Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL BANDWIDTH SELECTION IN HETEROSKEDASTICITY–AUTOCORRELATION ROBUST TESTING"

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THIS APPENDIX provides technical results and proofs for the paper.

A.1. *Technical Lemmas and Supplements*

LEMMA 1: *Define*

$$c_1^* = 4\int_{-\infty}^{\infty} |k(v)|\, dv.$$

Let Assumption 2 *hold*. *Then the cumulants* κ_m *of* $\Xi_b - \mu_b$ *satisfy*

$$|\kappa_m| \le 2^m (m-1)!(c_1^* b)^{m-1} \quad \text{for } m \ge 1 \tag{A.1}$$

and its moments $\alpha_m = E(\Xi_b - \mu_b)^m$ *satisfy*

$$|\alpha_m| \le 2^{2m} m!(c_1^* b)^{m-1} \quad \text{for } m \ge 1. \tag{A.2}$$

PROOF OF LEMMA 1: To find the cumulants of $\Xi_b - \mu_b$, we write

$$\Xi_b = \int_0^1 \int_0^1 k_b^*(r, s)\, dW(r)\, dW(s), \tag{A.3}$$

where $k_b^*(r, s)$ is defined by

$$k_b^*(r, s) = k_b(r-s) - \int_0^1 k_b(r-t)\, dt - \int_0^1 k_b(\tau - s)\, d\tau + \int_0^1 \int_0^1 k_b(t-\tau)\, dt\, d\tau.$$

It can be shown that the cumulant generating function of $\Xi_b - \mu_b$ is

$$\ln \phi(t) = \sum_{m=1}^{\infty} \kappa_m \frac{(it)^m}{m!}, \tag{A.4}$$

where $\kappa_1 = 0$ and for $m \ge 2$,

$$\kappa_m = 2^{m-1}(m-1)! \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^m k_b^*(\tau_j, \tau_{j+1}) \right) d\tau_1 \cdots d\tau_m, \tag{A.5}$$

with $\tau_1 = \tau_{m+1}$.

Now

$$
\begin{aligned}
&\left|\int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^m k_b^*(\tau_j, \tau_{j+1})\right) d\tau_1 \cdots d\tau_m\right| \\
&\le \int_0^1 \cdots \int_0^1 |k_b^*(\tau_1, \tau_2) k_b^*(\tau_2, \tau_3) \cdots k_b^*(\tau_{m-1}, \tau_m)| \\
&\qquad \times |k_b^*(\tau_m, \tau_1)|\, d\tau_1 \cdots d\tau_m \qquad \text{(A.6)}\\
&\le 2 \int_0^1 \cdots \int_0^1 |k_b^*(\tau_1, \tau_2) k_b^*(\tau_2, \tau_3) \cdots k_b^*(\tau_{m-1}, \tau_m)|\, d\tau_1 \cdots d\tau_m \qquad \text{(A.7)}\\
&\le 2 \sup_{\tau_2} \int_0^1 |k_b^*(\tau_1, \tau_2)|\, d\tau_1 \\
&\qquad \times \int_0^1 |k_b^*(\tau_2, \tau_3) k_b^*(\tau_3, \tau_4) \cdots k_b^*(\tau_{m-1}, \tau_m)|\, d\tau_2 \cdots d\tau_m \\
&\le 2 \sup_{\tau_2} \int_0^1 |k_b^*(\tau_1, \tau_2)|\, d\tau_1 \\
&\qquad \times \sup_{\tau_3} \int_0^1 |k_b^*(\tau_2, \tau_3)|\, d\tau_2 \cdots \sup_{\tau_m} \int_0^1 [k_b^*(\tau_{m-1}, \tau_m)]\, d\tau_{m-1} \\
&= 2\left(\sup_s \int_0^1 |k_b^*(r, s)|\, dr\right)^{m-1}. \qquad \text{(A.8)}
\end{aligned}
$$

But

$$
\begin{aligned}
\sup_s \int_0^1 |k_b^*(r, s)|\, dr &\le 4 \sup_s \int_0^1 |k_b(r - s)|\, dr \qquad \text{(A.9)}\\
&= 4 \sup_{s \in [0,1]} \left(\int_{-s}^{1-s} |k_b(v)|\, dv\right) \\
&\le 4 \int_{-\infty}^{\infty} |k_b(v)|\, dv \\
&= bc_1^*.
\end{aligned}
$$

As a result,

$$
\left|\int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^m k_b^*(\tau_j, \tau_{j+1})\right) d\tau_1 \cdots d\tau_m\right| \le 2(c_1^* b)^{m-1}, \qquad \text{(A.10)}
$$

so

$$|\kappa_m| \leq 2^m (m-1)!(c_1^* b)^{m-1}. \tag{A.11}$$

Note that the moments $\{\alpha_j\}$ and cumulants $\{\kappa_j\}$ satisfy the relationship

$$\alpha_m = \sum_{\pi} \frac{m!}{(j_1!)^{m_1}(j_2!)^{m_2}\cdots(j_\ell!)^{m_\ell}} \frac{1}{m_1!m_2!\cdots m_\ell!} \prod_{j\in\pi} \kappa_j, \tag{A.12}$$

where the sum is taken over the elements

$$\pi = [\underbrace{j_1, \ldots j_1}_{m_1 \text{ times}}, \underbrace{j_2, \ldots j_2}_{m_2 \text{ times}}, \ldots, \underbrace{j_\ell, \ldots, j_\ell}_{m_\ell \text{ times}}] \tag{A.13}$$

for some integer ℓ, sequence $\{j_i\}_{i=1}^{\ell}$ such that $j_1 > j_2 > \cdots > j_\ell$ and $m = \sum_{i=1}^{\ell} m_i j_i$.

Combining the preceding formula with (A.11) gives

$$\begin{aligned} |\alpha_m| &< 2^m m!(c_1^* b)^{m-1} \sum_{\pi} \frac{(j_1)^{-m_1}(j_2)^{-m_2}\cdots(j_\ell)^{-m_\ell}}{m_1!m_2!\cdots m_\ell!} \\ &\leq 2^{2m} m!(c_1^* b)^{m-1}, \end{aligned} \tag{A.14}$$

where the last line follows because

$$\sum_{\pi} \frac{(j_1)^{-m_1}(j_2)^{-m_2}\cdots(j_\ell)^{-m_\ell}}{m_1!m_2!\cdots m_\ell!} \leq \sum_{\pi} \frac{1}{m_1!m_2!\cdots m_\ell!} < 2^m. \tag{A.15}$$

Q.E.D.

LEMMA 2: *Let Assumptions* 2 *and* 3 *hold*. *When* $T \to \infty$ *for a fixed* b, *we have*

$$\mu_{bT} = \mu_b + O\left(\frac{1}{T}\right); \tag{A.16}$$

$$\kappa_{m,T} = \kappa_m + O\left\{\frac{m!2^m}{T^2}(c_1^* b)^{m-2}\right\}, \tag{A.17}$$

uniformly over $m \geq 1$;

$$\alpha_{m,T} = E(\varsigma_{bT} - \mu_{bT})^m = \alpha_m + O\left\{\frac{m!2^{2m}}{T^2}(c_1^* b)^{m-2}\right\} \tag{A.18}$$

uniformly over $m \geq 1$.

PROOF OF LEMMA 2: We first calculate $\mu_{bT} = (T\omega_T^2)^{-1}\,\mathrm{Trace}(\Omega_T A_T W_b A_T)$. Let $W_b^* = A_T W_b A_T$. Then the (i, j)th element of W_b^* is

$$\tilde{k}_b\left(\frac{i}{T}, \frac{j}{T}\right) = k_b\left(\frac{i-j}{T}\right) - \frac{1}{T}\sum_{p=1}^{T} k_b\left(\frac{i-p}{T}\right) - \frac{1}{T}\sum_{q=1}^{T} k_b\left(\frac{q-j}{T}\right) + \frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{p-q}{T}\right). \tag{A.19}$$

So

$$\begin{aligned}
&\mathrm{Trace}(\Omega_T A_T W_b A_T) \\
&\quad = \mathrm{Trace}(\Omega_T W_b^*) \\
&\quad = \sum_{1 \le r_1, r_2 \le T} \left\{\gamma(r_1 - r_2)\tilde{k}_b\left(\frac{r_1}{T}, \frac{r_2}{T}\right)\right\} \\
&\quad = \sum_{r_2=1}^{T}\sum_{h_1=1-r_2}^{T-r_2} \gamma(h_1)\tilde{k}_b\left(\frac{r_2+h_1}{T}, \frac{r_2}{T}\right) \\
&\quad = \left(\sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0}\sum_{r_2=1-h_1}^{T}\right)\gamma(h_1)\tilde{k}_b\left(\frac{r_2+h_1}{T}, \frac{r_2}{T}\right).
\end{aligned} \tag{A.20}$$

