Econometrica Supplementary Material

SUPPLEMENT TO "SOCIAL LEARNING EQUILIBRIA"

(*Econometrica*, Vol. 88, No. 3, May 2020, 1235–1267)

ELCHANAN MOSSEL
Department of Mathematics, MIT

MANUEL MUELLER-FRANK
Department of Economics, IESE Business School

ALLAN SLY
Department of Mathematics, Princeton University

OMER TAMUZ
Division of Humanities and Social Sciences, Caltech

S1. ADDITIONAL EXTENSIONS

THIS SUPPLEMENTARY APPENDIX PRESENTS additional extensions and results. The first concerns the case of heterogeneous types of agents, with the corresponding result following immediately from the results established in the paper. The second extension relaxes the assumption of binary states and actions inherent in the canonical setting.

S1.1. *Heterogeneous Preferences and Priors*

We relax the homogeneity assumption and consider agents who have different utility functions and/or prior beliefs (with full support). Assume that all agents share the same belief regarding the conditional signal distributions but there are finitely many different types. Agent i's type is determined by her full-support prior belief on the binary state space and her utility function for a binary action. We assume that for each utility function it is strictly preferable to match the action with the state than to mismatch.

Assume that the agents' types are common knowledge. The following can be established by making slight adjustments to our proofs.

PROPOSITION S1: *In a canonical setting with finitely many commonly known types and where signals are unbounded, every CSLE satisfies information aggregation.*

If signals are unbounded, then in a CSLE all agents agree on the same action, and additionally this action is optimal. Thus unbounded signals overcome heterogeneity in priors and payoffs. The result follows from the fact that types are commonly known and there exists at least one type with infinitely many agents. Proposition S1 is interesting to view in light of Aumann's "agreeing to disagree" result (1976). He showed that if agents share a common prior then common knowledge of posteriors implies agreement. Proposition S1 shows that if signals are unbounded then (common) knowledge of actions implies that agreement and information aggregation hold among an infinite group of agents, even if priors and utility functions differ.

Elchanan Mossel: mossel@mit.edu
Manuel Mueller-Frank: MMuellerfrank@iese.edu
Allan Sly: allansly@princeton.edu
Omer Tamuz: tamuz@caltech.edu

 https://doi.org/10.3982/ECTA16465

S1.2. *Beyond the Canonical Setting: Many States and Actions*

Another extension of our results beyond the canonical setting is to settings with more than two states and more than two actions. In this section, we consider social learning settings in which signals are still conditionally independent—as in the canonical setting—but the set of states can be of any (finite) size, as can the set of actions. We show that our agreement result for CSLEs still holds, under an additional condition on the structure of the utility function and the private signals; this condition rules out some pathological cases in which disagreement can arise.[1]

We consider a social learning setting $(N, A, \Theta, u, S, \mu)$ with N countably infinite, A and Θ finite, and conditionally independent private signals; we will refer to this as a *finite* setting. We say that private signals are *always useful* if, for any prior $p \in \Delta(\Theta)$ for which more than one action maximizes expected utility, observing a conditionally independent private signal (distributed as the agents' signals s_i) strictly increases the expected utility of a rational agent.[2] That is, if s_x is an additional conditionally independent private signal, distributed as s_i, and if D is any event that is conditionally independent of s_x, then whenever

$$\left|\operatorname*{argmax}_{a\in A} \mathbb{E}\big[u(a, \theta) \mid D\big]\right| \geq 2$$

(i.e., whenever conditioning on D results in more than one optimal action) it holds that

$$\mathbb{E}\Big[\max_{a\in A} \mathbb{E}\big[u(a, \theta) \mid D, s_x\big] \mid D\Big] > \max_{a\in A} \mathbb{E}\big[u(a, \theta) \mid D\big]$$

(i.e., learning s_x increases one's expected utility). It is easy to see that for the case of two states and two actions, this holds whenever signals are informative.

The assumption of always useful signals implies that it is impossible for a state θ_0 to have more than one optimal action, since otherwise, conditioning on the event that the state is θ_0, an additional private signal will not change the agent's belief, and thus cannot result in higher expected utility. Another implication for the case of many states is that signals cannot be restricted to some particular dimension and ignore others. For example, if $\Theta = \{0, 1\} \times \{0, 1\}$, the assumption of always useful signals rules out a signal that is informative with respect to the first coordinate, but provides no information regarding the second. Note that the assumption of always useful signals does not imply that signals are unbounded.

PROPOSITION S2: *Consider a finite setting with infinitely many agents. If signals are always useful, then every CSLE satisfies agreement.*

The information aggregation results of Proposition 1 and Theorem 2 also hold in this setting, under an appropriate definition of unbounded signals. In interest of brevity, we leave Proposition S2 as the only result that we extend in this direction.

[1]Clearly, there can be disagreement in equilibrium when signals are completely uninformative. Likewise, when two actions yield the same utility in some state then agents who learn the state can choose different actions in equilibrium. These issues are avoided in the canonical settings, where each state has a different uniquely optimal action and where signals are informative. In this more general setting, a more complicated assumption is required.

[2]Arieli and Mueller-Frank (2017) showed that in a general sequential social learning game, in every equilibrium signals are never useful at the limit belief.

The proof of Proposition S2 starts with the following “no trade” lemma, showing that disagreement implies indifference. Its proof, which we omit, follows the same argument as the proof of Lemma 2.

