Econometrica Supplementary Material

SUPPLEMENT TO "NONCONTRACTIBLE HETEROGENEITY IN DIRECTED SEARCH": APPENDIXES

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APPENDIX A

THIS SECTION SUPPLIES proofs of many of the assertions in the text, which are restated here. The meaning of the notation is given in the main text of the paper.

A.1. *The Main Characterization Theorem*

For reference, we start with the proposition that characterizes equilibrium. The proof is given in the main text; however, it is used in the proofs of the properties of the restated examples given below.

PROPOSITION 5.1: *Suppose that both the function $\iota(w,y,x)\equiv\frac{v(w,y,x)}{w}$ and its derivative with respect to w are nondecreasing in x. Then a pair (G,P) is an equilibrium if and only if there is a point y_0 and a pair of functions $\omega(y)$ and $h(y)$ satisfying $\omega(y_0)=\underline{w}$ and $G^-(\omega(y))=H(h(y))$ such that*

$$\omega(y)\frac{\tau}{1-H(h(y))}F'(y)=\omega'(y) \tag{5.1}$$

and

$$\begin{aligned} &v[\omega(y),y,h(y)] \\ &\quad=-\int_{\underline{y}}^{y}\left[v_w(\omega(y),y',h(y))-\frac{v(\omega(y),y',h(y))}{\omega(y)}\right]\omega'(y')\,dy'. \end{aligned}\tag{5.2}$$

APPENDIX B: PROPERTIES OF THE EXAMPLES

EXAMPLE 6.1: *If $v(w,y,x)=1+\alpha y-w$ for all $x\in[\underline{x},\overline{x}]$, then there is a degenerate single wage equilibrium with $\underline{w}=1+\alpha\overline{y}-\int_{\underline{y}}^{\overline{y}}(1+\alpha y')\,de^{-\int_{y'}^{\overline{y}}\tau F'(t)\,dt}$.*

PROOF—RESTATED: Proposition 5.1 applies to the degenerate case since we can set $y_0=\overline{y}$ and $h(\overline{y})=\underline{x}$. To see this, observe that if the wage distribution is degenerate at some wage w_0, then all workers apply at this wage. Since y_0 is the highest type worker who applies to the lowest wage in the support, the conclusion $y_0=\overline{y}$ follows. The function $h(\overline{y})$ is supposed to be the lowest firm type that offers worker $\overline{y}$ his reservation wage. Since all firms offer the same wage, this is $\underline{x}$. We then have trivially from (5.1) that $\omega(\overline{y})\tau F'(\overline{y})=\underline{w}\tau F'(\overline{y})=\omega'(\overline{y})$.

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The first order condition (5.2) then becomes (since $G^{-}(\omega(y)) = G^{-}(\underline{w}) = 0$)

$$v[\underline{w}, \overline{y}, \underline{x}] = -\int_{\underline{y}}^{\overline{y}} \left[v_w(\underline{w}, y', \underline{x}) - \frac{v(\underline{w}, y', \underline{x})}{\underline{w}} \right] \omega'(y')\, dy'$$
$$= \int_{\underline{y}}^{\overline{y}} [v(\underline{w}, y', \underline{x}) - v_w(\underline{w}, y', \underline{x})\underline{w}] \tau e^{-\int_{y'}^{\overline{y}} \tau F'(t)\, dt} F'(y')\, dy'$$
$$= \int_{\underline{y}}^{\overline{y}} [v(\underline{w}, y', \underline{x}) - v_w(\underline{w}, y', \underline{x})\underline{w}]\, de^{-\int_{y'}^{\overline{y}} \tau F'(t)\, dt}.$$

Substituting the specific profit function $1 + \alpha y - w$ then gives the first order condition

$$(6.1) \qquad 1 + \alpha \overline{y} - \underline{w} = \int_{\underline{y}}^{\overline{y}} (1 + \alpha y')\, de^{-\int_{y'}^{\overline{y}} \tau F'(t)\, dt},$$

which gives the result. *Q.E.D.*

EXAMPLE 6.2: *Suppose that $v(w, y, x) = (\alpha y - w)$ for all x. Then a wage distribution can be supported in equilibrium only if $F'(y)$ is decreasing.*

PROOF—Restated: Fix the lowest wage $\underline{w}$ in the support of the equilibrium distribution and let y_0 be the type for whom $\omega(y_0) = \underline{w}$. When firms have the same profit function, all wages in the support must yield the same profit. This is guaranteed by condition (5.2), which after substituting the special profit function becomes

$$\alpha y - \omega(y) = -\int_{\underline{y}}^{y} \left[1 - \frac{\alpha y' - \omega(y)}{\omega(y)} \right] \omega'(y')\, dy'.$$

Rewriting slightly gives

$$(\alpha y - \omega(y))\omega(y) = \int_{\underline{y}}^{y} \alpha y' \omega'(y')\, dy'.$$

Since this must hold uniformly in y, the derivatives of this expression with respect to y must also be the same, that is,

$$(\alpha y - \omega(y))\omega'(y) + \omega(y)(\alpha - \omega'(y)) = \alpha y \omega'(y).$$

This gives the simple condition $\omega'(y) = \frac{\alpha}{2}$. This is the condition that the market payoff function must have when $\omega(y)$ is in the support of the equilibrium wage distribution so that firms' profits are constant on the support of this distribution. From condition (5.1), it must be that

$$\omega(y)\frac{\tau F'(y)}{1-G(\omega(y))}=\frac{\alpha}{2}$$

along the support of the equilibrium wage distribution. Since $\frac{\omega(y)}{1-G(\omega(y))}$ is strictly increasing in y, this condition cannot be fulfilled unless $F'(y)$ is decreasing. *Q.E.D.*

EXAMPLE 6.3: *Suppose that $v(w,x,y)=y-w$ and that $F(y)=y(2-y)$ with $\underline{y}=0$ and $\overline{y}=1$. Then there is a worker–firm ratio $\tau_0<\frac{3}{2}$ such that a nondegenerate distribution of wages can be supported in equilibrium for the economy where the ratio of workers to firms is τ_0. The equilibrium wage distribution is convex and has support $[\frac{1}{2\tau_0},\frac{1}{2\tau_0}+\frac{1}{4}]$.*