But

$$\begin{aligned}
&\sum_{r_2=1}^{T-h_1} \tilde{k}_b\left(\frac{r_2+h_1}{T}, \frac{r_2}{T}\right) \\
&\quad = \sum_{r_2=1}^{T-h_1} k_b\left(\frac{h_1}{T}\right) - \frac{1}{T}\sum_{r_1=1+h_1}^{T}\sum_{p=1}^{T} k_b\left(\frac{r_1-p}{T}\right) \\
&\qquad - \frac{1}{T}\sum_{r_2=1}^{T-h_1}\sum_{q=1}^{T} k_b\left(\frac{q-r_2}{T}\right) + \sum_{r_2=1}^{T-h_1}\frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{p-q}{T}\right) \\
&\quad = -\frac{1}{T}\sum_{r_1=1}^{T}\sum_{p=1}^{T} k_b\left(\frac{r_1-p}{T}\right) - \frac{1}{T}\sum_{r_2=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{q-r_2}{T}\right) \\
&\qquad + \sum_{r_2=1}^{T}\frac{1}{T^2}\sum_{p=1}^{T}\sum_{q=1}^{T} k_b\left(\frac{p-q}{T}\right) + Tk_b\left(\frac{h_1}{T}\right) + C(h_1)
\end{aligned} \tag{A.21}$$

$$= -\frac{1}{T}\sum_{r=1}^{T}\sum_{s=1}^{T} k_b\left(\frac{r-s}{T}\right) + T k_b\left(\frac{h_1}{T}\right) + C(h_1)$$

$$= \sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + T\left\{k_b\left(\frac{h_1}{T}\right) - k_b(0)\right\} + C(h_1),$$

where $C(h_1)$ is a function of h_1 that satisfies $|C(h_1)| \le h_1$. Similarly,

$$\text{(A.22)} \quad \sum_{r_2=1-h_1}^{T} \tilde{k}_b\left(\frac{r_2+h_1}{T}, \frac{r_2}{T}\right)$$
$$= \sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + T\left\{k_b\left(\frac{h_1}{T}\right) - k_b(0)\right\} + C(h_1).$$

Therefore, $\mathrm{Trace}(\Omega_T A_T W_b A_T)$ is equal to

$$\text{(A.23)} \quad \sum_{h=-T+1}^{T-1} \gamma(h) \sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right) + T \sum_{h=-T+1}^{T-1} \gamma(h)\left\{k_b\left(\frac{h}{T}\right) - k_b(0)\right\}$$
$$+ O(1)$$
$$= \sum_{h=-T+1}^{T-1} \gamma(h) \sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right)$$
$$+ T(bT)^{-q} \sum_{h=-T+1}^{T-1} |h|^q \gamma(h)\left\{\frac{k(h/(bT)) - k(0)}{|h/(bT)|^q}\right\} + O(1)$$
$$= \sum_{h=-T+1}^{T-1} \gamma(h) \sum_{r_2=1}^{T} \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_2}{T}\right)$$
$$+ T(bT)^{-q} g_q \sum_{h=-\infty}^{\infty} |h|^q \gamma(h)(1 + o(1)) + O(1).$$

Using

$$\text{(A.24)} \quad \sum_{h=-T+1}^{T-1} \gamma(h) = \omega_T^2\left(1 + O\left(\frac{1}{T}\right)\right)$$

and

$$\frac{1}{T}\sum_{r_2=1}^{T}\tilde{k}_b\left(\frac{r_2}{T},\frac{r_2}{T}\right)=\int_0^1 k_b^*(r,r)\,dr+O\left(\frac{1}{T}\right), \tag{A.25}$$

we now have

$$\mu_{bT}=\int_0^1 k_b^*(r,r)\,dr-(bT)^{-q}g_q\left(\omega_T^{-2}\sum_{h=-\infty}^{\infty}|h|^q\gamma(h)\right)(1+o(1))+O\left(\frac{1}{T}\right). \tag{A.26}$$

By definition, $\mu_b=E\Xi_b=\int_0^1 k_b^*(r,r)\,dr$ and thus $\mu_{bT}=\mu_b+O(T^{-1})$ as desired.

We next approximate $\mathrm{Trace}[(\Omega_T A_T W_b A_T)^m]$ for $m>1$. The approach is similar to the case $m=1$ but notationally more complicated. Let $r_{2m+1}=r_1$, $r_{2m+2}=r_2$, and $h_{m+1}=h_1$. Then

$$\begin{aligned}
&\mathrm{Trace}[(\Omega_T A_T W_b A_T)^m] \\
&=\sum_{r_1,r_2,\ldots,r_{2m+1}=1}^{T}\prod_{j=1}^{m}\gamma(r_{2j-1}-r_{2j})\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+1}}{T}\right) \\
&=\sum_{r_2,r_4,\ldots,r_{2m}=1}^{T}\sum_{h_1=1-r_2}^{T-r_2}\sum_{h_2=1-r_4}^{T-r_4}\cdots\sum_{h_m=1-r_{2m}}^{T-r_{2m}}\prod_{j=1}^{m}\gamma(h_j)\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}+h_{j+1}}{T}\right) \\
&=\left(\sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T-h_1}+\sum_{h_1=1-T}^{0}\sum_{r_2=1-h_1}^{T}\right)\cdots\left(\sum_{h_m=1}^{T-1}\sum_{r_{2m}=1}^{T-h_m}+\sum_{h_m=1-T}^{0}\sum_{r_{2m}=1-h_m}^{T}\right) \\
&\quad\prod_{j=1}^{m}\gamma(h_j)\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}+h_{j+1}}{T}\right) \\
&=\mathrm{I}+\mathrm{II},
\end{aligned} \tag{A.27}$$

where

$$\begin{aligned}
\mathrm{I}&=\left(\sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T-h_1}+\sum_{h_1=1-T}^{0}\sum_{r_2=1-h_1}^{T}\right)\cdots\left(\sum_{h_m=1}^{T-1}\sum_{r_{2m}=1}^{T-h_m}+\sum_{h_m=1-T}^{0}\sum_{r_{2m}=1-h_m}^{T}\right) \\
&\quad\prod_{j=1}^{m}\gamma(h_j)\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)
\end{aligned} \tag{A.28}$$

and

$$\mathrm{II} = O\Biggl\{\Biggl(\sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0}\sum_{r_2=1-h_1}^{T}\Biggr)\cdots\Biggl(\sum_{h_m=1}^{T-1}\sum_{r_{2m}=1}^{T-h_m} + \sum_{h_m=1-T}^{0}\sum_{r_{2m}=1-h_m}^{T}\Biggr) \prod_{j=1}^{m}|\gamma(h_j)|\Bigl(\frac{|h_{j+1}|}{bT}\Bigr)\Biggr\}. \tag{A.29}$$

Here we have used

$$\left|\tilde{k}_b\Bigl(\frac{r_{2j}}{T}, \frac{r_{2j+2}+h_{j+1}}{T}\Bigr) - \tilde{k}_b\Bigl(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\Bigr)\right| = O\Bigl(\frac{|h_{j+1}|}{bT}\Bigr). \tag{A.30}$$

To show this, note that

$$\begin{aligned}\frac{1}{T}\sum_{p=1}^{T} k_b\Bigl(\frac{p-r_{2j+2}-h_{j+1}}{T}\Bigr) &= \frac{1}{T}\sum_{p=1-h_{j+1}}^{T-h_{j+1}} k_b\Bigl(\frac{p-r_{2j+2}}{T}\Bigr)\\ &= \frac{1}{T}\sum_{p=1}^{T} k_b\Bigl(\frac{p-r_{2j+2}}{T}\Bigr) + O\Bigl(\frac{|h_{j+1}|}{T}\Bigr)\end{aligned} \tag{A.31}$$

and

$$\left|k_b\Bigl(\frac{r_{2j}-r_{2j+2}-h_{j+1}}{T}\Bigr) - k_b\Bigl(\frac{r_{2j}-r_{2j+2}}{T}\Bigr)\right| = O\Bigl(\frac{|h_{j+1}|}{bT}\Bigr), \tag{A.32}$$

so that

$$\begin{aligned}\tilde{k}_b\Bigl(\frac{r_{2j}}{T}, \frac{r_{2j+2}+h_{j+1}}{T}\Bigr) &= \tilde{k}_b\Bigl(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\Bigr) + k_b\Bigl(\frac{r_{2j}-r_{2j+2}-h_{j+1}}{T}\Bigr)\\ &\quad - k_b\Bigl(\frac{r_{2j}-r_{2j+2}}{T}\Bigr) + O\Bigl(\frac{|h_{j+1}|}{T}\Bigr)\\ &= \tilde{k}_b\Bigl(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\Bigr) + O\Bigl(\frac{|h_{j+1}|}{bT}\Bigr).\end{aligned} \tag{A.33}$$

