Econometrica Supplementary Material

SUPPLEMENT TO "ROBUSTNESS AND SEPARATION IN MULTIDIMENSIONAL SCREENING"

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This supplement contains additional materials for the main paper. Section B contains proofs of auxiliary results not included in the main paper. Section C details how the generalized virtual values coincide with traditional (ironed) virtual values in the single-good monopoly problem. Theorems, equations, and sections in the main paper are referenced using the original numbering.

APPENDIX B: ADDITIONAL PROOFS

PROOF OF LEMMA 4.3 (ADAPTED FROM MADARÁSZ AND PRAT (2012)): It is easy to see that statement (b) of the lemma follows from (a) by integrating over each S_k in the partition; so it suffices to prove (a).

As in the proof of Lemma 4.2, we can take $\Delta = \max_{x,\theta} u(x, \theta) - \min_{x,\theta} u(x, \theta)$, and then in any mechanism, any two types' payments can differ by at most Δ. Also, put $\tau = \min\{\varepsilon/6\Delta, 1\}$.

By Lipschitz continuity, there exists δ such that whenever θ, θ' are two types with $d(\theta, \theta') < \delta$, then $|u(x, \theta) - u(x, \theta')| < \tau\varepsilon/6$ for all x. We show this δ has the desired property.

Let (x, t) be any given mechanism. Let $\underline{t} = \min_\theta t(\theta)$. Let $S \subseteq \Delta(X) \times \mathbb{R}$ be the set of values $(x(\theta), \tau\underline{t} + (1-\tau)t(\theta))$ for $\theta \in \Theta$ and let $\overline{S}$ be its closure, which is compact (by the above observation on payments). Then define $(\widetilde{x}, \widetilde{t})$ by simply assigning to each type $\theta \in \Theta$ the outcome in $\overline{S}$ that maximizes its payoff, $Eu(x, \theta) - t$. This exists by compactness. This $(\widetilde{x}, \widetilde{t})$ is a mechanism: IC is satisfied by definition and IR is satisfied since the payments have only been reduced relative to those in (x, t), so each type θ has the option of getting allocation $x(\theta)$ for a payment of less than $t(\theta)$, which gives nonnegative payoff.

Now let $d(\theta, \theta') < \delta$. We know that the outcome chosen by θ' in the new mechanism can be approximated arbitrarily closely by an element of S corresponding to some type θ''; in particular, there exists θ'' such that

$$\left|Eu\big(\widetilde{x}(\theta'), \theta\big) - Eu\big(x(\theta''), \theta\big)\right| < \frac{\tau\varepsilon}{6} \quad \text{and} \quad \left|\widetilde{t}(\theta') - \big[\tau\underline{t} + (1-\tau)t(\theta'')\big]\right| < \frac{\tau\varepsilon}{6}. \tag{B.1}$$

Now, we know from IC for the original mechanism that

$$Eu\big(x(\theta), \theta\big) - t(\theta) \geq Eu\big(x(\theta''), \theta\big) - t(\theta'') \tag{B.2}$$

and by the definition of the new mechanism $(\widetilde{x}, \widetilde{t})$ that

$$Eu\big(\widetilde{x}(\theta'), \theta'\big) - \widetilde{t}(\theta') \geq Eu\big(x(\theta), \theta'\big) - \big[\tau\underline{t} + (1-\tau)t(\theta)\big].$$

Using (twice) the fact that $d(\theta, \theta') < \delta$, the latter inequality turns into

$$Eu\big(\widetilde{x}(\theta'), \theta\big) - \widetilde{t}(\theta') \geq Eu\big(x(\theta), \theta\big) - \big[\tau\underline{t} + (1-\tau)t(\theta)\big] - \frac{\tau\varepsilon}{3}.$$

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Now combining with (B.1) we get

$$Eu\big(x\big(\theta''\big),\theta\big)-\big[\tau\underline{t}+(1-\tau)t\big(\theta''\big)\big]>Eu\big(x(\theta),\theta\big)-\big[\tau\underline{t}+(1-\tau)t(\theta)\big]-\frac{2\tau\varepsilon}{3}. \quad \text{(B.3)}$$