PROOF OF THE ASSERTIONS IN EXAMPLE 6.3: In this case, $F'(y)=2-2y$. We borrow from Example 6.2 the fact that $\omega'(y)=\frac{\alpha}{2}$ for every $y>y_0$ so that (5.2) is satisfied. Recall that $\underline{w}$ is the lowest wage in the support of the equilibrium distribution, while y_0 is the highest type who applies to the firm offering $\underline{w}$. The method will be to select a pair $(y_0,\underline{w})$ to anchor the bottom of the wage distribution, and then choose α and τ so that firms have no incentive to cut wages below $\underline{w}$. We then show how the wage distribution can be constructed above the point $\underline{w}$ so that (5.1) holds. The fact that this construction constitutes an equilibrium follows from the "if" part of Proposition 5.1.

Start with the choice of $(y_0,\underline{w})$. Whatever pair we choose, (5.1) must hold at y_0. Specifically

$$\underline{w}=\frac{\alpha}{2\tau(2-2y_0)}. \tag{B.1}$$

Since w_0 is to be the lowest wage and y_0 is to be the highest type who applies to it, the reservation wages of all types below y_0 are determined by $\omega(y)=w_0e^{-\int_y^{y_0}\tau(2-2t)\,dt}$, which means that the payoff function for the firm that offers a wage below $\underline{w}$ is also completely determined by the formula

$$\int_{\underline{y}}^{y}\frac{v(\omega(y),y',\underline{x})}{\omega(y)}\omega'(y')\,dy'=\int_0^{y}\frac{\alpha y'-\omega(y)}{\omega(y)}\omega'(y')\,dy'.$$

By Proposition 5.1, the pair $(y_0,\underline{w})$ will support an equilibrium distribution if this function attains its maximum at $y=y_0$. To see how to ensure this, think of this as a function $\int_{\underline{y}}^{y}\frac{\alpha y'-w}{w}\omega'(y')\,dy'$ of two variables, (y,w). The firm's problem is to maximize this function subject to the constraint that the pair (y,w) that it chooses provides the market payoff, that is, $w=\omega(y)$. In this sense, we want the isoprofit curve associated with the function $\int_{\underline{y}}^{y} \frac{\alpha y' - w}{w} \omega'(y')\, dy'$ to be tangent to the market return function $\omega(y)$ at the point y_0. The isoprofit curves for this function are readily verified to be concave and to have slope

$$\frac{(\alpha y - w)\omega'(y)}{\displaystyle\int_0^y \alpha y' \omega'(y')\, dy'}.$$

The requirement that an isoprofit line is tangent to the "market payoff function" ω at the point y_0 gives the condition

$$\underline{w} = \alpha\left(y_0 - \int_0^{y_0} y' \omega'(y')\, dy'\right).$$

Combining this with (B.1) provides the restriction

$$\frac{1}{2\tau(2-2y_0)} = \left(y_0 - \int_0^{y_0} y' \omega'(y')\, dy'\right). \tag{B.2}$$

Using the formula for $\omega(y)$ when $y < y_0$, the right hand side of the last equation is

$$\begin{aligned} &y_0 - \underline{w} \int_0^{y_0} y' \tau(2-2y') e^{-\int_{y'}^{y_0} \tau(2-2t)\, dt}\, dy' \\ &\quad = y_0 - y_0 \underline{w} + \int_0^{y_0} e^{-\int_{y'}^{y_0} \tau(2-2t)\, dt}\, dy', \end{aligned}$$

where the last equality follows from integration by parts.

With these preliminaries, we can now construct a solution. The pair $(y_0, \underline{w})$ that we choose to construct the equilibrium needs to satisfy

$$\frac{1}{2\tau(2-2y_0)} = y_0 - y_0 \underline{w} + \int_0^{y_0} e^{-\int_{y'}^{y_0} \tau(2-2t)\, dt}\, dy'.$$

Take $w_0 = \frac{1}{2\tau(2-y_0)}$ so that we can take $\alpha = 1$. Then we are trying to solve

$$\frac{1+y_0}{2\tau(2-2y_0)} = y_0 + \int_0^{y_0} e^{-\int_{y'}^{y_0} \tau(2-2t)\, dt}\, dy'.$$

By continuity of the expression in this equation and the intermediate value theorem,[1] there is a worker–firm ratio τ that satisfies this equation for any y_0 we choose, so take $y_0 = \frac{1}{2}$ and suppose that τ_0 is the worker–firm ratio that solves the last equation when $y_0 = \frac{1}{2}$. One bit of information that will be required

[1]The left hand side strictly exceeds the right hand side when $\tau = 0$; the converse is true as τ goes to infinity.

below is the fact that the solution for τ in the equation above cannot exceed $\frac{3}{2}$. To see why, observe that when $\tau = \frac{3}{2}$, the left hand side of the equation above is equal to $\frac{1}{2}$, while the right hand side evaluated at $y_0 = \frac{1}{2}$ strictly exceeds $\frac{1}{2}$.

What this construction has accomplished so far is to point out that there is a market with worker–firm ratio $\tau_0 < \frac{3}{2}$, where firms' profits are $1 - y$, such that if the lowest wage in the distribution of wages is $\underline{w} = \frac{1}{2\tau_0(2-2y_0)} = \frac{1}{2\tau_0}$, then no firm will want to cut their wage below $\frac{1}{2\tau_0}$ provided workers whose types are less than or equal to $\frac{1}{2}$ apply with positive probability.