The first term (I) can be written as

$$\begin{aligned}\mathrm{I} = &\Biggl(\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T} - \sum_{h_1=1}^{T-1}\sum_{r_2=T-h_1+1}^{T} - \sum_{h_1=1-T}^{0}\sum_{r_2=1}^{-h_1}\Biggr)\\ &\cdots\Biggl(\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T} - \sum_{h_m=1}^{T-1}\sum_{r_{2m}=T-h_m+1}^{T} - \sum_{h_m=1-T}^{0}\sum_{r_{2m}=1}^{-h_m}\Biggr)\end{aligned} \tag{A.34}$$

$$
\begin{aligned}
&\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right\}\\
&=\sum_{\pi}\sum_{h_1,r_2}\cdots\sum_{h_m,r_{2m}}\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right\},
\end{aligned}
$$

where $\sum_{h_j,r_{2j}}$ is one of the three choices $\sum_{h_j=1-T}^{T-1}\sum_{r_{2j}=1}^{T}$, $-\sum_{h_j=1}^{T-1}\sum_{r_{2j}=T-h_j+1}^{T}$, or $-\sum_{h_j=1-T}^{0}\sum_{r_{2j}=1}^{-h_j}$, and $\sum_{\pi}$ is the summation over all possible combinations of $(\sum_{h_1,r_2}\cdots\sum_{h_m,r_{2m}})$. The 3^m summands in (A.34) can be divided into two groups: the first group consists of the summands all of whose r indices run from 1 to T and the second group consists of the rest. It is obvious that the first group can be written as

$$
\left(\sum_{h}\prod_{j=1}^{m}\gamma(h_j)\right)\sum_{r}\left\{\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right\}.
$$

The dominating terms (in terms of the order of magnitude) in the second group are of the forms

$$
\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T}\cdots\sum_{h_p=1-T}^{T-1}\sum_{r_{2p}=T-h_p+1}^{T}\cdots\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T}\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right\}
$$

or

$$
\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T}\cdots\sum_{h_p=1-T}^{T-1}\sum_{r_{2p}=1}^{-h_p}\cdots\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T}\prod_{j=1}^{m}\gamma(h_j)\left\{\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right\}.
$$

These are the summands with only one r index not running from 1 to T. Both of the above terms are bounded by

$$
\begin{aligned}
&\sum_{h_1=1-T}^{T-1}\sum_{r_2=1}^{T}\cdots\sum_{h_p=1-T}^{T-1}\cdots\sum_{h_m=1-T}^{T-1}\sum_{r_{2m}=1}^{T}\prod_{j=1}^{m}|\gamma(h_j)||h_p|\prod_{j\neq p}\left|\tilde{k}_b\left(\frac{r_{2j}}{T},\frac{r_{2j+2}}{T}\right)\right|\\
&\leq\left[\sup_{r_4}\sum_{r_2=1}^{T}\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\right]^{m-2}\left(\sum_{h_j}|\gamma(h_j)|\right)^{m-1}\left(\sum_{h_p}|\gamma(h_p)||h_p|\right),
\end{aligned}
$$

using the same approach as in (A.6). Approximating the sum by an integral and noting that the second group contains $(m-1)$ terms, all of which are of the same order of magnitude as the above typical dominating terms, we conclude that the second group is $O[2mT^{m-2}(c_1^*b)^{m-2}]$ uniformly over m. As a consequence,

$$\text{(A.35)}\quad \mathrm{I} = \left(\sum_h \prod_{j=1}^m \gamma(h_j)\right) \sum_r \left\{\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right)\right\} + O\{2mT^{m-2}(c_1^*b)^{m-2}\}$$

uniformly over m.

The second term (II) is easily shown to be $o(2mT^{m-2}(c_1^*b)^{m-2})$ uniformly over m. Therefore,

$$\text{(A.36)}\quad \begin{aligned} &\mathrm{Trace}[(\Omega_T A_T W_b A_T)^m] \\ &\quad = \left(\sum_h \gamma(h)\right)^m \sum_r \left\{\tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right)\right\} + O\{2mT^{m-2}(c_1^*b)^{m-2}\} \end{aligned}$$

and

$$\text{(A.37)}\quad \begin{aligned} \kappa_{m,T} &= 2^{m-1}(m-1)!T^{-m}(\omega_T^2)^{-m}\,\mathrm{Trace}[(\Omega_T A_T W_b A_T)^m] \\ &= 2^{m-1}(m-1)!\left\{T^{-m}\sum_r \tilde{k}_b\left(\frac{r_{2j}}{T}, \frac{r_{2j+2}}{T}\right) + O\left[\frac{2m}{T^2}(c_1^*b)^{m-2}\right]\right\} \\ &= 2^{m-1}(m-1)! \\ &\quad \times \left\{\int \prod_{j=1}^m \int_0^1 k_b^*(\tau_j, \tau_{j+1})\,d\tau_j\,d\tau_{j+1} + O\left[\frac{2m}{T^2}(c_1^*b)^{m-2}\right]\right\} \\ &= \kappa_m + O\left\{\frac{m!2^m}{T^2}(c_1^*b)^{m-2}\right\} \end{aligned}$$

uniformly over m.

Finally, we consider $\alpha_{m,T}$. Note that $\alpha_{1,T} = E(\varsigma_{bT} - \mu_{bT}) = 0$ and

$$\text{(A.38)}\quad \alpha_{m,T} = \sum_\pi \frac{m!}{(j_1!)^{m_1}(j_2!)^{m_2}\cdots(j_k!)^{m_k}} \frac{1}{m_1!m_2!\cdots m_k!} \prod_{j\in\pi} \kappa_{j,T},$$

where the summation $\sum_\pi$ is defined in (A.12). Combining the preceding formula with (A.17) gives

$$\text{(A.39)}\quad \begin{aligned} \alpha_{m,T} &= \alpha_m + O\left\{\frac{2^m}{T^2}(c_1^*b)^{m-2}\sum_\pi \frac{m!}{m_1!m_2!\cdots m_k!}\right\} \\ &= \alpha_m + O\left\{\frac{m!2^{2m}}{T^2}(c_1^*b)^{m-2}\right\}, \end{aligned}$$

uniformly over m, where the last line follows because $\sum_{\pi} \frac{1}{m_1! m_2! \cdots m_k!} < 2^m$. *Q.E.D.*

LEMMA 3: *Let Assumptions* 2 *and* 3 *hold. If* $b \to 0$ *and* $T \to \infty$ *such that* $bT \to \infty$, *then*

$$\mu_{bT} = \int_0^1 k_b^*(r,r)\,dr - (bT)^{-q} g_q \left(\omega_T^{-2} \sum_{h=-\infty}^{\infty} |h|^q \gamma(h) \right) (1 + o(1)) + O\left(\frac{1}{T}\right); \tag{A.40}$$

$$\kappa_{2,T} = 2 \int_0^1 \int_0^1 (k_b^*(r,s))^2\,dr\,ds (1 + o(1)) + O\left(\frac{1}{T}\right); \tag{A.41}$$

for $m = 3$ *and* 4,

$$\kappa_{m,T} = O(b^{m-1}) + O\left(\frac{1}{T}\right). \tag{A.42}$$

PROOF OF LEMMA 3: We have proved (A.40) in the proof of Lemma 2 because equation (A.26) holds for both fixed b and decreasing b. It remains to consider $\kappa_{m,T}$ for $m = 2$, 3, and 4. We first consider $\kappa_{2,T} = 2T^{-2}(\omega_T^{-4}) \times \mathrm{Trace}[(\Omega_T A_T W_b A_T)^2]$. As a first step, we have

$$\begin{aligned}
&\mathrm{Trace}[(\Omega_T A_T W_b A_T)^2] \\
&= \sum_{r_1, r_2, r_3, r_4} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_3}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_1}{T}\right) \right\} \gamma(r_1 - r_2)\gamma(r_3 - r_4) \\
&= \sum_{r_2=1}^{T} \sum_{h_1=1-r_2}^{T-r_2} \sum_{r_4=1}^{T} \sum_{h_2=1-r_4}^{T-r_4} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4 + h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2 + h_1}{T}\right) \right\} \\
&\quad \times \gamma(h_1)\gamma(h_2) \\
&= \left(\sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} + \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T} \right) \left(\sum_{h_2=1}^{T-1} \sum_{r_4=1}^{T-h_2} + \sum_{h_2=1-T}^{0} \sum_{r_4=1-h_2}^{T} \right) \\
&\quad \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4 + h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2 + h_1}{T}\right) \right\} \gamma(h_1)\gamma(h_2) \\
&:= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{A.43}$$

