b\big(\tau'_j\big)\big] \times U\big(C\big(t_j, \tau'_j\big)\big) \\
&\quad + e^{-\rho\tau'_j} E_{t_j} V\big(e^{r_L\tau_j}\big(X'_{t_j^+} - C\big(t_j, \tau'_j\big)\big), R\big(t_j, \tau'_j\big)S'_{t_j^+}\big) - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain (B.3). To show (B.4), choose $\varepsilon > 0$ and take a policy $a'_{t'_j, t'_{j+1}, \ldots}$ such that $V(X_{t_j}, S_{t_j}) \leq \mathcal{U}(a'_{t'_j, t'_{j+1}, \ldots}) + \varepsilon$. Accordingly

$$\begin{aligned}
V(X_{t_j}, S_{t_j}) &\leq \mathcal{U}\big(\alpha'_{t'_j, t'_{j+1}, \ldots}\big) + \varepsilon \\
&= \frac{1}{1-\alpha}\bigg\{\big[1-(1-\alpha)\kappa b\big(\tau'_j\big)\big] \int_{t_j}^{t_j+\tau'_j} \big(c'_t\big)^{1-\alpha} e^{-\rho(t-t_j)}\, dt\bigg\} \\
&\quad + e^{-\rho\tau'_j} E_{t_j} \mathcal{U}\big(\alpha'_{t'_{j+1}, t'_{j+2}, \ldots}\big) + \varepsilon \\
&\leq \big[1-(1-\alpha)\kappa b\big(\tau'_j\big)\big] \times U\big(C\big(t_j, \tau'_j\big)\big) \\
&\quad + e^{-\rho\tau'_j} E_{t_j} V\big(e^{r_L\tau_j}\big(X'_{t_j^+} - C\big(t_j, \tau'_j\big)\big), R\big(t_j, \tau'_j\big)S'_{t_j^+}\big) + \varepsilon.
\end{aligned} \tag{B.5}$$

Q.E.D.

Lemma 13 shows that the value function satisfies (20). The next lemma shows that the reverse conclusion holds, subject to two additional conditions.

LEMMA 14: *If* $\widehat{V}(X_{t_j}, S_{t_j})$ *satisfies* (20) *with the supremum on the right hand side of* (20) *attained for some policy, and if* $\lim_{t_k\to\infty} e^{-\rho t_k} E_{t_j}\widehat{V}(X_{t_k}, S_{t_k}) = 0$ *for all* $(X_{t_j}, S_{t_j}) \in R^2_+$ *and for all feasible* $a_{j=1,\dots,\infty}$, *then* $\widehat{V} = V$ *and the policy that attains the supremum on the right hand side of* (20) *is an optimal policy for the intertemporal optimization problem*.

PROOF: The proof closely follows Stokey and Lucas (1989), so we give a brief sketch of some minor adaptations that are required so as to deal with the specifics of our setup. Iterating on (20) implies that if we adopt any feasible policy tuple a_j, we obtain

$$\text{(B.6)}\qquad \widehat{V}(X_{t_j}, S_{t_j}) \geq E_{t_j} \sum_{i=j,\dots,k} \big[1-(1-\alpha)\kappa b(\tau_i)\big] e^{-\rho(t_i-t_j)} U\big(C(t_i,\tau_i)\big) + e^{-\rho(t_{k+1}-t_j)} E_{t_j}\widehat{V}(X_{t_{k+1}}, S_{t_{k+1}}) \quad \text{for any } k \geq j.$$

Now if the feasible policy a_j involves a finite number of observations (so that $\tau_{k+1} = \infty$), then (B.6) shows that $\widehat{V}(X_{t_j}, S_{t_j})$ is an upper bound to the payoff of a_j since $\widehat{V}(X_{t_{k+1}}, S_{t_{k+1}}) \geq \widetilde{V}(X_{t_{k+1}}, S_{t_{k+1}}; \tau_j = \infty) \geq \mathcal{U}(a_{k+1})$, where $\widetilde{V}(X_{t_{k+1}}, S_{t_{k+1}}; \tau_j = \infty)$ is the maximized value of the right hand side of (20) restricted by $\tau_{k+1} = \infty$ and $\mathcal{U}(a_{k+1})$ denotes the payoff from following the strategy a_{k+1} for t_{k+1} onward. If the feasible policy involves an infinite number of observations, then taking $t_{k+1} \to \infty$ and using $\lim_{t_{k+1}\to\infty} e^{-\rho t_{k+1}} E_{t_j}\widehat{V}(X_{t_{k+1}}, S_{t_{k+1}}) = 0$, we once again conclude that $\widehat{V}(X_{t_j}, S_{t_j})$ provides an upper bound to the payoff from following $a_{j=1,\dots,\infty}$. Furthermore, the inequality in (B.6) becomes an equality for the policy that attains the maximum on the right hand side of (20). Accordingly, that policy is optimal and $\widehat{V}(X_{t_j}, S_{t_j})$ is the value function. *Q.E.D.*

The next lemma shows that the value function is homogeneous of degree $1-\alpha$.

LEMMA 15: *Letting* $x_t \equiv \frac{X_t}{S_t}$, *the value function satisfies* (21).