LEMMA S1: *Fix a social learning setting with A and Θ finite, and let $(\bar{\ell}, \bar{a})$ be a SLE defined on this setting. Let A_0 be a subset of A, and let D be the event that for each $a \in A_0$ there is an agent who chooses the action a:*

$$D = \{\textit{for each } a \in A_0 \textit{ there exists an } i \in N \textit{ such that } a_i = a\}.$$

If the probability of D is positive, then for any $a, b \in A_0$ it holds that

$$\mathbb{E}\big[u(a, \theta) \mid D\big] = \mathbb{E}\big[u(b, \theta) \mid D\big].$$

Given this lemma, we turn to the proof of Proposition S2.

PROOF OF PROPOSITION S2: Let A_0 be a subset of A of size at least 2, and let D be the disagreement event that for each $a \in A_0$ there is an agent who chooses the action a:

$$D = \{\text{for each } a \in A_0 \text{ there exists an } i \in N \text{ such that } a_i = a\}.$$

We assume by contradiction that D has positive probability. By Lemma S1,

$$\mathbb{E}\big[u(a, \theta) \mid D\big] = \mathbb{E}\big[u(b, \theta) \mid D\big]$$

for all $a, b \in A_0$, and we denote this quantity by $U(A_0 \mid D)$. Using this notation, we can write the expected utility of agent i as

$$\mathbb{E}\big[u(a_i, \theta)\mathbb{1}_{\{D\}}\big] + \mathbb{E}\big[u(a_i, \theta)(1 - \mathbb{1}_{\{D\}})\big] = U(A_0 \mid D) \cdot \mathbb{P}[D] + \mathbb{E}\big[u(a_i, \theta)(1 - \mathbb{1}_{\{D\}})\big] \quad \text{(S1)}$$

As in the proof of Theorem 1, let

$$b_x = b(s_x) \in \operatorname*{argmax}_{a \in A_0} \mathbb{E}\big[u(a, \theta) \mid D, s_x\big],$$

denote $b_i = b(s_i)$, and consider a deviation by agent i in which she chooses b_i whenever D occurs, and a_i otherwise. Then the expected utility of this deviation is

$$\mathbb{E}\big[u(b_i, \theta)\mathbb{1}_{\{D\}}\big] + \mathbb{E}\big[u(a_i, \theta)(1 - \mathbb{1}_{\{D\}})\big]. \quad \text{(S2)}$$

Now,

$$\begin{aligned}
\mathbb{E}\big[u(b_i, \theta)\mathbb{1}_{\{D\}}\big] &= \sum_{\omega \in \Theta} \mathbb{E}\big[u(b_i, \theta)\mathbb{1}_{\{D\}} \mid \theta = \omega\big] \cdot \mathbb{P}[\theta = \omega] \\
&= \sum_{\omega \in \Theta} \mathbb{E}\big[u(b_i, \omega)\mathbb{1}_{\{D\}} \mid \theta = \omega\big] \cdot \mathbb{P}[\theta = \omega] \\
&= \sum_{\omega \in \Theta} \sum_{a \in A} u(a, \omega)\mathbb{P}[b_i = a, D \mid \theta = \omega] \cdot \mathbb{P}[\theta = \omega]. \quad \text{(S3)}
\end{aligned}$$

By the concentration of dependence principle,

$$\lim_{i \to \infty} \mathbb{P}[b_i = a, D \mid \theta = \omega] = \mathbb{P}[b_i = a \mid \theta = \omega] \cdot \mathbb{P}[D \mid \theta = \omega].$$

Note that the right-hand side holds for any i, since signals are conditionally identically distributed. Hence it also holds for agent x, and thus

$$\lim_{i\to\infty} \mathbb{P}[b_i = a, D \mid \theta = \omega] = \mathbb{P}[b_x = a \mid \theta = \omega] \cdot \mathbb{P}[D \mid \theta = \omega].$$

Substituting this back into (S3) yields

$$\begin{aligned}\lim_{i\to\infty} \mathbb{E}\big[u(b_i, \theta)\mathbb{1}_{\{D\}}\big] &= \sum_{\omega\in\Theta}\sum_{a\in A} u(a, \omega)\mathbb{P}[b_x = a \mid \theta = \omega] \cdot \mathbb{P}[D \mid \theta = \omega] \cdot \mathbb{P}[\theta = \omega] \\ &= \sum_{\omega\in\Theta}\sum_{a\in A} u(a, \omega)\mathbb{P}[b_x = a, D \mid \theta = \omega] \cdot \mathbb{P}[\theta = \omega] \\ &= \mathbb{E}\big[u(b_x, \theta) \mid D\big] \cdot \mathbb{P}[D].\end{aligned}$$

By our assumption that the signals are always useful,

$$\mathbb{E}\big[u(b_x, \theta) \mid D\big] > U(A_0 \mid D).$$

Substituting this back into (S2) and comparing to (S1) shows that b_i is a profitable deviation for some i large enough, and so we have reached a contradiction with our equilibrium assumption. *Q.E.D.*

REFERENCES

ARIELI, I., AND M. MUELLER-FRANK (2017): "A General Analysis of Sequential Learning," IESE Business School Working Paper WP-1119-E. [2]

AUMANN, R. J. (1976): "Agreeing to Disagree," *The Annals of Statistics*, 4 (6), 1236–1239. [1]

Co-editor Joel Sobel handled this manuscript.

Manuscript received 25 June, 2018; final version accepted 16 December, 2019; available online 28 January, 2020.