where

$$I_1 = \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-h_1} \sum_{r_4=1}^{T-h_2} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4+h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2+h_1}{T}\right) \right\} \gamma(h_1)\gamma(h_2),$$

$$I_2 = \sum_{h_1=1}^{T-1} \sum_{r_2=1}^{T-h_1} \sum_{h_2=1-T}^{0} \sum_{r_4=1-h_2}^{T} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4+h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2+h_1}{T}\right) \right\} \times \gamma(h_1)\gamma(h_2),$$

$$I_3 = \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T} \sum_{r_2=1}^{T-h_1} \sum_{r_4=1}^{T-h_2} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4+h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2+h_1}{T}\right) \right\} \times \gamma(h_1)\gamma(h_2),$$

and

$$I_4 = \sum_{h_1=1-T}^{0} \sum_{r_2=1-h_1}^{T} \sum_{h_2=1-T}^{0} \sum_{r_4=1-h_2}^{T} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4+h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2+h_1}{T}\right) \right\} \times \gamma(h_1)\gamma(h_2).$$

We now consider each term in turn. Using equation (A.33), we have

$$\tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4+h_2}{T}\right) = \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4}{T}\right) + O\left(\frac{b|h_2|}{T}\right) \tag{A.44}$$

and

$$\tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2+h_1}{T}\right) = \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2}{T}\right) + O\left(\frac{b|h_1|}{T}\right). \tag{A.45}$$

It follows from (A.44) and (A.45) that

$$\begin{aligned} I_1 &= \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-1} \sum_{r_4=1}^{T-1} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4+h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2+h_1}{T}\right) \right\} \gamma(h_1)\gamma(h_2) \\ &\quad + O\left(\sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \left[T(|h_1|+|h_2|) + |h_1 h_2| \right] |\gamma(h_1)\gamma(h_2)| \right) \\ &= \sum_{h_1=1}^{T-1} \sum_{h_2=1}^{T-1} \sum_{r_2=1}^{T-1} \sum_{r_4=1}^{T-1} \left\{ \tilde{k}_b\left(\frac{r_2}{T}, \frac{r_4+h_2}{T}\right) \tilde{k}_b\left(\frac{r_4}{T}, \frac{r_2+h_1}{T}\right) \right\} \gamma(h_1)\gamma(h_2) \\ &\quad + O(T) \end{aligned} \tag{A.46}$$

$$
\begin{aligned}
&= \sum_{h_1=1}^{T-1}\sum_{h_2=1}^{T-1}\sum_{r_2=1}^{T-1}\sum_{r_4=1}^{T-1}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_2}{T}\right)\right\}\gamma(h_1)\gamma(h_2)+O(T) \\
&\quad + O\left\{\sum_{h_1=1}^{T-1}\sum_{h_2=1}^{T-1}\sum_{r_2=1}^{T-1}\sum_{r_4=1}^{T-1}\left|\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\right|\left(\frac{b(|h_1|+|h_2|)}{T}\right)\right. \\
&\qquad\qquad \left.\times|\gamma(h_1)\gamma(h_2)|\right\} \\
&= \sum_{h_1=1}^{T-1}\sum_{h_2=1}^{T-1}\sum_{r_2=1}^{T-1}\sum_{r_4=1}^{T-1}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_2}{T}\right)\right\}\gamma(h_1)\gamma(h_2)+O(T).
\end{aligned}
$$

Following the same procedure, we can show that

$$I_2 = \sum_{h_1=1}^{T-1}\sum_{r_2=1}^{T}\sum_{h_2=1-T}^{0}\sum_{r_4=1}^{T}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_2}{T}\right)\right\}\gamma(h_1)\gamma(h_2)+O(T), \tag{A.47}$$

$$I_3 = \sum_{h_1=1-T}^{0}\sum_{r_2=1}^{T}\sum_{r_2=1}^{T}\sum_{r_4=1}^{T}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_2}{T}\right)\right\}\gamma(h_1)\gamma(h_2)+O(T), \tag{A.48}$$

and

$$
\begin{aligned}
I_4 = {} & \sum_{h_1=1-T}^{0}\sum_{r_2=1}^{T}\sum_{h_2=1-T}^{0}\sum_{r_4=1}^{T}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\tilde{k}_b\left(\frac{r_4}{T},\frac{r_2}{T}\right)\right\}\gamma(h_1)\gamma(h_2) \\
& + O(T).
\end{aligned} \tag{A.49}
$$

As a consequence,

$$\operatorname{Trace}[(\Omega_T A_T W_b A_T)^2] = \sum_{r_2,r_4}\left\{\tilde{k}_b\left(\frac{r_2}{T},\frac{r_4}{T}\right)\right\}^2\left(\sum_{h=1-T}^{T-1}\gamma(h_1)\right)^2+O(T) \tag{A.50}$$

and

$$
\begin{aligned}
\kappa_{2,T} &= 2T^{-2}(\omega_T^{-4})\operatorname{Trace}[(\Omega_T A_T W_b A_T)^2] \\
&= 2\int_0^1\int_0^1 (k_b^*(r,s))^2\,dr\,ds\,(1+o(1))+O\left(\frac{1}{T}\right).
\end{aligned} \tag{A.51}
$$

The proof for $\kappa_{m,T}$ for $m=3$ and 4 is essentially the same except that we use Lemma 1 to obtain the first term $O(b^{m-1})$. The details are omitted. *Q.E.D.*

A.2. *Proofs of the Main Results*

PROOF OF THEOREM 1: Using the independence between $W(1)$ and Ξ_b, we have

$$F_\delta(z) = P\{|(W(1)+\delta)\Xi_b^{-1/2}| < z\} = E\{G_\delta(z^2\Xi_b)\}. \tag{A.52}$$

Taking a fourth-order Taylor expansion of $G_\delta(z^2\Xi_b)$ around $\mu_b z^2$ yields

$$\begin{aligned} G_\delta(z^2\Xi_b) &= G_\delta(\mu_b z^2) + \frac{1}{2}(G''_\delta(\mu_b z^2)z^4)(\Xi_b - \mu_b)^2 \\ &\quad + \frac{1}{6}(G'''_\delta(\mu_b z^2)z^6)(\Xi_b - \mu_b)^3 \\ &\quad + \frac{1}{24}(G^{(4)}_\delta(\tilde{\mu}_b z^2)z^8)(\Xi_b - \mu_b)^4, \end{aligned} \tag{A.53}$$

where $\tilde{\mu}_b$ lies on the line segment between μ_b and Ξ_b. Taking expectation on both sides of the equation and using the fact that $|G^{(4)}_\delta(\tilde{\mu}_b z^2)z^8| \le C$ for some constant C, we have

$$\begin{aligned} EG_\delta(z^2\Xi_b) &= G_\delta(\mu_b z^2) + \frac{1}{2}G''_\delta(\mu_b z^2)E(\Xi_b - \mu_b)^2 z^4 \\ &\quad + \frac{1}{6}G'''_\delta(\mu_b z^2)\alpha_3 z^6 + O(|\alpha_4|) \end{aligned} \tag{A.54}$$

as $b \to 0$, where the $O(\cdot)$ term holds uniformly over $z \in \mathbb{R}^+$.