PROOF: Consider an optimal policy $a^A_{j=1,\dots,\infty}$ associated with the initial state variables $(X^A_{t_j}, S^A_{t_j})$. Now suppose that we consider the initial state variables $(X^B_{t_j}, 1)$ and, additionally, we assume that $x^A_{t_j} = x^B_{t_j}$. Construct a policy $a^B_{j=1,\dots,\infty}$ as follows. For all $j = 1, \dots, \infty$, let $\tau^B_j = \tau^A_j$ and $\phi^B_j = \phi^A_j$, and also let

$C^B(t_j, \tau_j^B) = \frac{1}{S_{t_j}^A} C^A(t_j, \tau_j^A)$, $y^{b,B}(t_j) = \frac{1}{S_{t_j}^A} y^{b,A}(t_j)$, and $y^{s,B}(t_j) = \frac{1}{S_{t_j}^A} y^{s,A}(t_j)$. Using (4), (5), and (19), and the fact that policy A is feasible, it is straightforward to verify that policy B is feasible and implies a consumption process that is equal to $c_t^B = \frac{c_t^A}{S_{t_j}^A}$ for all $t \in (t_j, t_{j+1}]$, all t_j, and all realizations of uncertainty. Accordingly, $V(X_{t_j}^B, 1) = V(x_{t_j}^B, 1) = V(x_{t_j}^A, 1) \geq \frac{1}{(S_{t_j}^A)^{1-\alpha}} V(X_{t_j}^A, S_{t_j}^A)$.

Similarly, consider a path that is optimal for $(X_{t_j}^B, 1)$. Repeating the same arguments as above, the policy defined by $\tau_j^A = \tau_j^B$, $\phi_j^A = \phi_j^B$, $C^A(t_j, \tau_j^A) = S_{t_j}^A C^B(t_j, \tau_j^B)$, $y^{b,A}(t_j) = S_{t_j}^A y^{b,B}(t_j)$, and $y^{s,A}(t_j) = S_{t_j}^A y^{s,B}(t_j)$ is feasible starting from $(X_{t_j}^A, S_{t_j}^A)$, assuming always that $x_{t_j}^A = x_{t_j}^B$. Moreover, this policy implies that $c_t^A = S_{t_j}^A c_t^B$ for all $t \in (t_j, t_{j+1}]$, all t_j, and all realizations of uncertainty. Accordingly, $V(X_{t_j}^A, S_{t_j}^A) \geq (S_{t_j}^A)^{1-\alpha} V(X_{t_j}^B, 1) = (S_{t_j}^A)^{1-\alpha} V(x_{t_j}^A, 1)$.

Now letting $v(x_{t_j}) \equiv (1-\alpha)V(x_{t_j}, 1)$ and using $\frac{v(x_{t_j}^A)}{1-\alpha} = V(x_{t_j}^A, 1) \geq \frac{1}{(S_{t_j}^A)^{1-\alpha}} \times V(X_{t_j}^A, S_{t_j}^A)$ together with $\frac{v(x_{t_j}^A)}{1-\alpha}(S_{t_j}^A)^{1-\alpha} = V(x_{t_j}^A, 1)(S_{t_j}^A)^{1-\alpha} \leq V(X_{t_j}^A, S_{t_j}^A)$ yields (21). *Q.E.D.*

In preparation for the main proposition, we also introduce the norm

$$\text{(B.7)} \qquad \|f\| \equiv \max_{X_t, S_t \in R_+^2 \text{ s.t. } X_t + S_t = 1} \left| f(X_t, S_t) \right|.$$

We let $\mathcal{B}$ denote the set of functions that map $R_+^2 \to R_+$ if $\alpha < 1$ (respectively, $R_+^2 \to R_-$ if $\alpha > 1$) that are homogeneous of degree $1-\alpha$ and bounded in the norm defined in (B.7). Similarly, we let $\mathcal{H}$ denote the set of functions that belong in $\mathcal{B}$ and additionally are continuous. Finally, define the operator T applied to function f as

$$\text{(B.8)} \qquad Tf \equiv \sup_{C(t_j,\tau_j), y^b(t_j), y^s(t_j), \phi_j, \tau_j} \Big\{ \big[1 - (1-\alpha)\kappa b(\tau_j)\big] U\big(C(t_j, \tau_j)\big) + e^{-\rho\tau_j} E_{t_j}\big[f(X_{t_{j+1}}, S_{t_{j+1}})\big] \Big\}.$$

The next proposition contains our main result.

PROPOSITION 7: *The operator Tf maps $\mathcal{H}$ into $\mathcal{H}$ and has a fixed point in $\mathcal{H}$, which is the value function V. Moreover, for $f = V$, there exist policies that attain the optimum on the right hand side of* (B.8) *and these policies are optimal.*