Adding (B.2) and (B.3), and canceling common terms, we get

$$\tau t\big(\theta''\big)>\tau t(\theta)-\frac{2\tau\varepsilon}{3}$$

or

$$t\big(\theta''\big)>t(\theta)-\frac{2\varepsilon}{3}.$$

Hence, from (B.1),

$$\begin{aligned}\tilde{t}\big(\theta'\big)&>t\big(\theta''\big)-\tau\big(t\big(\theta''\big)-\underline{t}\big)-\frac{\tau\varepsilon}{6}\\&>\left(t(\theta)-\frac{2\varepsilon}{3}\right)-\tau\Delta-\frac{\tau\varepsilon}{6}\\&\geq t(\theta)-\frac{2\varepsilon}{3}-\frac{\varepsilon}{6}-\frac{\varepsilon}{6},\end{aligned}$$

which is the desired statement (a). *Q.E.D.*

PROOF OF PROPOSITION 5.1: Write $\underline{\theta}^g=\min(\Theta^g)$ and $\overline{\theta}^g=\max(\Theta^g)$. Consider first the pricing problem in each component separately. The profit from setting price p^g is $p^g(1-F^g(p^g))$, where F^g is the cumulative distribution function for θ^g. Since the optimal price p^{*g} is in the interior of Θ^g, the first-order condition must hold there:

$$1-F^g\big(p^{*g}\big)-p^{*g}f^g\big(p^{*g}\big)=0.$$

Now, for sufficiently small $\varepsilon>0$, we must have

$$\begin{aligned}&-\varepsilon\big(1-F^1\big(p^{*1}\big)\big)\big(1-F^2\big(p^{*2}\big)\big)\\&\quad+\big(p^{*1}-\varepsilon\big)\big(F^1\big(p^{*1}\big)-F^1\big(p^{*1}-\varepsilon\big)\big)\big(1-F^2\big(p^{*2}\big)\big)\\&\quad+\big(p^{*2}-\varepsilon\big)\big(F^2\big(p^{*2}\big)-F^2\big(p^{*2}-\varepsilon\big)\big)\big(1-F^1\big(p^{*1}\big)\big)>0.\end{aligned} \quad \text{(B.4)}$$

Indeed, when $\varepsilon=0$, this expression equals 0, and its derivative with respect to ε is

$$\begin{aligned}&-\big(1-F^1\big(p^{*1}\big)\big)\big(1-F^2\big(p^{*2}\big)\big)+p^{*1}f^1\big(p^{*1}\big)\big(1-F^2\big(p^{*2}\big)\big)+p^{*2}f^2\big(p^{*2}\big)\big(1-F^*\big(p^{*1}\big)\big)\\&\quad=\big(1-F^1\big(p^{*1}\big)\big)\big(1-F^2\big(p^{*2}\big)\big)\\&\qquad-\big(1-F^2\big(p^{*2}\big)\big)\big[1-F^1\big(p^{*1}\big)-p^{*1}f^1\big(p^{*1}\big)\big]\\&\qquad-\big(1-F^1\big(p^{*1}\big)\big)\big[1-F^2\big(p^{*2}\big)-p^{*2}f^2\big(p^{*2}\big)\big]\\&\quad=\big(1-F^1\big(p^{*1}\big)\big)\big(1-F^2\big(p^{*2}\big)\big)\\&\quad>0.\end{aligned}$$

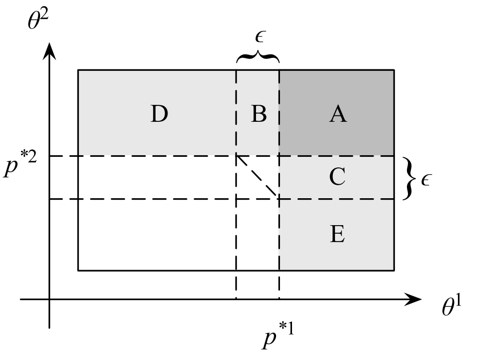

FIGURE B.1.—Buyer behavior under separate sales and bundling.

(Here the second equality comes from the first-order condition for each p^{*g}.) Write Δ for the left-hand side of (B.4) (which depends on ε). Let $\underline{\varepsilon}$ be a small value for which (B.4) holds.

Consider the behavior of various buyer types under the separate-price mechanism, illustrated in Figure B.1. Buyer types in region A buy both goods; those in regions B and D buy only good 2, while those in regions C and E buy only good 1.