In view of Lemma 1, we have

$$|\alpha_4| = O(b^3), \quad |\alpha_3| \le |\alpha_4|^{3/4} = O(b^{9/4}) = o(b^2). \tag{A.55}$$

As a consequence,

$$\begin{aligned} F_\delta(z) &= P\{|(W(1)+\delta)\Xi_b^{-1/2}| < z\} \\ &= G_\delta(\mu_b z^2) + \frac{1}{2}(G''_\delta(\mu_b z^2)z^4)\alpha_2 + o(b^2) \end{aligned} \tag{A.56}$$

uniformly over $z \in z \in \mathbb{R}^+$, where

$$\mu_b = E\Xi_b = \int_0^1 k_b^*(r,r)\,dr = 1 - \int_0^1\int_0^1 k_b(r-s)\,dr\,ds \tag{A.57}$$

and

$$\alpha_2 = 2\left(\int_0^1\int_0^1 k_b(r-s)\,dr\,ds\right)^2 + 2\int_0^1\int_0^1 k_b^2(r-s)\,dr\,ds \tag{A.58}$$

$$-4\int_0^1\int_0^1\int_0^1 k_b(r-p)k_b(r-q)\,dr\,dp\,dq.$$

We proceed to develop an asymptotic expansion of μ_b and α_2 as $b\to 0$. Let

$$\mathcal{K}_1(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}k(x)\exp(-i\lambda x)\,dx,\tag{A.59}$$
$$\mathcal{K}_2(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}k^2(x)\exp(-i\lambda x)\,dx.$$

Then

$$k(x)=\int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\exp(i\lambda x)\,d\lambda,\quad k^2(x)=\int_{-\infty}^{\infty}\mathcal{K}_2(\lambda)\exp(i\lambda x)\,d\lambda.\tag{A.60}$$

For the integral that appears in both μ_b and α_2, we have

$$\begin{aligned}
&\int_0^1\int_0^1 k_b(r-s)\,dr\,ds\\
&=\int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\int_0^1\int_0^1\exp\left(\frac{i\lambda(r-s)}{b}\right)dr\,ds\,d\lambda\\
&=\int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\left(\int_0^1\exp\left(\frac{i\lambda r}{b}\right)dr\right)\left(\int_0^1\exp\left(-\frac{i\lambda s}{b}\right)ds\right)\\
&=\int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)\left(\frac{b^2}{\lambda^2}\left(\left(1-\cos\left(\frac{\lambda}{b}\right)\right)^2+\left(\sin\left(\frac{\lambda}{b}\right)\right)^2\right)\right)d\lambda\\
&=b\int_{-\infty}^{\infty}\mathcal{K}_1(\lambda)b\left(\frac{\sin\frac{\lambda}{2b}}{\frac{\lambda}{2}}\right)^2d\lambda\\
&=2\pi b\mathcal{K}_1(0)+4b^2\int_{-\infty}^{\infty}\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\left(\sin\frac{\lambda}{2b}\right)^2d\lambda,
\end{aligned}\tag{A.61}$$

where the last equality holds because

$$\int_{-\infty}^{\infty}\left(\frac{\lambda}{2b}\right)^{-2}\left(\sin\frac{\lambda}{2b}\right)^2d\lambda=2\int_{-\infty}^{\infty}x^{-2}\sin^2 x\,dx=2\pi.\tag{A.62}$$

Now,

$$\int_{-\infty}^{\infty}\frac{\mathcal{K}_1(\lambda)-\mathcal{K}_1(0)}{\lambda^2}\left(\sin\frac{\lambda}{2b}\right)^2d\lambda\tag{A.63}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\mathcal{K}_1(\lambda) - \mathcal{K}_1(0)}{\lambda^2} \left(\left(\sin \frac{\lambda}{2b} \right)^2 - \frac{1}{2} \right) d\lambda \\
&\quad + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mathcal{K}_1(\lambda) - \mathcal{K}_1(0)}{\lambda^2} \, d\lambda \\
&= -\frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\mathcal{K}_1(\lambda) - \mathcal{K}_1(0)}{\lambda^2} \right) \left(\cos \frac{1}{b} \lambda \right) d\lambda \\
&\quad + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mathcal{K}_1(\lambda) - \mathcal{K}_1(0)}{\lambda^2} \, d\lambda \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\mathcal{K}_1(\lambda) - \mathcal{K}_1(0)}{\lambda^2} \right) d\lambda + o(1)
\end{aligned}
$$

as $b \to 0$, where we have used the Riemann–Lebesgue lemma. In view of the symmetry of $k(x)$, $\mathcal{K}_1(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(x) \cos(\lambda x) \, dx$ and, therefore, (A.61) and (A.63) lead to

$$
\begin{aligned}
\text{(A.64)} \quad & \int_0^1 \int_0^1 k_b(r - s) \, dr \, ds \\
&= 2\pi b \mathcal{K}_1(0) + 2b^2 \int_{-\infty}^{\infty} \left(\frac{\mathcal{K}_1(\lambda) - \mathcal{K}_1(0)}{\lambda^2} \right) d\lambda \\
&= 2\pi b \mathcal{K}_1(0) + b^2 \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(x) \frac{\cos \lambda x - 1}{\lambda^2} \, dx \, d\lambda \\
&= 2\pi b \mathcal{K}_1(0) - b^2 \int_{-\infty}^{\infty} k(x) |x| \, dx. \\
&= bc_1 + b^2 c_3 + o(b^2).
\end{aligned}
$$

Similarly,

$$
\text{(A.65)} \quad \int_0^1 \int_0^1 k_b^2(r - s) \, dr \, ds = bc_2 + b^2 c_4 + o(b^2).
$$

Next,

$$
\begin{aligned}
\text{(A.66)} \quad & \int_0^1 k_b(r - s) \, ds \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \mathcal{K}(\lambda) \int_0^1 \left\{ \exp\left(\frac{i\lambda(r - s)}{b} \right) + \exp\left(-\frac{i\lambda(r - s)}{b} \right) \right\} ds \, d\lambda \\
&= \int_{-\infty}^{\infty} \mathcal{K}(\lambda) \int_0^1 \cos\left(\frac{\lambda(r - s)}{b} \right) ds \, d\lambda
\end{aligned}
$$

$$= b \int_{-\infty}^{\infty} \mathcal{K}(\lambda) \frac{1}{\lambda} \left(\left\{ \sin\left(\frac{\lambda(r-1)}{b} \right) - \sin\left(\frac{\lambda r}{b} \right) \right\} \right) d\lambda$$
$$= b \int_{-\infty}^{\infty} \mathcal{K}(xb) \frac{1}{x} \left(\left\{ \sin(x(r-1)) - \sin(xr) \right\} \right) dx,$$

so

$$\int_0^1 \int_0^1 \int_0^1 k_b(r-p) k_b(r-q)\, dr\, dp\, dq \tag{A.67}$$
$$= b^2 \int_0^1 \left[\int_{-\infty}^{\infty} \mathcal{K}(xb) \frac{1}{x} \left(\left\{ \sin(x(r-1)) - \sin(xr) \right\} \right) dx \right]^2 dr$$
$$= b^2 \mathcal{K}^2(0) \int_0^1 \left(\int_{-\infty}^{\infty} \frac{1}{x} \sin(x(r-1))\, dx - \int_{-\infty}^{\infty} \frac{1}{x} \sin(xr)\, dx \right)^2 dr$$
$$= b^2 \mathcal{K}^2(0) \int_0^1 \left(\int_{-\infty}^{\infty} \frac{\sin(x(r-1))}{x(r-1)}\, d(x(r-1)) \right.$$
$$\left. - \int_{-\infty}^{\infty} \frac{1}{xr} \sin(xr)\, dx\, r \right)^2 dr$$
$$= b^2 \mathcal{K}^2(0) \int_0^1 \left(2 \int_{-\infty}^{\infty} \frac{1}{y} \sin(y)\, dy \right)^2 dr = c_1^2 b^2.$$

Combining (A.64), (A.65), and (A.67) yields

$$\mu_b = 1 - bc_1 - b^2 c_3 + o(b^2) \tag{A.68}$$

and

$$\alpha_2 = 2bc_2 + b^2(c_4 - 2c_1^2) + o(b^2). \tag{A.69}$$

Now

$$F_\delta(z) = G_\delta(\mu_b z^2) + \frac{1}{2}(G''_\delta(\mu_b z^2) z^4)\alpha_2 + o(b^2) \tag{A.70}$$
$$= G_\delta(z^2) - G'_\delta(z^2) z^2 bc_1 - G'_\delta(z^2) z^2 b^2 c_3 + \frac{1}{2} G''_\delta(z^2) z^4 c_1^2 b^2$$
$$+ G''_\delta(z^2) z^4 bc_2 + \frac{1}{2} G''_\delta(z^2) z^4 b^2 (c_4 - 2c_1^2)$$
$$+ \frac{1}{2} G'''_\delta(z^2) z^6 (-bc_1)(2bc_2) + o(b^2)$$
$$= G_\delta(z^2) + [c_2 G''_\delta(z^2) z^4 - c_1 G'_\delta(z^2) z^2] b$$

$$-\left(G'_\delta(z^2)z^2c_3 - \frac{1}{2}G''_\delta(z^2)z^4(c_4 - c_1^2) + G'''_\delta(z^2)z^6c_1c_2\right)b^2$$
$$+ o(b^2)$$

uniformly over $z \in \mathbb{R}^+$, where the uniformity holds because $|G'_\delta(z^2)z^2| \le C$, $|G''_\delta(z^2)z^4| < C$, and $|G''_\delta(z^2)z^6| < C$ for some constant C. This completes the proof of the theorem. *Q.E.D.*