PROOF: First, we prove that Tf maps $\mathcal{H}$ into $\mathcal{H}$. Using the definition of $U(C(t_j, \tau_j))$ and inspection of (B.8), it is immediate that if f is homogeneous of degree $1-\alpha$ (so that it can be expressed as $\frac{1}{1-\alpha}(X_t+S_t)^{1-\alpha}\widetilde{f}(\widetilde{x}_t)$), then so is Tf. Next we observe that if f is bounded in the norm (B.7), so is Tf. In the case $\alpha > 1$, the result is immediate, since Tf is bounded above by zero and below by the feasible policy that sets $\tau_1 = \infty$, $y_s(t_j) = -[1-\theta_S]S_{t_j}$, and $C(t_j, \infty) = X_{t_j^+}$, which implies the discounted utility $\underline{U} \equiv \frac{1}{1-\alpha}(X_{t_j} + S_{t_j})^{1-\alpha}\chi^{-\alpha}[(1-\theta_X)\widetilde{x}_{t_j} + (1-\psi_s)(1-\theta_S)(1-\widetilde{x}_{t_j})]^{1-\alpha}$, where $\chi \equiv \frac{\rho-(1-\alpha)r_L}{\alpha} > 0$. Clearly, this feasible policy is bounded in the norm (B.7), and thus Tf is bounded in the norm (B.7). In the case where $\alpha < 1$, we note that $\underline{U}$ still provides a lower bound. To derive an upper bound, let $l(\tau_j) \equiv [1-(1-\alpha)\kappa b(\tau_j)] \times [h(\tau_j)]^\alpha$, define $G \equiv \sup_{\tau_j>0} l(\tau_j)$, and observe that the assumptions of Lemma 1 imply that G is finite.[41] In turn, this implies that $[1-(1-\alpha)\kappa b(\tau_j)] \times U(C(t_j, \tau_j)) = [1-(1-\alpha)\kappa b(\tau_j)] \times \frac{1}{1-\alpha}[h(\tau_j)]^\alpha[C(t_j, \tau_j)]^{1-\alpha} \leq \frac{1}{1-\alpha}G[X_{t_j} + S_{t_j}]^{1-\alpha}$ is bounded in the norm (B.7). Moreover, $\|e^{-\rho\tau_j}E_{t_j}f(X_{t_{j+1}}, S_{t_{j+1}})\| \leq \|f(X_t, S_t)\|$ is bounded in the norm (B.7), since f is bounded in the norm (B.7). Accordingly, Tf is bounded in the norm (B.7) for both $\alpha < 1$ and $\alpha > 1$. Finally, Tf maps continuous functions to continuous functions. (To see this, note that the right hand side of (B.8) can be expressed as the maximum of three functions, namely the maximal value conditional on $y^b > 0$, conditional on $y^s < 0$, and conditional on $y^b = y^s = 0$. Each of these functions is continuous by a version of the theorem of the maximum (see in particular Alvarez and Stokey (1998)[42]), and hence so is the maximum of the three functions.) We conclude that Tf maps $\mathcal{H}$ into $\mathcal{H}$.

To show that Tf has a fixed point in $\mathcal{H}$, we adapt the arguments in Alvarez and Stokey (1998). Specifically, we distinguish two cases, depending on whether $\alpha < 1$ or $\alpha > 1$. The case $\alpha < 1$ allows a relatively straightforward proof based on a contraction mapping argument. The case $\alpha > 1$ requires a different set of arguments. It is useful to note that the proof that we develop for the case $\alpha > 1$ would provide an alternate proof (with obvious modifications) for the case $\alpha < 1$, but the reverse is not true.

We start with the case $\alpha < 1$. For this case, we start by proving the following implication of assumption (7).

LEMMA 16: *For all $\tau_j > 0$ and all $\phi_j \in [0, 1]$, assumption (7) implies that*

$$e^{-\rho\tau_j}E_{t_j}\left\{\left[R(t_j, \tau_j)\right]^{1-\alpha}\right\} < 1. \tag{B.9}$$

[41]To see this, note that $l(\tau_j)$ is continuous, $\lim_{\tau_j\to 0} l(\tau_j) \leq 0$, and $\lim_{\tau_j\to\infty} l(\tau_j) = \frac{1}{\chi^\alpha} < \infty$.

[42]Notice in particular that Tf is always bounded below by $\underline{U}$ for any $f \in \mathcal{H}$.

PROOF: To simplify notation, we fix some t_j and t_{j+1}, and for any $t \in [t_j, t_{j+1}]$, we let $R_t \equiv R(t_j, t - t_j) = \phi_j \frac{P_t}{P_{t_j}} + (1 - \phi_j)e^{r_f(t-t_j)}$. Applying Ito's lemma gives

$$dR_t = \phi_j \frac{P_t}{P_{t_j}} \mu \, dt + (1 - \phi_j)e^{r_f(t-t_j)} r_f \, dt + \phi_j \frac{P_t}{P_{t_j}} \sigma \, dz_t. \tag{B.10}$$

Dividing both sides of (B.10) by R_t and letting $\overline{\pi}_t \equiv \frac{\phi_j(P_t/P_{t_j})}{\phi_j(P_t/P_{t_j}) + (1-\phi_j)e^{r_f(t-t_j)}}$ gives

$$\frac{dR_t}{R_t} = \overline{\pi}_t \mu \, dt + (1 - \overline{\pi}_t) r_f \, dt + \overline{\pi}_t \sigma \, dz_t. \tag{B.11}$$

The unique solution of the linear stochastic differential equation (B.11) for $t \in [t_j, t_{j+1}]$ subject to the initial condition $R_{t_j} = R(t_j, 0) = 1$ is given by

$$R_t = e^{\int_{t_j}^{t} [\overline{\pi}_t \mu - (1/2)\overline{\pi}_t^2 \sigma^2 + (1-\overline{\pi}_t) r_f] \, dt + \int_{t_j}^{t} \overline{\pi}_t \sigma \, dz_t}. \tag{B.12}$$