We can now construct the wage distribution above $\frac{1}{2\tau_0}$ that satisfies (5.1). As we derived in Example 6.2, $\omega(y) = w_0 + \frac{\alpha_0}{2}(y - y_0) = \frac{1}{2\tau_0} + \frac{1}{2}(y - \frac{1}{2})$ when $y > y_0$. The equilibrium wage distribution can now be computed using

$$\omega(y)\frac{\tau_0(2-2y)}{1-G(\omega(y))} = \frac{1}{2} \tag{B.3}$$

simply by solving for $G(\omega(y))$, and then changing variables by replacing $\omega(y)$ by w and y by $\omega^{-1}(w)$. This gives the condition

$$\begin{aligned} G(\omega(y)) &= 1 - 2\tau_0\omega(y)(2-2y) \\ &= 1 - 2\tau_0\left(\frac{1}{2\tau_0} + \frac{1}{2}(y - y_0)\right)(2-2y) \end{aligned}$$

or

$$G(w) = 1 - w(2\tau_0 + 4) + w^2 8\tau_0.$$

The derivative of this expression with respect to w is $-(2\tau_0 + 4) + w16\tau$. The lowest value that w can take is $\frac{1}{2\tau_0}$, so the second term is no smaller than 8. As we established above, τ_0 can be no larger than $\frac{3}{2}$, so the negative term can be no larger than 7. As a consequence, this expression is increasing in wages as required.

Finally note that $G(w)$ is a convex function and that the upper bound to wages is $\frac{1}{2\tau_0} + \frac{1}{2} < 1$. *Q.E.D.*

APPENDIX C: THE RELATIONSHIP BETWEEN THE WAGE OFFER DISTRIBUTION AND THE ACCEPTED WAGE DISTRIBUTION

PROPOSITION 7.3: *The wage offer distribution G and the accepted wage distribution are related by*

$$G(w) = G^*(w) + \omega(\underline{y}) \int_{\underline{w}}^{w} \frac{G^{*\prime}(w')}{w'\left(1 - \frac{\omega(\underline{y})}{w'}\right)} dw'.$$

PROOF: Recall that a worker of type y who applies to some wage w that exceeds his reservation wage is hired with probability

$$e^{-\int_{\underline{y}}^{y^*(w)} k(y')\,dF(y')}.$$

The number of workers who have this type is $\tau f(y)$, while the probability that they apply at a firm whose wage is w is $\frac{G'(w)}{1-G(\omega(y))}$. The number of jobs that are filled at wage w is then given by

$$\int_{\underline{y}}^{y^*(w)} e^{-\int_{\underline{y}}^{y^*(w)} k(y')\,dF(y')} \frac{\tau f(y) G'(w)}{1-G(\omega(y))}\,dy.$$

This gives the observed wage distribution as

$$G^*(w) = \int_{\underline{w}}^{w} \int_{\underline{y}}^{y^*(w')} e^{-\int_{\underline{y}}^{y^*(w')} k(y')\,dF(y')} \frac{\tau f(y) G'(w')}{1-G(\omega(y))}\,dy\,dw'.$$

Since each worker's expected payoff is constant at every wage above his or her reservation wage , $w' e^{-\int_{\underline{y}}^{y^*(w')} k(y')\,dF(y')} = \omega(y)$, so that

$$G^*(w) = \int_{\underline{w}}^{w} \int_{\underline{y}}^{y^*(w')} \frac{\omega(y)}{w'} \frac{\tau f(y) G'(w')}{1-G(\omega(y))}\,dy\,dw'.$$

Now using (5.1) from Proposition 5.1, this simplifies to

$$\begin{aligned}
G^*(w) &= \int_{\underline{w}}^{w} \int_{\underline{y}}^{y^*(w')} \frac{\omega'(y) G'(w')}{w'}\,dy\,dw' \\
&= \int_{\underline{w}}^{w} \frac{G'(w')}{w'} \int_{\underline{y}}^{y^*(w')} \omega'(y)\,dy\,dw' \\
&= \int_{\underline{w}}^{w} \frac{G'(w')(w' - \omega(\underline{y}))}{w'}\,dw' \\
&= G(w) - \omega(\underline{y}) \int_{\underline{w}}^{w} \frac{G'(w')}{w'}\,dw'.
\end{aligned}$$

Differentiating with respect to w gives the relationship

$$\frac{G^{*\prime}(w)}{\left(1 - \frac{\omega(\underline{y})}{w}\right)} = G'(w),$$

so that

$$G(w) = G^*(w) + \omega(\underline{y}) \int_{\underline{w}}^{w} \frac{G^{*\prime}(w')}{w'\left(1 - \frac{\omega(\underline{y})}{w'}\right)}\, dw'.$$

Q.E.D.

APPENDIX D: THE CHARACTERIZATION THEOREM FOR THE FINITE GAME

LEMMA 8.1: *For any array of wages $w_1, \dots, w_m$ offered by firms for which $w_1 > 0$, there is a partition $\{y_K, \dots, y_m\}$ containing no more than m intervals, and a set $\{\pi_j^k\}_{k \geq K; j \geq k}$ of probabilities satisfying $\pi_j^k > 0$ and $\sum_{j=k}^m \pi_j^k = 1$ for each k and such that the strategy*

$$\pi_j(y) = \begin{cases} \pi_j^k, & \text{if } j \geq k;\ y \in [y_k, y_{k+1}), \\ 0, & \text{otherwise}, \end{cases}$$

is almost everywhere a unique (symmetric) continuation equilibrium application strategy. The probabilities π_j^i satisfy

$$\left(\frac{\pi_j^i}{\pi_i^i}\right)^{n-1} = \frac{w_i}{w_j} \tag{8.2}$$

for each $j > i$. Furthermore, the numbers $\{y_k\}$ and $\{\pi_j^k\}$ depend continuously on the wages offered by firms.