PROOF OF COROLLARY 2: Using a power series expansion, we have

$$\begin{aligned}
F_0(z_{\alpha,b}) &= D(z_{\alpha,b}^2) + [c_2D''(z_{\alpha,b}^2)z_{\alpha,b}^4 - c_1D'(z_{\alpha,b}^2)z_{\alpha,b}^2]b \\
&\quad - \left(D'(z_{\alpha,b}^2)z_{\alpha,b}^2c_3 - \frac{1}{2}D''(z_{\alpha,b}^2)z_{\alpha,b}^4(c_4 - c_1^2)\right. \\
&\qquad \left. + D'''(z_{\alpha,b}^2)z_{\alpha,b}^6c_1c_2\right)b^2 + o(b^2) \\
&= D(z_\alpha^2) + D'(z_\alpha^2)(z_{\alpha,b}^2 - z_\alpha^2) + \frac{1}{2}D''(z_\alpha^2)(z_{\alpha,b}^2 - z_\alpha^2)^2 \\
&\quad + [c_2D''(z_\alpha^2)z_\alpha^4 - c_1D'(z_\alpha^2)z_\alpha^2]b \\
&\quad + [c_2D'''(z_\alpha^2)z_\alpha^4 + 2c_2D''(z_\alpha^2)z_\alpha^2 - c_1D''(z_\alpha^2)z_\alpha^2 - c_1D'(z_\alpha^2)] \\
&\qquad \times (z_{\alpha,b}^2 - z_\alpha^2)b \\
&\quad - \left(D'(z_\alpha^2)z_\alpha^2c_3 - \frac{1}{2}D''(z_\alpha^2)z_\alpha^4(c_4 - c_1^2) + D'''(z_\alpha^2)z_\alpha^6c_1c_2\right)b^2 \\
&\quad + o(b^2),
\end{aligned} \tag{A.71}$$

that is,

$$\begin{aligned}
&D'(z_\alpha^2)(z_{\alpha,b}^2 - z_\alpha^2) + \frac{1}{2}D''(z_\alpha^2)(z_{\alpha,b}^2 - z_\alpha^2)^2 \\
&\quad + [c_2D''(z_\alpha^2)z_\alpha^4 - c_1D'(z_\alpha^2)z_\alpha^2]b \\
&\quad + [c_2D'''(z_\alpha^2)z_\alpha^4 + 2c_2D''(z_\alpha^2)z_\alpha^2 - c_1D''(z_\alpha^2)z_\alpha^2 - c_1D'(z_\alpha^2)] \\
&\qquad \times [(z_{\alpha,b}^2 - z_\alpha^2)]b \\
&\quad - \left(D'(z_\alpha^2)z_\alpha^2c_3 - \frac{1}{2}D''(z_\alpha^2)z_\alpha^4(c_4 - c_1^2) + D'''(z_\alpha^2)z_\alpha^6c_1c_2\right)b^2 + o(b^2) \\
&= 0.
\end{aligned} \tag{A.72}$$

Let

$$z_{\alpha,b}^2 = z_\alpha^2 + k_1b + k_2b^2 + o(b^2). \tag{A.73}$$

Then

$$
\begin{aligned}
\text{(A.74)}\quad & [c_2D''(z_\alpha^2)z_\alpha^4 - c_1D'(z_\alpha^2)z_\alpha^2]b + D'(z_\alpha^2)k_1b \\
& - \left(D'(z_\alpha^2)z_\alpha^2c_3 - \frac{1}{2}D''(z_\alpha^2)z_\alpha^4(c_4 - c_1^2) + D'''(z_\alpha^2)z_\alpha^6c_1c_2\right)b^2 \\
& + [c_2D'''(z_\alpha^2)z_\alpha^4 + 2c_2D''(z_\alpha^2)z_\alpha^2 - c_1D''(z_\alpha^2)z_\alpha^2 - c_1D'(z_\alpha^2)]k_1b^2 \\
& + \frac{1}{2}D''(z_\alpha^2)k_1^2b^2 + D'(z_\alpha^2)k_1b + o(b^2) = 0.
\end{aligned}
$$

This implies that

$$
\text{(A.75)}\quad k_1 = -\frac{1}{D'(z_\alpha^2)}[c_2D''(z_\alpha^2)z_\alpha^4 - c_1D'(z_\alpha^2)z_\alpha^2]
$$

and

$$
\begin{aligned}
\text{(A.76)}\quad k_2 = -\frac{1}{D'(z_\alpha^2)}\Bigg[& -\left(D'(z_\alpha^2)z_\alpha^2c_3 - \frac{1}{2}D''(z_\alpha^2)z_\alpha^4(c_4 - c_1^2) + D'''(z_\alpha^2)z_\alpha^6c_1c_2\right) \\
& + (c_2D'''(z_\alpha^2)z_\alpha^4 + 2c_2D''(z_\alpha^2)z_\alpha^2 - c_1D''(z_\alpha^2)z_\alpha^2 \\
& - c_1D'(z_\alpha^2))k_1 + \frac{1}{2}D''(z_\alpha^2)k_1^2\Bigg].
\end{aligned}
$$

Now

$$
\text{(A.77)}\quad D'(z) = \frac{z^{-1/2}e^{-z/2}}{\Gamma(1/2)\sqrt{2}}, \quad D''(z) = \frac{1}{4\sqrt{\pi}z^2}(-\sqrt{2z}e^{-z/2} - z^{3/2}\sqrt{2}e^{-z/2}),
$$

$$
\text{(A.78)}\quad D'''(z) = \frac{1}{8\sqrt{\pi}z^{7/2}}(3z\sqrt{2}e^{-z/2} + 2z^2\sqrt{2}e^{-z/2} + z^3\sqrt{2}e^{-z/2}),
$$

and thus

$$
\text{(A.79)}\quad \frac{D''(z^2)}{D'(z^2)} = \frac{1}{4z^3}(-2z - 2z^3), \quad \frac{D'''(z^2)}{D'(z^2)} = \frac{1}{4z^4}(2z^2 + z^4 + 3).
$$

Hence

$$
\text{(A.80)}\quad k_1 = \left(c_1 + \frac{1}{2}c_2\right)z_\alpha^2 + \frac{1}{2}c_2z_\alpha^4
$$

and

$$
\text{(A.81)}\quad k_2 = \left(\frac{1}{2}c_1^2 + \frac{3}{2}c_1c_2 + \frac{3}{16}c_2^2 + c_3 + \frac{1}{4}c_4\right)z_\alpha^2
$$

$$+\left(-\frac{1}{2}c_1+\frac{3}{2}c_1c_2+\frac{9}{16}c_2^2+\frac{1}{4}c_4\right)z_\alpha^4+\left(\frac{5}{16}c_2^2\right)z_\alpha^6-\left(\frac{1}{16}c_2^2\right)z_\alpha^8$$

as desired.

It follows from $z_{\alpha,b}^2 = z_\alpha^2 + k_1 b + k_2 b^2 + o(b^2)$ that

$$\text{(A.82)}\quad \begin{aligned} z_{\alpha,b} &= z_\alpha\left(1+\frac{1}{2}\frac{k_1b+k_2b^2}{z_\alpha^2}-\frac{1}{8}\frac{k_1^2b^2}{z_\alpha^4}\right)+o(b^2)\\ &= z_\alpha+\frac{1}{2}\frac{k_1}{z_\alpha}b+\left(\frac{1}{2}\frac{k_2}{z_\alpha}-\frac{1}{8}\frac{k_1^2}{z_\alpha^3}\right)b^2+o(b^2)\\ &= z_\alpha+k_3b+k_4b^2+o(b^2), \end{aligned}$$

where

$$\text{(A.83)}\quad k_3=\frac{1}{2}\left[\left(c_1+\frac{1}{2}c_2\right)z_\alpha+\frac{1}{2}c_2z_\alpha^3\right]$$

and

$$\text{(A.84)}\quad \begin{aligned} k_4 &= \left(\frac{1}{2}c_3+\frac{1}{8}c_4+\frac{5}{8}c_1c_2+\frac{1}{8}c_1^2+\frac{1}{16}c_2^2\right)z_\alpha\\ &\quad+\left(-\frac{1}{4}c_1+\frac{1}{8}c_4+\frac{5}{8}c_1c_2+\frac{7}{32}c_2^2\right)z_\alpha^3+\frac{1}{8}c_2^2z_\alpha^5-\frac{1}{32}c_2^2z_\alpha^7. \end{aligned}$$

Q.E.D.