Using (B.12) and recalling that $R_{t_{j+1}} = R(t_j, t_{j+1} - t_j) = R(t_j, \tau_j)$, we obtain

$$\begin{aligned} & e^{-\rho\tau_j} E\{[R(t_j, \tau_j)]^{1-\alpha}\} \\ & \quad = E\{e^{-\rho\tau_j + (1-\alpha)\{\int_{t_j}^{t_{j+1}} [\overline{\pi}_t \mu - (1/2)\overline{\pi}_t^2 \sigma^2 + (1-\overline{\pi}_t) r_f] \, dt + \int_{t_j}^{t_{j+1}} \overline{\pi}_t \sigma \, dz_t\}}\} \\ & \quad \le \max_{\pi_t} E\{e^{-\rho\tau_j + (1-\alpha)\{\int_{t_j}^{t_{j+1}} [\pi_t \mu - (1/2)\pi_t^2 \sigma^2 + (1-\pi_t) r_f] \, dt + \int_{t_j}^{t_{j+1}} \pi_t \sigma \, dz_t\}}\}. \end{aligned} \tag{B.13}$$

In light of (B.11) and (B.12), the maximization problem in (B.13) is identical to the Merton-type problem of maximizing $\max_{\overline{\pi}_t} e^{-\rho\tau_j} E\{R_{t_{j+1}}^{1-\alpha}\}$ subject to the constant-investment-opportunity-set dynamics (B.11), which has the well known constant rebalancing solution $\pi_t = \pi = \frac{\mu - r_f}{\alpha\sigma^2}$. Substituting this solution into (B.13), letting

$$\nu \equiv (1 - \alpha)\left[r_f + \frac{1}{2\alpha}\left(\frac{\mu - r_f}{\sigma}\right)^2\right],$$

and utilizing properties of the log-normal distribution gives

$$\max_{\pi_t \in [0,1]} E\{e^{-\rho\tau_j + (1-\alpha)\{\int_{t_j}^{t_{j+1}} [\pi_t \mu - (1/2)\pi_t^2 \sigma^2 + (1-\pi_t) r_f] \, dt + \int_{t_j}^{t_{j+1}} \pi_t \sigma \, dz_t\}}\} = e^{(\nu-\rho)\tau_j} < 1. \tag{B.14}$$

Combining (B.14) with (B.13) and noting that ϕ_j, τ_j are arbitrary implies (B.9). *Q.E.D.*

We next define $\widetilde{x}_t \equiv \frac{X_t}{X_t+S_t} = \frac{x_t}{x_t+1}$ and observe that $x_t = \frac{\widetilde{x}_t}{1-\widetilde{x}_t}$. Because of (21), we obtain that

$$\begin{aligned} V(X_{t_j}, S_{t_j}) &= \frac{1}{1-\alpha}(X_t+S_t)^{1-\alpha}\left(\frac{S_t}{X_t+S_t}\right)^{1-\alpha} v(x_t) \\ &= \frac{1}{1-\alpha}(X_t+S_t)^{1-\alpha}(1-\widetilde{x}_t)^{1-\alpha} v\left(\frac{\widetilde{x}_t}{1-\widetilde{x}_t}\right) \\ &= \frac{1}{1-\alpha}(X_t+S_t)^{1-\alpha} v^*(\widetilde{x}_t), \end{aligned}$$

where $v^*(\widetilde{x}_t) \equiv (1-\widetilde{x}_t)^{1-\alpha} v(\frac{\widetilde{x}_t}{1-\widetilde{x}_t})$.

In that case, we obtain that

$$\begin{aligned} &E_{t_j}\left\{e^{-\rho\tau_j}\left(\frac{X_{t_{j+1}}+S_{t_{j+1}}}{X_{t_j}+S_{t_j}}\right)^{1-\alpha}\right\} \\ &\quad = E_{t_j}\left\{e^{-\rho\tau_j}\left(\frac{e^{r_L\tau_j}(X_{t_j^+}-C(t_j,\tau_j))+R(t_j,\tau_j)S_{t_j^+}}{X_{t_j}+S_{t_j}}\right)^{1-\alpha}\right\} \\ &\quad \le \left(\frac{X_{t_j^+}+S_{t_j^+}}{X_{t_j}+S_{t_j}}\right)^{1-\alpha} e^{-\rho\tau_j} E_{t_j}\left\{\left(e^{r_L\tau_j}\widetilde{x}_{t_j^+}+R(t_j,\tau_j)(1-\widetilde{x}_{t_j^+})\right)^{1-\alpha}\right\}, \end{aligned} \tag{B.15}$$

where the inequality follows from $C(t_j,\tau_j)\ge 0$ and the definition of $\widetilde{x}_{t_j^+}$. Next we show that

$$\begin{aligned} &e^{-\rho\tau_j} E_{t_j}\left\{\left(e^{r_L\tau_j}\widetilde{x}_{t_j^+}+R(t_j,\tau_j)(1-\widetilde{x}_{t_j^+})\right)^{1-\alpha}\right\} \\ &\quad \le \max_{\phi_j\in[0,1]} e^{-\rho\tau_j} E_{t_j}\left\{R(t_j,\tau_j)^{1-\alpha}\right\}. \end{aligned} \tag{B.16}$$