PROOF: The proof is inductive. Evidently a worker with the highest type will be hired with probability 1 wherever he applies, so every equilibrium strategy must have the highest type worker apply to one of the firms that offer the highest wage. If $w_{m-1} = w_m$, set $y_m = 1$ and $\pi_m^m = 1$. In this case, observe that a worker of type y_m is just indifferent between applying to firm m and $m-1$; otherwise, fix an open interval $(y_m, \overline{y})$. The expected payoff to worker i with a type in this interval who applies to firm m is

$$\left[1 - \int_y^{\overline{y}} \pi_m(y')\, dF(y')\right]^{n-1} w_m.$$

The expected payoff to applying to any firm j whose wage is $w_j < w_m$ is

$$\left[1 - \int_y^{\overline{y}} \pi_j(y')\, dF(y')\right]^{n-1} w_j.$$

Now observe that for y_m close enough to $\overline{y}$, workers will strictly prefer applying to firm m than applying to firm j, even if all the workers whose types are higher apply to firm m with probability 1. In other words, for workers whose type is close enough to $\overline{y}$, applying to one of the firms whose wage is highest strictly dominates any other choice. Thus there is some interval near $\overline{y}$ such that workers whose types are in this interval apply to firm m with probability 1 in every Bayesian equilibrium. The lowest type for which this is true is the type y_m such that

$$\left[1-\int_{y_m}^{\overline{y}} dF(y')\right]^{n-1} w_m = w_{m-1}$$

or the type y that satisfies

$$[F(y)]^{n-1} w_m = w_{m-1}. \tag{D.1}$$

Then $\pi_m^i(y) = 1 \equiv \pi_m^m$ for every i and for every $y \in (y_m, \overline{y}]$ must be true in every Bayesian equilibrium of this subgame.

Note that y_m is a continuous function of w_m and w_{m-1}, and that $y_m \to \overline{y}$ as $w_{m-1} \to w_m$. Since π_m^m is constant, it is trivially a continuous function of w_m and w_{m-1}. Furthermore, note that a worker of type y_m gets the same payoff from every firm whose index is greater than or equal to $m-1$.

Now suppose that we have defined cutoff valuations $\{y_{k+1}, \dots, y_m\}$ and probabilities $\pi_{j'}^{k'}$ for $k' = k+1, \dots, m$ and $j' \geq k'$, satisfying $\sum_{j' \geq k'} \pi_{j'}^{k'} = 1$ for each k'. Suppose that these satisfy the following conditions.

CONDITION 1: *$\pi_{j'}(y) = \pi_{j'}^{k'}$ for each $y \in (y_{k'}, y_{k'+1})$ and $\pi_{j'} = 0$ otherwise, in every symmetric Bayesian equilibrium.*

CONDITION 2: *A worker of type $y_{k'}$, where $y_{k'} \in \{y_{k+1}, \dots, y_m\}$, gets the same payoff from every firm whose index is at least $k'-1$.*

CONDITION 3: *Each of these numbers is a continuous function of wages $w_k, \dots, w_m$.*

PROOF OF CONDITIONS 1–3: If $y_{k+1} = \underline{y}$, then we have shown that the Bayesian continuation equilibrium for this subgame is almost everywhere uniquely defined (the exceptions are the cutoff values y_k). So suppose $y_{k+1} > \underline{y}$. We now show that Conditions 1–3 can be extended to some interval $[y_k, y_{k+1})$ which will be nondegenerate provided $w_k < w_{k-1}$.

If $w_k = w_{k+1}$ or $w_{k-1} = w_k$, set $y_k = y_{k+1}$, $\pi_k^k = 0$, and $\pi_j^k = \pi_j^{k+1}$ for each $j > k$. It is straightforward that valuations $\{y_k, \dots, y_m\}$ and probabilities $\pi_{j'}^{k'}$ for $k' = k, \dots, m$ satisfy Conditions 1–3 of the induction hypothesis.

Otherwise either $w_{k-1} < w_k < w_{k+1}$ or $k = 1$. Each of these cases can be analyzed the same way. In the former case, observe that in this construction, worker types larger than y_{k+1} will never apply to firm k. Thus for y close enough to y_{k+1}, applying to *any* firm with wage rate below w_k will be strictly dominated by applying to firm k no matter what workers with types in the interval (y, y_{k+1}) choose to do. In the case where $k = 1$, firm k is already the lowest wage firm. In either case, we conclude that there is an interval of types (y_k, y_{k+1}), with y_k possibly equal to $\underline{y}$, such that workers with types in this interval will apply with positive probability only to firms with wages at least w_k in every Bayesian equilibrium.

By the induction hypothesis, a worker of type y_{k+1} will receive the same payoff from each firm $k+1$ through m. This payoff is given by

$$\left[1 - \sum_{i=1}^{j} \pi_{k+j}^{k+i}[F(y_{k+i+1}) - F(y_{k+i})]\right]^{n-1} w_{k+j}$$

when this worker applies to firm $k+j$. By the induction hypothesis, this payoff is equal to w_k for each $j \geq 1$. Notice that this payoff is independent of what workers whose types are in the interval (y_k, y_{k+1}) choose to do. A worker i of type $y \in (y_k, y_{k+1})$ who applies to firm $k+j$ receives payoff

$$\text{(D.2)} \qquad \left[1 - \int_{y}^{y_{k+1}} \pi_{k+j}(y)\, dF(y) - \sum_{i=1}^{j} \pi_{k+j}^{k+i}[F(y_{k+i+1}) - F(y_{k+i})]\right]^{n-1} w_{k+j},$$

while the same worker who applies to firm k gets

$$\text{(D.3)} \qquad \left[1 - \int_{y}^{y_{k+1}} \pi_k(y)\, dF(y)\right]^{n-1} w_k.$$

The function described in (D.3) is nondecreasing in y and has a limit from the left at y_{k+1} equal to w_k. Since applying to firms whose wages are lower than w_k is a strictly dominated strategy of a worker of type y close enough to y_{k+1}, it must be the case that for every i, $\int_{y}^{y_{k+1}} \pi_{k+j}^{i}(y)\, dy$ is strictly positive for some j. Then from (D.2) and (D.3), $\int_{y}^{y_{k+1}} \pi_{k+j}(y)\, dy$ must be strictly positive for all j.