PROOF OF COROLLARY 3: For notational convenience, let

$$\text{(A.85)}\quad p^{(1)}(z_\alpha^2)=\left(c_1+\frac{c_2}{2}\right)z_\alpha^2+\left(\frac{c_2}{2}\right)z_\alpha^4,$$

and then $z_{\alpha,b}^2 = z_\alpha^2 + p^{(1)}(z_\alpha^2)b + o(b)$. We have

$$\text{(A.86)}\quad \begin{aligned} &1-EG_\delta(z_{\alpha,b}^2\Xi_b)\\ &= 1-G_\delta(z_\alpha^2+p^{(1)}(z_\alpha^2)b)\\ &\quad-\left\{c_2G_\delta''(z_\alpha^2+p^{(1)}(z_\alpha^2)b)[z_\alpha^2+bp^{(1)}(z_\alpha^2)]^2\right\}b\\ &\quad+c_1\left\{G_\delta'(z_\alpha^2+p^{(1)}(z_\alpha^2)b)(z_\alpha^2+p^{(1)}(z_\alpha^2)b)\right\}b+o(b)\\ &= 1-G_\delta(z_\alpha^2)-G_\delta'(z_\alpha^2)p^{(1)}(z_\alpha^2)b-[c_2G_\delta''(z_\alpha^2)z_\alpha^4-c_1G_\delta'(z_\alpha^2)z_\alpha^2]b\\ &\quad+o(b)\\ &= 1-G_\delta(z_\alpha^2)-G_\delta'(z_\alpha^2)\left[\left(c_1+\frac{c_2}{2}\right)z_\alpha^2+\left(\frac{c_2}{2}\right)z_\alpha^4\right]b \end{aligned}$$

$$
\begin{aligned}
&\quad - [c_2 G_\delta''(z_\alpha^2) z_\alpha^4 - c_1 G_\delta'(z_\alpha^2) z_\alpha^2] b + o(b) \\
&= 1 - G_\delta(z_\alpha^2) - G_\delta'(z_\alpha^2)\left[\left(c_1 + \frac{c_2}{2}\right) z_\alpha^2 + \left(\frac{c_2}{2}\right) z_\alpha^4\right] b \\
&\quad - [c_2 G_\delta''(z_\alpha^2) z_\alpha^4 - c_1 G_\delta'(z_\alpha^2) z_\alpha^2] b + o(b) \\
&= 1 - G_\delta(z_\alpha^2) - \left(\frac{c_2}{2} G_\delta'(z_\alpha^2) z_\alpha^4 + c_2 G_\delta''(z_\alpha^2) z_\alpha^4 + \frac{c_2}{2} G_\delta'(z_\alpha^2) z_\alpha^2\right) b \\
&\quad + o(b) \\
&= 1 - G_\delta(z_\alpha^2) - c_2\left(\frac{1}{2} G_\delta'(z_\alpha^2) z_\alpha^4 + G_\delta''(z_\alpha^2) z_\alpha^4 + \frac{1}{2} G_\delta'(z_\alpha^2) z_\alpha^2\right) b + o(b).
\end{aligned}
$$

Note that

$$
G_\delta'(z) = \sum_{j=0}^{\infty} \frac{(\delta^2/2)^j}{j!} e^{-\delta^2/2} \frac{z^{j-1/2} e^{-z/2}}{\Gamma(j+1/2) 2^{j+1/2}} \tag{A.87}
$$

and

$$
\begin{aligned}
G_\delta''(z) &= \sum_{j=0}^{\infty} \frac{(\delta^2/2)^j}{j!} e^{-\delta^2/2} \left(\left(j - \frac{1}{2}\right) \frac{1}{z} \frac{z^{j-1/2} e^{-z/2}}{\Gamma(j+1/2) 2^{j+1/2}} \right. \\
&\qquad \left. - \frac{1}{2} \frac{z^{j-1/2} e^{-z/2}}{\Gamma(j+1/2) 2^{j+1/2}}\right) \\
&= \left(-\frac{1}{2z} - \frac{1}{2}\right) G_\delta'(z) + \sum_{j=0}^{\infty} \frac{(\delta^2/2)^j}{j!} e^{-\delta^2/2} \frac{z^{j-1/2} e^{-z/2}}{\Gamma(j+1/2) 2^{j+1/2}} \frac{j}{z} \\
&= -\frac{1}{2} G_\delta'(z) \left(\frac{1}{z} + 1\right) + K_\delta(z),
\end{aligned} \tag{A.88}
$$

so that

$$
\frac{1}{2} G_\delta'(z_\alpha^2) z_\alpha^4 + G_\delta''(z_\alpha^2) z_\alpha^4 + \frac{1}{2} G_\delta'(z_\alpha^2) z_\alpha^2 = z_\alpha^4 K_\delta(z_\alpha^2) \tag{A.89}
$$

and

$$
1 - E G_\delta(z_{\alpha,b}^2 \Xi_b) = 1 - G_\delta(z_\alpha^2) - c_2 z_\alpha^4 K_\delta(z_\alpha^2) b + o(b), \tag{A.90}
$$

completing the proof of the corollary. *Q.E.D.*

PROOF OF THEOREM 4: It follows from Lemma 3 that when $b \to 0$,

$$
\alpha_{2,T} = \kappa_{2,T} = 2bc_2(1 + o(1)) + O(T^{-1}), \tag{A.91}
$$

$$\alpha_{3,T} = \kappa_{3,T} = O(b^2) + O(T^{-1}),$$
$$\alpha_{4,T} = \kappa_{4,T} + 3\kappa_{2,T}^2 = O(b^2) + O(T^{-1}),$$

and

$$\mu_{bT} = \mu_b - (bT)^{-q} g_q \left(\omega_T^{-2} \sum_{h=-\infty}^{\infty} |h|^q \gamma(h) \right)(1 + o(1)) + O(T^{-1}). \tag{A.92}$$

Thus, as $b \to 0$,

$$\begin{aligned} F_{T,\delta}(z) &= P\{|\sqrt{T}(\hat{\beta} - \beta_0)/\hat{\omega}_b| \le z\} \\ &= E\{G_\delta(z^2 \varsigma_{bT})\} + O(T^{-1}) \\ &= G_\delta(\mu_{bT} z^2) + \frac{1}{2} G_\delta''(\mu_{bT} z^2) z^4 \alpha_{2,T} + o(b) \\ &= G_\delta(\mu_{bT} z^2) + \frac{1}{2} G_\delta''(\mu_{bT} z^2) z^4 (2bc_2) + o(b) + O(T^{-1}) \\ &= G_\delta(\mu_b z^2) + G_\delta'(\mu_b z^2) z^2 (\mu_{bT} - \mu_b) + bc_2 G_\delta''(\mu_b z^2) z^4 \\ &\quad + o(b) + O(T^{-1}) \end{aligned} \tag{A.93}$$

uniformly over $z \in \mathbb{R}^+$, using (A.91) and (A.92), but

$$\begin{aligned} G_\delta(\mu_b z^2) &= G_\delta(z^2) + G_\delta'(z^2) z^2 (\mu_b - 1) + o(b) \\ &= G_\delta(z^2) - bc_1 G_\delta'(z^2) z^2 + o(b) \end{aligned} \tag{A.94}$$

uniformly over $z \in \mathbb{R}^+$ and

$$\begin{aligned} &G_\delta'(\mu_b z^2) z^2 (\mu_{bT} - \mu_b) \\ &\quad = (G_\delta'(z^2) + O(b)) z^2 \\ &\qquad \times \left\{ -g_q \left(\omega_T^{-2} \sum_{h=-\infty}^{\infty} |h|^q \gamma(h) \right) (bT)^{-q} (1 + o(1)) + O(T^{-1}) \right\} \\ &\quad = -g_q \left(\omega_T^{-2} \sum_{h=-\infty}^{\infty} |h|^q \gamma(h) \right) G_\delta'(z^2) z^2 (bT)^{-q} (1 + o(1)) \\ &\qquad + o(b) + O(T^{-1}) \end{aligned} \tag{A.95}$$

uniformly over $z \in \mathbb{R}^+$. So

$$\begin{aligned} F_{T,\delta}(z) &= G_\delta(z^2) + (c_2 G_\delta''(\mu_b z^2) z^4 - c_1 G_\delta'(z^2) z^2) b \\ &\quad - g_q d_{qT} G_\delta'(z^2) z^2 (bT)^{-q} + o(b + (bT)^{-q}) + O(T^{-1}) \end{aligned} \tag{A.96}$$

uniformly over $z \in \mathbb{R}^+$, as desired. *Q.E.D.*

PROOF OF COROLLARY 5: (a) Using Theorem 4, we have, as $b + 1/T + 1/(bT) \to 0$,

$$
\begin{aligned}
\text{(A.97)}\quad & F_{T,0}(z_{\alpha,b}) \\
&= D(z_{\alpha,b}^2) + [c_2 D''(z_{\alpha,b}^2) z_{\alpha,b}^4 - c_1 D'(z_{\alpha,b}^2) z_{\alpha,b}^2] b \\
&\quad - g_q d_{qT} D'(z_{\alpha,b}^2) z_{\alpha,b}^2 (bT)^{-q} + O(T^{-1}) + o(b + (bT)^{-q}) \\
&= F_0(z_{\alpha,b}) - g_q d_{qT} D'(z_{\alpha,b}^2) z_{\alpha,b}^2 (bT)^{-q} + O(T^{-1}) + o(b + (bT)^{-q}) \\
&= 1 - \alpha - g_q d_{qT} D'(z_\alpha^2) z_\alpha^2 (bT)^{-q} + O(T^{-1}) + o(b + (bT)^{-q}),
\end{aligned}
$$

so

$$
\text{(A.98)}\quad 1 - F_{T,0}(z_{\alpha,b}) - \alpha = g_q d_{qT} D'(z_\alpha^2) z_\alpha^2 (bT)^{-q} + O(T^{-1}) + o(b + (bT)^{-q}).
$$