To see why (B.16) holds, note that for any $\phi_j \in [0,1]$ and $\widetilde{x}_{t_j^+} \in [0,1]$, we obtain $e^{r_L\tau_j}\widetilde{x}_{t_j^+} + R(t_j,\tau_j;\phi_j)\times(1-\widetilde{x}_{t_j^+}) = e^{r_L\tau_j}\widetilde{x}_{t_j^+} + (\phi_j\frac{P_{t_j+\tau_j}}{P_{t_j}} + (1-\phi_j))\times(1-\widetilde{x}_{t_j^+}) \le \phi_j(1-\widetilde{x}_{t_j^+})\frac{P_{t_j+\tau_j}}{P_{t_j}} + ((1-\phi_j)(1-\widetilde{x}_{t_j^+})+\widetilde{x}_{t_j^+})e^{r_f\tau_j} = \phi_j(1-\widetilde{x}_{t_j^+})\frac{P_{t_j+\tau_j}}{P_{t_j}} + (1-\phi_j(1-\widetilde{x}_{t_j^+}))e^{r_f\tau_j} = R(t_j,\tau_j;\phi_j(1-\widetilde{x}_{t_j^+}))$. Therefore, $E_{t_j}(e^{r_L\tau_j}\widetilde{x}_{t_j^+} + R(t_j,\tau_j;\phi_j)(1-\widetilde{x}_{t_j^+}))^{1-\alpha} \le E_{t_j}(R(t_j,\tau_j;\phi_j(1-\widetilde{x}_{t_j^+})))^{1-\alpha} \le \max_{\phi_j\in[0,1]} E_{t_j}\{R(t_j,\tau_j)^{1-\alpha}\}$.

Using (B.16) inside (B.15), noting that $\frac{X_{t_j^+}+S_{t_j^+}}{X_{t_j}+S_{t_j}} \le 1$, and using (B.9), implies that $E_{t_j}\{e^{-\rho\tau_j}(\frac{X_{t_{j+1}}+S_{t_{j+1}}}{X_{t_j}+S_{t_j}})^{1-\alpha}\} < 1$. Suppose next that we choose some (arbitrarily small) $\varepsilon > 0$ and we confine attention to choices $\tau_j \ge \varepsilon$. (Also define $T^{(\varepsilon)}$ to equal T subject to $\tau_j \ge \varepsilon$.) Then we obtain that there exists

$\beta < 1$ such that $e^{-\rho\tau_j} E_{t_j} R(t_j, \tau_j)^{1-\alpha} \le \beta$.[43] Therefore, for any constant η and any function $f \in \mathcal{H}$, the operator $T^{(\varepsilon)}$ satisfies the "discounting" property $T^{(\varepsilon)}(f + \eta(X_{t_j} + S_{t_j})^{1-\alpha}) \le T^{(\varepsilon)} f + \eta\beta(X_{t_j} + S_{t_j})^{1-\alpha}$. Furthermore, the operator $T^{(\varepsilon)}$ satisfies the monotonicity property $f \le g \Rightarrow T^{(\varepsilon)} f \le T^{(\varepsilon)} g$. Accordingly, the operator $T^{(\varepsilon)}$ is a contraction by Lemma 1 in Alvarez and Stokey (1998) (Boyd's lemma) and possesses a unique fixed point $V^{(\varepsilon)}$. Since this fixed point is in $\mathcal{H}$ (so that, in particular, $TV^{(\varepsilon)}$ is bounded below and above in the norm (B.7) and continuous), it implies that the supremum on the right hand side of (B.8) is attained. Furthermore, the fixed point $V^{(\varepsilon)}$ is in $\mathcal{H}$ and hence in $\mathcal{B}$. But note that for any function $f \in \mathcal{B}$, we obtain $0 \le \lim_{t_k\to\infty} e^{-\rho t_k} E_{t_j} f(X_{t_k}, S_{t_k}) = \lim_{t_k\to\infty} e^{-\rho t_k} E_{t_j} (X_{t_k} + S_{t_k})^{1-\alpha} \frac{f(X_{t_k}, S_{t_k})}{(X_{t_k}+S_{t_k})^{1-\alpha}} \le (X_{t_j} + S_{t_j})^{1-\alpha} \|f\| \times \lim_{t_k\to\infty} e^{-\rho t_k} E_{t_j} (\frac{X_{t_k}+S_{t_k}}{X_{t_j}+S_{t_j}})^{1-\alpha} = 0$. Accordingly, by Lemma 14, $V^{(\varepsilon)}$ is the value function subject to the additional constraint $\tau_j \ge \varepsilon$.