The payoff must be the same at firm k and $k+j$ for each $j > 0$ and for every $y \in (y_k, y_{k+1})$. This requires that (D.2) and (D.3) must be equal identically in y. Differentiating this identity repeatedly gives

$$\text{(D.4)} \qquad \left(\frac{\pi_{k+j}^{k}(y)}{\pi_k^{k}(y)}\right)^{n-1} = \frac{w_k}{w_{k+j}},$$

implying that π_{k+j}^{k} are constant.

They can all be determined from the condition

$$\text{(D.5)} \qquad \sum_{j=0}^{m-k} \pi_{k+j}^{k} = 1.$$

Notice that by the induction hypothesis, the limits from the left of (D.2) and (D.3) at y_{k+1} must both be equal to w_k. Thus (D.4) and (D.5) are also sufficient for identity of the payoffs.

Having found the value for π_k^k, we can determine the lower bound y_k. Since workers with higher types and higher investments only apply to firms whose wages are at least w_k, this worker is sure to be hired if he applies to the $(k-1)$st firm, assuming that there is one. On the other hand, since he is the lowest type who applies to the kth firm, he will be hired by the kth firm only if no other worker with a higher type applies. Then define y_k as follows: if $k=1$, then $y_k = y_1 = \underline{y}$; otherwise, if

$$\text{(D.6)} \qquad \left[1 - \pi_k^k(F(y_{k+1}) - F(y))\right]^{n-1} w_k = w_{k-1}$$

has a solution that exceeds $\underline{y}$, set y_k equal to this solution; otherwise set $y_k = \overline{y}$.

This argument extends Conditions 1 and 2 by construction. Condition 3 is readily verified using the maximum theorem, since $w_{k+j} > 0$ by assumption. *Q.E.D.*

This completes the proof of Lemma 8.1. *Q.E.D.*

THEOREM D.1: *Let G be a distribution of wages, w be an arbitrary wage offered by a firm of type x, and w^- be the largest wage in the support of G that is less than or equal to w. Let G_n be a sequence of distributions that converges weakly to G. Let j_n be the corresponding sequence of indices of firm x's wage (i.e., such that w is the j_nth lowest wage in the distribution associated with G_n). There is a nondecreasing right continuous function $\omega(y)$ and a nondecreasing right continuous function $y^*(w)$ (both of which depend on G) such that for almost every $y \in [\underline{y}, \overline{y}]$,*

$$\lim_{n\to\infty}\left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y)\,dF(y)\right]^{n-1} = \frac{w^-}{w} e^{-\int_y^{y^*(w^-)} k(y')\,dF(y')}$$

and

$$\lim_{n\to\infty}\int_{\underline{y}}^{\overline{y}} v(w,y,x)\,d\phi_{j_n}^n(y)$$
$$= \frac{w^-}{w}\int_{\underline{y}}^{y^*(w^-)} k(y)v(w,y,x)e^{-\int_y^{y^*(w^-)} k(y')\,dF(y')}F'(y)\,dy$$
$$+ v(w, y^*(w^-), x)\left(1 - \frac{w^-}{w}\right)$$

whenever $w \geq w_0$, and

$$\lim_{n\to\infty}\left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y)\,dF(y)\right]^{n-1} = \min\left[1, \frac{w_0}{w} e^{-\int_y^{y^*(w_0)} \tau\,dF(y')}\right]$$

and

$$\lim_{n\to\infty}\int_{\underline{y}}^{\overline{y}} v(w,y,x)\,d\phi_{j_n}^{n}(y)$$
$$=\int_{\underline{y}}^{y(w)} \tau v(w,y,x)\frac{w_0}{w}e^{-\int_y^{y^*(w_0)}\tau\,dF(y')}F'(y)\,dy$$

otherwise. *In these expressions*, $y(w)$ *is the solution to*

$$\frac{w_0}{w}e^{-\int_y^{y^*(w_0)}\tau\,dF(y')}=1$$

and

$$k(y)=\frac{\tau}{1-G^{-}(\omega(y))}.$$

Furthermore, $y^*(w)=\sup\{y:\omega(y)\le w\}$.

A Preliminary Result

LEMMA D.1: *For any sequence* G_n, *there is a subsequence such that* $\omega_n(y)$ *converges weakly to a right continuous nondecreasing function* $\omega(y)$. *Along this sequence*, *define* $y_n^*(w)=\sup\{y:\omega_n(y)\le w\}$. *The sequence* $y_n^*(\cdot)$ *converges weakly to a right continuous nondecreasing function* $y^*(\cdot)$.

PROOF: By construction, each $\omega_n(y)$ is right continuous and nondecreasing, and for each n, $\int_{\underline{w}}^{\overline{w}} d\omega_n(y)\le\overline{w}_G-\underline{w}_G$, where $\overline{w}_G$ and $\underline{w}_G$ are the maximum and minimum points in the support of G, respectively. Hence by the Helly compactness theorem, $\omega_n(y)$ has a subsequence that converges weakly to a nondecreasing right continuous function. Let $y_n^*(\cdot)$ be the sequence associated with $\omega_n(y)$. It is also nondecreasing and right continuous, and so there is a subsequence such that it has a weak limit $y^*(\cdot)$ by the same reasoning. Since $\omega_n(y)$ converges weakly, it converges weakly on any subsequence. So there is a sequence along which both ω_n and its inverse y_n^* converge weakly. *Q.E.D.*

The Main Convergence Lemma

Define w^s as the largest wage in the support of G that is less than or equal to w, or if no such wage exists, let w^s be the smallest wage in the support of G that is at least as large as w. For convenience, choose the approximations G_n in such a way that the lowest wage w_0 in each approximation is the lowest wage in the support of G. Similarly, suppose that the highest wage w_m in each approximation is also the highest wage in the support of G.

The first lemma is a result used in the proof of Lemma D.4.