(b) Plugging $z_{\alpha,b}^2$ into

$$
\begin{aligned}
\text{(A.99)}\quad F_{T,\delta}(z) &= G_\delta(z^2) + [c_2 G_\delta''(\mu_b z^2) z^4 - c_1 G_\delta'(z^2) z^2] b \\
&\quad - g_q d_{qT} G_\delta'(z^2) z^2 (bT)^{-q} + o\{b + (bT)^{-q}\} + O(T^{-1})
\end{aligned}
$$

yields

$$
\begin{aligned}
\text{(A.100)}\quad & P\left(\left| \frac{\sqrt{T}(\hat{\beta} - \beta_0)}{\hat{\omega}_b} \right|^2 \geq z_{\alpha,b}^2 \right) \\
&= 1 - G_\delta(z_{\alpha,b}^2) - [c_2 G_\delta''(z_{\alpha,b}^2) z_{\alpha,b}^4 - c_1 G_\delta'(z_{\alpha,b}^2) z_{\alpha,b}^2] b \\
&\quad + g_q d_{qT} G_\delta'(z_{\alpha,b}^2) z_{\alpha,b}^2 (bT)^{-q} + O(T^{-1}) + o(b + (bT)^{-q}) \\
&= 1 - G_\delta(z_\alpha^2) - c_2 z_\alpha^4 K_\delta(z_\alpha^2) b \\
&\quad + g_q d_{qT} G_\delta'(z_\alpha^2) z_\alpha^2 (bT)^{-q} + O(T^{-1}) + o(b + (bT)^{-q}),
\end{aligned}
$$

where the last equality follows as in the proof of Corollary 3. *Q.E.D.*

PROOF OF THEOREM 6: First, since $D(\cdot)$ is a bounded function, we have

$$
\begin{aligned}
\text{(A.101)}\quad & P\{ |W(1) \Xi_b^{-1/2}| \leq z \} \\
&= \lim_{B \to \infty} E D(z^2 \Xi_b) 1\{ |\Xi_b - \mu_b| \leq B \} \\
&= \lim_{B \to \infty} E \sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2) (\Xi_b - \mu_b)^m z^{2m} 1\{ |\Xi_b - \mu_b| \leq B \}
\end{aligned}
$$

$$= \lim_{B\to\infty} \sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\alpha_m z^{2m} 1\{|\Xi_b - \mu_b| \le B\},$$

where the last line follows because the infinite sum $\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\alpha_m z^{2m}$ converges uniformly to $D(z^2\Xi_b)$ when $|\Xi_b - \mu_b| \le B$. Uniformity holds because $D(\cdot)$ is infinitely differentiable with bounded derivatives.

Since $D(z^2)$ decays exponentially as $z^2 \to \infty$, there exists a constant C such that $|D^{(m)}(\mu_b z^2) z^{2m}| < C$ for all m. Using this and Lemma 1, we have

$$\text{(A.102)}\quad \left|\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\alpha_m z^{2m}\right| \le C \sum_{m=1}^{\infty} \frac{1}{m!} |\alpha_m| \le C \sum_{m=1}^{\infty} \frac{1}{m!} 2^{2m} m! (c_1^* b)^{m-1}$$
$$= C(c_1^* b)^{-1} \sum_{m=1}^{\infty} (4c_1^* b)^m < \infty,$$

provided that $b < 1/(4c_1^*)$. As a consequence, the operation $\lim_{B\to\infty}$ can be moved inside the summation sign in (A.101), giving

$$\text{(A.103)}\quad P\big\{|W(1)\Xi_b^{-1/2}| \le z\big\} = \sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_b z^2)\alpha_m z^{2m},$$

when $b < 1/(4c_1^*)$.

Second, it follows from

$$\text{(A.104)}\quad F_{T,\delta}(z) = P\big\{|\sqrt{T}(\hat\beta - \beta_0)/\hat\omega_b| \le z\big\} = E\{G_\delta(z^2 \varsigma_{bT})\} + O(T^{-1}),$$

that, when $c = 0$, we have

$$\text{(A.105)}\quad P\big\{|\sqrt{T}(\hat\beta - \beta_0)/\hat\omega_b| \le z\big\} = E\{D(z^2 \varsigma_{bT})\} + O(T^{-1}).$$

But

$$\text{(A.106)}\quad E\{D(z^2\varsigma_{bT})\} = \sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_{bT} z^2)\alpha_{m,T} z^{2m},$$

where the right hand side converges to $E\{D(z^2\varsigma_{bT})\}$ uniformly over T because

$$\alpha_{m,T} = \alpha_m + O\left\{\frac{2^{2m} m!}{T^2}(c_1^* b)^{m-2}\right\}$$

uniformly over m by Lemma 2, $|D^{(m)}(\mu_{bT} z^2) z^{2m}| < C$ for some constant C, and thus

$$\left|\sum_{m=1}^{\infty} \frac{1}{m!} D^{(m)}(\mu_{bT} z^2)\alpha_{m,T} z^{2m}\right| \le C \sum_{m=1}^{\infty} \frac{1}{m!} |\alpha_m| + \frac{C}{T^2} \sum_{m=1}^{\infty} 2^{2m} (c_1^* b)^{m-2}$$
$$< \infty$$

when $b < 1/(4c_1^*)$. Therefore,

$$P\left\{\left|\sqrt{T}\frac{\hat{\beta}-\beta_0}{\hat{\omega}_b}\right| \leq z\right\} = \sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_{bT}z^2)\alpha_{m,T}z^{2m} + O\left(\frac{1}{T}\right) \tag{A.107}$$

uniformly over $z \in \mathbb{R}^+$.

It follows from (A.103) and (A.107) that

$$\begin{aligned}
&|F_{T,0}(z) - F_0(z)| \\
&\quad = \left|P\left\{\left|\sqrt{T}\frac{\hat{\beta}-\beta}{\hat{\omega}_b}\right| \leq z\right\} - P\left\{|W(1)\Xi_b^{-1/2}| < z\right\}\right| \\
&\quad = \left|\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_{bT}z^2)\alpha_{m,T}z^{2m} - \sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\alpha_m z^{2m}\right| + O\left(\frac{1}{T}\right) \\
&\quad = \left|\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\alpha_{m,T}z^{2m} - \sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)\alpha_m z^{2m}\right| + O\left(\frac{1}{T}\right)
\end{aligned} \tag{A.108}$$

uniformly over $z \in \mathbb{R}$, where the second equality holds because

$$D^{(m)}(\mu_{bT}z^2) = D^{(m)}(\mu_b z^2) + O(D^{(m+1)}(\mu_b z^2)z^2/T)$$

uniformly over $z \in \mathbb{R}$ and

$$\left|\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m+1)}(\mu_b z^2)\alpha_{m,T}z^{2m+2}\right| < \infty.$$

Therefore,

$$\begin{aligned}
|F_{T,0}(z) - F_0(z)| &= \left|\sum_{m=1}^{\infty}\frac{1}{m!}D^{(m)}(\mu_b z^2)(\alpha_{m,T} - \alpha_m)z^{2m}\right| + O\left(\frac{1}{T}\right) \\
&= O\left\{\frac{1}{T^2}\sum_{m=1}^{\infty}2^{2m}(c_1^* b)^{m-2}\right\} + O\left(\frac{1}{T}\right) \\
&= O\left(\frac{1}{T}\right)
\end{aligned} \tag{A.109}$$

uniformly over $z \in \mathbb{R}^+$ as desired. *Q.E.D.*

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