Now consider the change in expected profit when the mechanism is changed to offering either separate prices (p^{*1}, p^{*2}) or $p^{*1} + p^{*2} - \varepsilon$ for the bundle. Buyers whose value for both goods g is above p^{*g} (region A in the figure) now buy the bundle, paying ε less than before. Buyers with $\theta^1 \in [p^{*1} - \varepsilon, p^{*1})$ and $\theta^2 \geq p^{*2}$ (region B) formerly bought only good 2 but now buy the bundle, paying $p^{*1} - \varepsilon$ more than before. And buyers with $\theta^2 \in [p^{*2} - \varepsilon, p^{*2})$ and $\theta^1 \geq p^{*1}$ (region C) switch to buying the bundle, paying $p^{*2} - \varepsilon$ more than before. These changes constitute a lower bound on the net change in profit. (In addition, some types who formerly bought nothing now buy the bundle; we ignore them.)

Thus, writing $\pi(A)$, $\pi(B)$, and $\pi(C)$ for the measures of these regions under joint distribution π, our change in profit is at least

$$-\varepsilon\pi(A) + \left(p^{*1} - \varepsilon\right)\pi(B) + \left(p^{*2} - \varepsilon\right)\pi(C). \tag{B.5}$$

Now, for any negatively affiliated π, we have

$$\begin{aligned}
\pi(B) &\geq \frac{\pi\left(\left[p^{*1} - \varepsilon, p^{*1}\right] \times \Theta^2\right)}{\pi\left(\left[p^{*1}, \overline{\theta}^2\right] \times \Theta^2\right)} \times \pi(A) \\
&= \frac{F^1\left(p^{*1}\right) - F^1\left(p^{*1} - \varepsilon\right)}{1 - F^1\left(p^{*1}\right)} \times \pi(A)
\end{aligned}$$

and similarly

$$\pi(C) \geq \frac{F^2\left(p^{*2}\right) - F^2\left(p^{*2} - \varepsilon\right)}{1 - F^2\left(p^{*2}\right)} \times \pi(A).$$

Plugging in to (B.5), our change in profit in going to the bundled mechanism is at least

$$\pi(A) \times \left[-\varepsilon + (p^{*1} - \varepsilon)\frac{F^1(p^{*1}) - F^1(p^{*1} - \varepsilon)}{1 - F^1(p^{*1})} + (p^{*2} - \varepsilon)\frac{F^2(p^{*2}) - F^2(p^{*2} - \varepsilon)}{1 - F^2(p^{*2})}\right]$$

$$= \pi(A) \times \frac{\Delta}{(1 - F^1(p^{*1}))(1 - F^2(p^{*2}))},$$

with Δ given by the left-hand side of (B.4).

Take $\varepsilon = \underline{\varepsilon}$ in the bundled mechanism. Recall that in this case, the corresponding value of Δ was positive; call this value $\underline{\Delta}$. This shows that the bundling mechanism, with price $p^{*1} + p^{*2} - \underline{\varepsilon}$ for the bundle, earns expected profit at least as high as the R^* from separate sales, proving part (a).

All of this basically follows McAfee, McMillan, and Whinston (1989) (who considered the independent case). In our case, we obtain only a weak improvement from this bundling, because with negative affiliation, $\pi(A)$ may be zero, or arbitrarily close. This is why we must randomize the bundle price to obtain a strict improvement; we now detail this adjustment.

Take $\overline{\varepsilon} = \min\{p^{*1} - \underline{\theta}^1, p^{*2} - \underline{\theta}^2\}$; without loss of generality, $\overline{\varepsilon} = p^{*2} - \underline{\theta}^2$. Then, in the mechanism with bundle price $p^{*1} + p^{*2} - \overline{\varepsilon}$, region E in Figure B.1 disappears, and regions A and C constitute all of the area to the right of the line $\theta^1 = p^{*1}$, implying

$$\pi(C) = (1 - F^1(p^{*1})) - \pi(A).$$

Therefore, expression (B.5) is at least

$$-\overline{\varepsilon}\pi(A) + (p^{*2} - \overline{\varepsilon})\left[(1 - F^1(p^{*1})) - \pi(A)\right]$$
$$= \underline{\theta}^2(1 - F^1(p^{*1})) - p^{*2}\pi(A).$$

Consequently, if ε is chosen to equal $\underline{\varepsilon}$ with probability $\underline{q}$ and $\overline{\varepsilon}$ with probability $1 - \underline{q}$, the expected gain in profit relative to separate prices is at least

$$\underline{q}\frac{\underline{\Delta}}{(1 - F^1(p^{*1}))(1 - F^2(p^{*2}))}\pi(A) + (1 - \underline{q})\left(\underline{\theta}^2(1 - F^1(p^{*1})) - p^{*2}\pi(A)\right)$$
$$= (1 - \underline{q})\underline{\theta}^2(1 - F^1(p^{*1})) + \pi(A)\left[\underline{q}\frac{\underline{\Delta}}{(1 - F^1(p^{*1}))(1 - F^2(p^{*2}))} - (1 - \underline{q})p^{*2}\right].$$