LEMMA D.2: *Let $\overline{v}(y)$ be any pointwise limit for the equilibrium payoff to a worker of type y as n goes to infinity. Then $\overline{v}(y) > 0$ for each $y \in Y$.*

PROOF: Choose any wage w' such that $G(w') < 1$. Since worker y attains the same payoff no matter where he applies by Lemma 8.1, the limit of his equilibrium payoff is given by

$$\begin{aligned}
&\lim_{n\to\infty}\left(1-\int_y^{y_n^*(w')}\pi_{j^n}^n(y')\,dF(y')\right)^{n-1}w' \\
&\quad\geq \lim_{n\to\infty}\left(1-\int_y^{y_n^*(w')}\pi_{j^n}^n(y^*(w'))\,dF(y')\right)^{n-1}w' \\
&\quad= \lim_{n\to\infty}\left(1-\int_y^{y_n^*(w')}\pi_{m}^n(y^*(w'))\left(\frac{w_m}{w'}\right)^{1/(n-1)}dF(y')\right)^{n-1}w' \\
&\quad\geq \lim_{n\to\infty}\left(1-\frac{1}{(1-G_n^-(w'))m}\left(\frac{w_m}{w'}\right)^{1/(n-1)}\big[F(y^*(w')-F(y))\big]\right)^{n-1}w' \\
&\quad= e^{-(F(y^*(w'))-F(y))/(1-G^-(w'))}w' > 0.
\end{aligned}$$

The first inequality follows from (D.4) and the fact that higher types have higher reservation wages so that they allocate their application probabilities over fewer firms. The second equality simply substitutes using (D.4). The third inequality comes from the fact that the sum of the application probabilities over all firms whose wage is w' or higher must be equal to 1. The limit is standard in directed search. *Q.E.D.*

Next we verify the property described in the text, that workers application strategies are such that they apply with equal probability to all firms whose wages are above their reservation wage. This result in turn is used to calculate the limit of workers' payoffs in Lemma D.4.

LEMMA D.3: *Let j_n be the index of w in the nth approximation to G. Let $\omega(y)$ be a limit of the sequence $\omega_n(y)$ as defined in Lemma D.1. Then for any y such that $\pi_{j_n}^n > 0$ for infinitely many n, $\lim_{n\to\infty}\pi_{j_n}^n(y)(n-1) = \frac{\tau}{1-G^-(\omega(y))} \equiv k(y)$.*

PROOF: From (D.4),

$$\pi_{j_n}^n(y)(n-1) = \left(\frac{w_m}{w}\right)^{1/(n-1)}\pi_m^n(y)(n-1) \tag{D.7}$$

whenever $\pi^n_{j_n}(y) > 0$, where $\pi^n_m(y)$ is the probability with which a worker of type y applies to the firm with the highest wage. By Lemma 8.1, $\pi^n_m(y) > 0$ for every worker type y. Taking limits in (D.7) with respect to n gives

$$\lim_{n\to\infty} \pi^n_{j_n}(y)(n-1) = \lim_{n\to\infty} \pi^n_m(y)(n-1).$$

Recall that $\omega_n(y)$ is the lowest wage to which a worker of type y applies with positive probability in the continuation equilibrium with n workers. From (D.7)

$$\sum_{j:w_j\geq\omega_n(y)} \pi^n_j(y)(n-1) = \pi^n_m(y)(n-1) \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{1/(n-1)}. \tag{D.8}$$

The sum on the left hand side of this last equation is $n-1$ since the application probabilities sum to 1. On the right hand side, observe that

$$\sum_{j:w_j\geq\omega_n(y)} 1 \leq \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{1/(n-1)} \leq \left(\frac{w_m}{w_{j^*_n}}\right)^{1/(n-1)} \sum_{j:w_j\geq\omega_n(y)} 1,$$

where j^*_n is the index of the lowest wage that a worker of type y applies to with positive probability (we suppress the dependence on y since it is constant in this argument). Dividing this by m gives

$$\begin{aligned}\left(1-G^-_n(\omega_n(y))\right) &\leq \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{1/(n-1)} \Big/ m \\ &\leq \left(\frac{w_m}{w_{j^*_n}}\right)^{1/(n-1)} \left(1-G^-_n(\omega_n(y))\right).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \sum_{j:w_j\geq\omega_n(y)} \left(\frac{w_m}{w_j}\right)^{1/(n-1)} \Big/ m = 1-G^-(\omega(y)),$$

where $w(y)$ is the right continuous nondecreasing function identified in Lemma D.1. Then from (D.8),

$$\begin{aligned}\lim_{n\to\infty} \pi^n_{j_n}(y)(n-1) &= \lim_{n\to\infty} \frac{n-1}{1-G^-_n(\omega_n(y))\cdot m} \\ &= \frac{\tau}{1-G^-(\omega(y))} = k(y),\end{aligned} \tag{D.9}$$

which gives the result. *Q.E.D.*

LEMMA D.4: *Let j_n be the index of w in the nth approximation to G. Let $\omega(y)$ be a pointwise limit of the sequence $\omega_n(y)$ as defined in Lemma* D.1. *Then for almost every y,*

$$\lim_{n\to\infty}\left(1-\int_y^{y_n^*(w)}\pi_{j_n}^n(y')\,dF(y')\right)^{n-1}=\frac{w^-}{w}e^{-\int_y^{y^*(w^-)}\tau/(1-G^-(\omega(y')))\,dF(y')}$$

when $w \ge w_0$, and

$$\min\left[1,\frac{w^0}{w}e^{-\int_y^{y^*(w_0)}\tau/(1-G^-(\omega(y')))\,dF(y')}\right]$$

otherwise.