Next consider a sequence of $\varepsilon_k > 0$ such that $\lim_{k\to\infty} \varepsilon_k = 0$. The associated sequence $V^{(\varepsilon_k)}(X_{t_j}, S_{t_j})$ is a nondecreasing sequence of functions, which is bounded above by $V^{(0)}(X_{t_j}, S_{t_j})$. Hence this sequence of functions converges pointwise to a limit $\overline{V} = \lim_{\varepsilon_k\to 0} V^{(\varepsilon_k)}$. The completeness of $\mathcal{H}$ implies that $\overline{V} \in \mathcal{H}$. Moreover, $\overline{V} = \lim_{\varepsilon_k\to 0} V^{(\varepsilon_k)} = \lim_{\varepsilon_k\to 0} T^{(\varepsilon_k)} V^{(\varepsilon_k)} = T\overline{V}$, where the last equality follows upon applying the theorem of the maximum to

$$\lim_{\varepsilon_k\to 0} \sup_{C(t_j,\tau_j), y^b(t_j), y^s(t_j), \phi_j, \tau_j \ge \varepsilon_k} \big[1 - (1-\alpha)\kappa b(\tau_j)\big] U\big(C(t_j, \tau_j)\big) + e^{-\rho\tau_j} E_{t_j}\big\{V^{(\varepsilon_k)}(X_{t_{j+1}}, S_{t_{j+1}})\big\}$$

and observing that the monotone convergence theorem implies that $\lim_{\varepsilon_k\to 0} E_{t_j}\{V^{(\varepsilon_k)}(X_{t_{j+1}}, S_{t_{j+1}})\} = E_{t_j}\overline{V}(X_{t_{j+1}}, S_{t_{j+1}})$. Accordingly, $\overline{V} \in \mathcal{H}$ is a fixed point of (B.8). And since $\overline{V} \in \mathcal{H}$ (so that, in particular, it is continuous and bounded in the norm (B.7)), it satisfies the rest of the requirements[44] of

[43]The fact that there exists such β follows from the fact that $e^{-\rho\tau_j} E_{t_j} R(t_j, \tau_j)^{1-\alpha} \le \sup_{\tau_j \ge \varepsilon > 0, \phi_j \in [0,1]} e^{-\rho\tau_j} E_{t_j} R(t_j, \tau_j)^{1-\alpha} = \max_{\tau_j \ge \varepsilon > 0, \phi_j \in [0,1]} e^{-\rho\tau_j} E_{t_j} R(t_j, \tau_j)^{1-\alpha}$. To see why the supremum is attained, we note that a continuous function on a closed set attains its maximum, so that on any set $[\varepsilon, \overline{\overline{\tau}}]$, the function $e^{-\rho\tau_j} E_{t_j} R(t_j, \tau_j)^{1-\alpha}$ attains a maximum. Moreover, since $\lim_{\tau_j\to\infty} e^{-\rho\tau_j} E_{t_j} R(t_j, \tau_j)^{1-\alpha} = 0$, there exists $\overline{\overline{\tau}}$ and $\widehat{\tau} \ge \overline{\overline{\tau}}$ such that $e^{-\rho\overline{\overline{\tau}}_j} E_{t_j} R(t_j, \overline{\overline{\tau}})^{1-\alpha} \ge e^{-\rho\tau} E_{t_j} R(t_j, \tau)^{1-\alpha}$ for all $\tau > \widehat{\tau}$. Accordingly, we can confine attention to closed sets of τ. Finally, by (B.9), $\max_{\tau_j \ge \overline{\tau} > 0, \phi_j \in [0,1]} e^{-\rho\tau_j} E_{t_j} R(t_j, \tau_j)^{1-\alpha} < 1$.

[44]We note that even if we remove the requirement that $\tau_j \ge \varepsilon$, it is still the case that $\lim_{t_k\to\infty} e^{-\rho t_k} E_{t_j} (\frac{X_{t_k}+S_{t_k}}{X_{t_j}+S_{t_j}})^{1-\alpha} = 0$. Indeed, the proof of Lemma 16 implies that

$$\lim_{t_k\to\infty} E_{t_j} \prod_{k=1,\dots,\infty} e^{-\rho\tau_k} \big(\phi_k e^{r^f \tau_k} + (1-\phi_j) e^{(\mu - 0.5\sigma^2)\tau_k + \sigma B_{\tau_k}}\big)^{1-\alpha} \le \lim_{t_k\to\infty} e^{(\nu-\rho)(t_k - t_j)} = 0.$$

Lemma 14 and, hence, is the value function. Moreover, the policies that attain the maximum on the right hand side of (B.8) are optimal.

We next consider the case $\alpha > 1$. In this case a contraction mapping argument does not necessarily apply (see, e.g., Alvarez and Stokey (1998)) and hence we need to take a more direct approach. The value function V satisfies (B.8) by Lemma 13. Hence, it is a fixed point of (B.8). So it suffices to show that $V \in \mathcal{H}$. By Lemma 14, V is homogeneous of degree $1-\alpha$. Also by the arguments given as part of the proof of Lemma 13, V is bounded above by zero and below by the (homogeneous of degree $1-\alpha$) function $\frac{1}{1-\alpha}(X_{t_j}+S_{t_j})^{1-\alpha}\chi^{-\alpha}[(1-\theta_X)\widetilde{x}_{t_j}+(1-\psi_s)(1-\theta_S)(1-\widetilde{x}_{t_j})]^{1-\alpha}$, which corresponds to the feasible policy $\tau_1=\infty$, $y_s(t_j)=-[1-\theta_S]S_{t_j}$, and $C(t_j,\infty)=X_{t_j^+}$. Accordingly, $V\in\mathcal{B}$.