PROOF: Suppose first that $w \ge w_0$. Then

$$\text{(D.10)}\quad \lim_{n\to\infty}\left(1-\int_y^{y_n^*(w)}\pi_{j_n}^n(y')\,dF(y')\right)^{n-1}$$
$$=\lim_{n\to\infty}\left(1-\int_y^{y_n^*(w_{jn-1})}\pi_{j_n}^n(y')\,dF(y')-\int_{y_n^*(w_{j_n-1})}^{y_n^*(w)}\pi_{j_n}^n(y')\,dF(y')\right)^{n-1}.$$

From Lemma D.2, this limit is strictly positive. By the definition of $y_n^*(w_{j_n-1})$, a worker of this type who applies to the firm offering wage w will be hired with probability

$$\left(1-\int_{y_n^*(w_{j_n-1})}^{y_n^*(w)}\pi_{j_n}^n(y')\,dF(y')\right)^{n-1}.$$

He will be hired for sure if he applies to the firm offering w_{j_n-1}. So

$$\int_{y_n^*(w_{j_n-1})}^{y_n^*(w)}\pi_{j_n}^n(y')\,dF(y')=1-\left(\frac{w_{j_n-1}}{w}\right)^{1/(n-1)}.$$

Substitute this into (D.10) above to get

$$\lim_{n\to\infty}\left(\left(\frac{w_{j_n-1}}{w}\right)^{1/(n-1)}-\int_y^{y_n^*(w_{j_n-1})}\pi_{j_n}^n(y')\,dF(y')\right)^{n-1}$$
$$=\lim_{n\to\infty}\exp\left\{(n-1)\log\left(\left(\frac{w_{j_n-1}}{w}\right)^{1/(n-1)}\right.\right.$$
$$\left.\left.-\frac{1}{n-1}\int_y^{y_n^*(w_{j_n-1})}\pi_{j_n}^n(y')(n-1)\,dF(y')\right)\right\}.$$

The exponential function is continuous provided its argument is finite. Using Lemma D.2, the limit inside the log function is strictly positive, so the limit can be moved inside the first bracket to get

$$\lim_{n\to\infty} \frac{\log\left(\left(\frac{w_{j_{n-1}}}{w}\right)^{1/(n-1)} - \frac{1}{n-1}\left\{\int_y^{y_n^*(w_{j_{n-1}})} \pi_{j_n}^n(y')(n-1)\,dF(y')\right\}\right)}{\frac{1}{n-1}}, \tag{D.11}$$

which can be written as

$$\lim_{x\to 0, t\to\gamma, z\to\zeta} \frac{\log(t^x - xz)}{x},$$

where $\gamma = \lim_{n\to\infty} \frac{w_{j^{n-1}}}{w}$ is 1 if w is in the support of G and is equal to $\frac{w^s}{w}$ otherwise. The value of the constant ζ follows from the bounded convergence theorem and Lemma D.3. ζ is equal to $\int_y^{y^*(w)} k(y')\,dF(y')$ when w is in the support of G and to $\int_y^{y^*(w^s)} k(y')\,dF(y')$ when w lies above the support of G. Now apply l'Hôpital's rule to get the limit of (D.11) as

$$\frac{w^-}{w} e^{-\int_y^{y^*(w^-)} k(y')\,dF(y')}$$

when w is above the support of G and as $e^{-\int_y^{y^*(w)} k(y')\,dF(y')}$ when w is in the support of G.

When $w < w_0$, the argument is similar. The limit of interest is

$$\begin{aligned}
&\lim_{n\to\infty}\left(1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y')\,dF(y')\right)^{n-1} \\
&\quad = \lim_{n\to\infty} \min\left[1, \frac{w_0}{w}\left(1 - \int_y^{y_n^*(w_0)} \pi_1^n(y')\,dF(y')\right)^{n-1}\right] \\
&\quad = \min\left[1, \frac{w_0}{w} \lim_{n\to\infty}\left(1 - \frac{1}{n-1}\int_y^{y_n^*(w_0)} (n-1)\pi_1^n(y')\,dF(y')\right)^{n-1}\right].
\end{aligned}$$

The equality follows from the fact that for any worker who applies at both wages w and $w_0$with positive probability,

$$\left(1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y')\,dF(y')\right)^{n-1} w = \left(1 - \int_y^{y_n^*(w_0)} \pi_1^n(y')\,dF(y')\right)^{n-1} w_0.$$

The min operator appears because types close to $y_n^*(w_0)$ apply at wage w with probability 0, so every such type would be hired with probability 1 if they did apply. Now evaluating the limit as above gives

$$\min\left[1, \frac{w_0}{w} e^{-\int_{\underline{y}}^{y^*(w_0)} k(y')\,dF(y')}\right]. \qquad \textit{Q.E.D.}$$

APPENDIX E: THE LIMIT THEOREM

From the argument in Section 4 of the main text, the worker's payoff is given by

$$we^{-\int_{\underline{y}}^{y^*(w)} k(y')\,dF(y')}, \tag{4.1}$$

where we substitute

$$k(y) \equiv \frac{\tau}{1 - G^{-}(\omega(y))}.$$

The firm's payoff for wages in the support of G is

$$\int_{\underline{y}}^{y^*(w)} k(y) v(w, y, x) e^{-\int_{\underline{y}}^{y^*(w)} k(y')\,dF(y')}\,dF(y). \tag{4.2}$$

For wages w' below the support of G, the firm's payoff is

$$\int_{\underline{y}}^{y(w')} \tau v(w', y, x) \frac{\underline{w}}{w} e^{-\int_{\underline{y}}^{y^*(\underline{w})} \tau\,dF(y')} F'(y)\,dy. \tag{4.3}$$

Finally, for wages that lie above the support of G, the payoff is

$$\frac{\overline{w}}{w'} \int_{\underline{y}}^{\overline{y}} k(y) v(w', y, x) e^{-\int_{\underline{y}}^{\overline{y}} k(y')\,dF(y')} F'(y)\,dy + v(w', \overline{y}, x)\left(1 - \frac{\overline{w}}{w'}\right). \tag{4.4}$$

THEOREM 9.1: *Let G be a distribution of wages, w a wage in the support of G offered by a firm of type x. Let G_n be a sequence of distributions with finite support that converges weakly to G. Then worker and firm payoffs in the continuation equilibrium in which other firms offer wages given by the mass points in G_n converge to the payoff functions given by* (4.1), (4.2), (4.3), *and* (4.4).

PROOF: The proof of Theorem 9.1 now follows from Lemmas D.1 and D.4. A firm of type x that offers wage w has profit

$$\int_{\underline{y}}^{\overline{y}} v(w, y, x)\,d\phi_{j_n}^n(y).$$

The argument now depends on whether $w \geq w_1$ (i.e., whether or not there is a wage below w in the support of G). Suppose first that $w \geq w_1$ and let j_n be the index of the wage w in the distribution G_n associated with the nth approximation. Substituting for ϕ gives

$$\begin{aligned}
&\int_{\underline{y}}^{y_n^*(w)} v(w, y, x)\, d\left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y')\, dF(y')\right]^n \\
&= \int_{\underline{y}}^{y_n^*(w_{jn-1})} v(w, y, x)\, d\left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y')\, dF(y')\right]^n \\
&\quad + \int_{y_n^*(w_{jn-1})}^{y_n^*(w)} v(w, y, x)\, d\left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y')\, dF(y')\right]^n \\
&= \int_{\underline{y}}^{y_n^*(w_{jn-1})} v(w, y, x) n \pi_{j_n}^n(y) \left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y')\, dF(y')\right]^{n-1} \\
&\quad \times F'(y)\, dy + v(w, y_n^*(w), x) \\
&\quad - v(w, y_n^*(w_{j_n-1}), x)\left[1 - \int_{y_n^*(w_{j_n-1})}^{y_n^*(w)} \pi_{j_n}^n(y')\, dF(y')\right]^n \\
&\quad + \int_{y_n^*(w_{j_n-1})}^{y_n^*(w)} \left[1 - \int_y^{y_n^*(w)} \pi_{j_n}^n(y')\, dF(y')\right]^n \frac{\partial v(w, y, x)}{\partial y}\, dy.
\end{aligned}$$

That last two terms in this expression are derived by integrating by parts. Now observe that a worker of type $y_n^*(w_{j_n-1})$ is just indifferent between applying at the wage w_{j_n-1} and being hired for sure, or applying at wage w and being hired with probability

$$\left[1 - \int_{y_n^*(w_{j_n-1})}^{y_n^*(w)} \pi_{j_n}^n(y')\, dF(y')\right]^{n-1}.$$

So substitute $\frac{w_{j_n-1}}{w}$ for this probability in the second term and take limits using the results of Lemma D.4 to get

$$\begin{aligned}
&\int_{\underline{y}}^{y^*(w^-)} v(w, y, x) k(y) \frac{w^-}{w} e^{-\int_y^{y^*(w^-)} k(y')\, dF(y')} F'(y)\, dy \\
&\quad + v(w, y^*(w), x)\left(1 - \frac{w^-}{w}\right).
\end{aligned}$$

The first term follows from the bounded convergence theorem and Lemma D.3. The second term follows from the substitution made above and from the fact that $y_n^*(w) - y_n^*(w_{j_n-1})$ converges to zero with n (if not, the probability of being hired at wage w for traders between $y_n^*(w)$ and $y_n^*(w_{j_n-1})$ goes to zero). The convergence of $y_n^*(w) - y_n^*(w_{j_n-1})$ to zero also reduces the last term in the expansion to zero because the derivative of v with respect to y is bounded (and the term multiplying it is less than 1).

Now consider the case where $w < w_0$. The firm's profit is

$$\begin{aligned}
&\int_{\underline{y}}^{y_n^*(w)} v(w, y, x)\, d\left[1 - \int_y^{y_n^*(w)} \pi_1^n(y')\, dF(y')\right]^n \\
&\quad = \int_{\underline{y}}^{y_n^*(w)} v(w, y, x) n\pi_1^n(y)\left[1 - \int_y^{y_n^*(w)} \pi_1^n(y')\, dF(y')\right]^{n-1} F'(y)\, dy \\
&\quad = \frac{n}{n-1}\int_{\underline{y}}^{y_n^*(w)} v(w, y, x)(n-1)\pi_1^n(y)\frac{w_0}{w} \\
&\qquad \times \left[1 - \int_y^{y_n^*(w_0)} \pi_1^n(y')\, dF(y')\right]^{n-1} F'(y)\, dy.
\end{aligned}$$

Now apply Lemmas D.3 and D.4, and use the bounded convergence theorem to take limits of this expression, yielding

$$\int_{\underline{y}}^{y(w)} v(w, y, x)\tau\frac{w_1}{w} e^{-\int_y^{y^*(w_1)} \tau\, dF(y')} F'(y)\, dy,$$

where $y(w)$ is either $\underline{y}$ or the solution to

$$\frac{w_0}{w} e^{-\int_y^{y^*(w_0)} \tau\, dF(y')} = 1,$$

whichever is higher.

The last part of the argument is to show that

$$y^*(w) = \sup\{y : \omega(y) \le w\}.$$

Suppose to the contrary that for some w, $y^*(w) > \sup\{y : \omega_n(y) \le w\} = y_n^*(w)$ for all large n. Observe that for each n, $\omega_n(y_n^*(w)) \ge w$. Furthermore, note that a worker of type $y_n^*(w)$ has a type that is at least as high as any other worker who applies at wage w. So such a worker is hired for sure at wage w. Let $y_0 = \lim_{n\to\infty} \sup\{y : \omega_n(y) \le w\} < y^*(w)$.

At the other extreme, if $y^*(w)$ is not a continuity point of ω, then since the latter function is right continuous and nondecreasing, there is a point $y_0 < y_1 < y^*(w)$ at which ω is continuous (and $\omega(y_1) \le w$). For large n, it must be that $\omega_n(y_1) > w$ since otherwise $y_n^*(w)$ would be at least as large as y_1. Yet since y_1 is a continuity point of ω and ω_n converges weakly to ω, then $\omega_n(y_1) \to \omega(y_1)$.

Then using Lemma D.3, the payoff to a worker of type $y_n^*(w)$ who applies at the wage $\omega_n(y)$ is converging to

$$we^{-\int_{y_0}^{y_1} k(y')\,dF(y')} < w.$$

This contradicts the property that workers should receive the same expected payoff by applying to all wages that are at least as large as their reservation wage.

A similar argument establishes a contradiction when $y_0 = \lim_{n\to\infty} \sup\{y : \omega_n(y) \leq w\} > y^*(w)$. *Q.E.D.*

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