We next show that V is continuous. To that end we start by introducing some notation. Let $W_{t_j}\equiv(X_{t_j},S_{t_j})$ denote a two-dimensional vector with X_{t_j} and S_{t_j} its two elements. We also let $\|W_{t_j}\|_d\equiv\max(X_{t_j},S_{t_j})$ and let $a_j=\{C(t_j,\tau_j),y^b(t_j),y^s(t_j),\phi_j,\tau_j\}$ denote some optimal policies starting from W_{t_j}. We next show that for any $\eta>0$, there exists $\Delta>0$, such that $\|W_{t_j}-\widehat{W}_{t_j}\|_d<\Delta$ implies $V(W_{t_j})<V(\widehat{W}_{t_j})-\eta$. To see this, fix $\eta>0$. We next show that it is possible to choose $\varepsilon>0$, $\delta>0$, $\widehat{y}^s\le 0$, $\widehat{y}^b\ge 0$ so that

$$\text{(B.17)}\quad \big[1-(1-\alpha)\kappa b(\tau_j)\big]\times\big|U\big(C(t_j,\tau_j)\big)-U\big(\widehat{C}(t_j,\tau_j)\big)\big| < \frac{\eta}{2}\quad\text{for all } \|W_{t_j}-\widehat{W}_{t_j}\|_d<\delta,$$

where $\widehat{C}(t_j,\tau_j)=\widehat{X}_{t_j^+}-e^{-r_L\tau_j}(1-\varepsilon)X_{t_{j+1}}$,

$$\text{(B.18)}\quad \widehat{S}_{t_j^+}\ge(1-\varepsilon)S_{t_j^+},$$

and

$$\text{(B.19)}\quad \big|1-(1-\varepsilon)^{1-\alpha}\big|E_{t_j}\sum_{i=j+1,\dots,\infty}\big[1-(1-\alpha)\kappa b(\tau_i)\big]e^{-\rho(t_i-t_j)}\big|U\big(C(t_i,\tau_i)\big)\big| < \frac{\eta}{2}.$$

To show that it is possible to find such ε, δ, $\widehat{y}^s\le 0$, $\widehat{y}^b\ge 0$, we start by observing that it is clearly possible to find sufficiently small $\varepsilon>0$ that satisfies (B.19). To show the existence of $\delta>0$, $\widehat{y}^s\le 0$, $\widehat{y}^b\ge 0$ satisfying (B.17) and (B.18), we distinguish three cases, namely (i) $y^s=y^b=0$, (ii) $y^s<0$, and (iii) $y^b>0$. (Because of Lemma 4, it is never optimal to set $y^s<0$ and simultaneously $y^b>0$.) If $y^s=0$ and $y^b=0$, then setting $\widehat{y}^s=0$, $\widehat{y}^b=0$ implies

Hence, using (B.15) and (B.16), it follows that $\lim_{t_k\to\infty}e^{-\rho t_k}E_{t_j}(\frac{X_{t_k}+S_{t_k}}{X_{t_j}+S_{t_j}})^{1-\alpha}=0$.

that $\widehat{X}_{t_j^+} = \widehat{X}_{t_j}, \widehat{S}_{t_j^+} = \widehat{S}_{t_j}$. In turn, for any $\varepsilon > 0$, there exists sufficiently small $\overline{\delta}(\varepsilon) > 0$, so that for all $\widehat{S}_{t_j}$ with $|\widehat{S}_{t_j} - S_{t_j}| < \overline{\delta}(\varepsilon)$, condition (B.18) holds, since $\widehat{S}_{t_j^+} - S_{t_j^+} = \widehat{S}_{t_j} - S_{t_j}$. Moreover, for a sufficiently small $\varepsilon > 0$, there exists $\overline{\overline{\delta}}(\varepsilon)$ so that condition (B.17) holds for $|\widehat{X}_{t_j} - X_{t_j}| < \overline{\overline{\delta}}(\varepsilon)$. Accordingly, there exists sufficiently small ε and sufficiently small $\delta(\varepsilon) < \min(\overline{\delta}(\varepsilon), \overline{\overline{\delta}}(\varepsilon))$ such that conditions (B.17), (B.18), and (B.19) hold. Next suppose that $y^s < 0$. In that case, set $-\widehat{y}^s = -y^s + \varepsilon S_{t_j^+} - (1-\theta_S)[S_{t_j} - \widehat{S}_{t_j}]$, and note that for (sufficiently small) $\varepsilon > 0$ and $\widetilde{\delta}(\varepsilon)$, we obtain that $-\widehat{y}^s > 0$ as long as $|\widehat{S}_{t_j} - S_{t_j}| < \widetilde{\delta}(\varepsilon)$. By construction, this choice of $-\widehat{y}^s$ satisfies constraint (B.18) with equality. Furthermore, for this choice of $-\widehat{y}^s$ and sufficiently small $\varepsilon > 0$, there exists $\widetilde{\widetilde{\delta}}(\varepsilon) < \widetilde{\delta}(\varepsilon)$ such that for all $\|W_{t_j} - \widehat{W}_{t_j}\|_d < \widetilde{\widetilde{\delta}}(\varepsilon)$, condition (B.17) also holds. Finally, if $\widehat{y}^b > 0$, then set $\widehat{y}^b = y^b - \varepsilon S_{t_j^+} - (1-\theta_S)[S_{t_j} - \widehat{S}_{t_j}]$. Once again observe that for (sufficiently small) $\varepsilon > 0$ and $\widetilde{\delta}(\varepsilon)$, we obtain that $y^b > 0$ and $\widehat{X}_{t_j^+} > 0$ as long as $\|W_{t_j} - \widehat{W}_{t_j}\|_d < \widetilde{\widetilde{\delta}}(\varepsilon)$. Additionally, this choice of $\widehat{y}^b$ satisfies constraint (B.18) with equality. Furthermore, for this choice of $\widehat{y}^b$ and sufficiently small $\varepsilon > 0$, there exists $\widetilde{\widetilde{\delta}}(\varepsilon) < \widetilde{\delta}(\varepsilon)$ such that for all $\|W_{t_j} - \widehat{W}_{t_j}\|_d < \widetilde{\widetilde{\delta}}(\varepsilon)$, condition (B.17) holds.

From this point onward, the proof follows from Alvarez and Stokey (1998, p. 177) and we repeat their argument for completeness. Specifically choose $\varepsilon > 0$, $\delta > 0$, $\widehat{y}^s \le 0$, $\widehat{y}^b \ge 0$ so that conditions (B.17), (B.18), and (B.19) hold, and take some $\widehat{W}_{t_j}$ with $\|W_{t_j} - \widehat{W}_{t_j}\|_d < \delta$. Consider the following policy $a_{j=1,\dots,\infty}$ with initial conditions $\widehat{W}_{t_j}$. Set $\widehat{\tau}_j = \tau_j$, $\widehat{\phi}_j = \phi_j$ for all $j \ge 1$. Furthermore, at t_j choose $\widehat{y}^s \le 0, \widehat{y}^b \ge 0$ consistent with (B.17), (B.18), and (B.19), and set $\widehat{C}(t_j, \tau_j) = \widehat{X}_{t_j^+} - e^{-r_L \tau_j}(1-\varepsilon)X_{t_{j+1}}$. From t_{j+1} onward, set $\widehat{y}^s = (1-\varepsilon)y^s$, $\widehat{y}^b = (1-\varepsilon)y^b$, and $\widehat{C}(t_{j+k}, \tau_{j+k}) = (1-\varepsilon)C(t_{j+k}, \tau_{j+k})$ for all $k \ge 1$. By construction, this policy satisfies $\widehat{W}_{t+1} \ge (1-\varepsilon)W_{t+1}$ and, hence, it is feasible. Furthermore, we obtain

$$
\begin{aligned}
&\left|V(W_{t_j}) - \mathcal{U}(\widehat{W}_{t_j}; a_{j=1,\dots,\infty})\right| \\
&\quad \le \left[1 - (1-\alpha)\kappa b(\tau_j)\right] \times \left|U\big(C(t_j, \tau_j)\big) - U\big(\widehat{C}(t_j, \tau_j)\big)\right| \\
&\qquad + \left|1 - (1-\varepsilon)^{1-\alpha}\right| \\
&\qquad \times E_{t_j} \sum_{i=j+1,\dots,\infty} \left[1 - (1-\alpha)\kappa b(\tau_i)\right] e^{-\rho(t_i - t_j)} \left|U\big(C(t_i, \tau_i)\big)\right| \\
&\quad < \eta.
\end{aligned}
$$

Accordingly, $V(\widehat{W}_{t_j}) \ge \mathcal{U}(\widehat{W}_{t_j}; a_{j=1,\dots,\infty}) \ge V(W_{t_j}) - \eta$.

By similar arguments, letting $\widehat{a}_j = \{\widehat{C}(t_j, \tau_j), \widehat{y}^b(t_j), \widehat{y}^s(t_j), \widehat{\phi}_j, \widehat{\tau}_j\}$ denote an optimal policy starting from $\widehat{W}_{t_j}$, and reversing the roles of (W_{t_j}, a_j) and $(\widehat{W}_{t_j}, \widehat{a}_j)$ in (B.17)–(B.19) implies the existence of small enough $\delta_1 > 0$, such that for all $\widehat{W}_{t_j}$ with $\|W_{t_j} - \widehat{W}_{t_j}\|_d < \delta_1$, we also obtain $V(W_{t_j}) \geq V(\widehat{W}_{t_j}) - \eta$. We conclude that there exists small enough $\Delta = \min\{\delta, \delta_1\}$ such that for all $\|W_{t_j} - \widehat{W}_{t_j}\|_d < \Delta$, we obtain $|V(W_{t_j}) - V(\widehat{W}_{t_j})| < \eta$, proving the continuity of V.

Since, as we showed above, $V \in \mathcal{H}$ and there always exists a choice (namely $\tau_1 = \infty$, $y_s(t_j) = -[1 - \theta_S]S_{t_j}$) that provides a lower bound to the expression inside curly brackets in (B.8), it follows that the supremum in (B.8) is attained when $f = V$. Furthermore, Theorem 4.5 in Stokey and Lucas (1989) implies that the policy that maximizes the right hand side of (B.8) for $f = V$ is optimal. *Q.E.D.*

REMARK 1: We note that since the value function is unique, an implication of Lemma 14 and Proposition 7 is that V is the unique fixed point of T in $\mathcal{H}$ satisfying the condition $\lim_{t_k \to \infty} e^{-\rho t_k} E_{t_j} V(X_{t_k}, S_{t_k}) = 0$.

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