Evidently, if $\underline{q}$ is chosen close enough to 1, the expression in brackets on the right will be positive. Then, for any negatively affiliated distribution π, the profit from the randomized bundling mechanism will be at least $R^* + (1 - \underline{q})\underline{\theta}^2(1 - F^1(p^{*1}))$, which is strictly above R^*, proving part (b). *Q.E.D.*

PROOF OF LEMMA 5.2: We first prove the lemma in the special case where each Θ^g consists of at most two values. Write them as $\Theta^g = \{\theta_1^g, \theta_2^g\}$ with $\theta_1^g < \theta_2^g$ (or $\{\theta_1^g\}$ if there is just one value), and write (x^*, t^*) for the optimal mechanism that sells each good separately.

In the standard analysis of the one-good problem (as detailed in Appendix C below), the virtual values associated to these two types are

$$\theta_1^g - \frac{\pi^g(\theta_2^g)}{\pi^g(\theta_1^g)} \times (\theta_2^g - \theta_1^g) \quad \text{and} \quad \theta_2^g.$$

If the former virtual value is negative, then the optimal price for the single good g is θ_2^g; if it is positive, the optimal price is θ_1^g. The virtual value cannot be zero, because there would then not be a unique optimal price, contrary to assumption. In either case, the constraints that receive positive weight in the dual problem are the IR constraint for θ_1^g and the IC constraint for θ_2^g to imitate θ_1^g. (If there is just one type θ_1^g, then its virtual value is the positive number θ_1^g and its IR constraint is binding.)

The distribution π constructed in Lemma 4.4 has full support on Θ. This also implies that all of the multipliers on the adjacent downward constraints constructed in the proof of Theorem 2.1, that is to say all of the $\lambda[(\theta_2^g, \theta^{-g}) \to (\theta_1^g, \theta^{-g})]$, are strictly positive, in view of their definition (4.17). Likewise, the lowest type $\theta_1 = (\theta_1^1, \dots, \theta_1^G)$ has a strictly positive weight on its IR constraint (namely $\kappa[\theta_1] = 1$).

Now, we saw in the proof of Theorem 2.1 that any mechanism (x, t) must satisfy

$$\sum_\theta \pi(\theta)t(\theta) \leq \sum_\theta \sum_g \pi(\theta)E\overline{u}^g(x^g(\theta), \theta^g) \leq \sum_\theta \sum_g \pi(\theta)\overline{u}^g_{\max}(\theta^g) = R^* \tag{B.6}$$

with equality for the mechanism (x^*, t^*). Since the $\pi(\theta)$ are all strictly positive, a mechanism can be optimal only if it satisfies $E\overline{u}^g(x^g(\theta), \theta^g) = \overline{u}^g_{\max}(\theta^g)$ for every θ and every g, that is, it maximizes the virtual value in every component for every type. Since the virtual value from allocating good g is always positive or negative, never 0 (while the virtual value from not allocating is 0), the virtual value maximizer is unique. Thus we must have $x(\theta) = x^*(\theta)$ for every θ. Moreover, we can see from the proof of Lemma 4.1 that the first inequality in (B.6) can be an equality only if all the IC and IR constraints that have positive multipliers hold with equality. Given that x and x^* coincide, this equality uniquely determines the payment of the lowest type $t(\theta_1)$ by its IR, and then uniquely determines the payment of each other type by upward induction using the ICs. Hence we have $t(\theta) = t^*(\theta)$ for every θ as well.

This shows that the only possible optimal mechanism is $(x, t) = (x^*, t^*)$, proving the lemma in the case where each Θ^g has at most two elements.

Now we can prove the lemma in general. For each good g, write $\Theta^g = \{\theta_1^g, \dots, \theta_{J^g}^g\}$, with the values listed in increasing order, $\theta_1^g < \cdots < \theta_{J^g}^g$. By assumption, the optimal price to sell good g is unique, and clearly it must equal one of the values θ_j^g; write j^{*g} for the index, so that the optimal price is $\theta_{j^{*g}}^g$. Write (x^*, t^*) for the mechanism that sells each good g separately at price $\theta_{j^{*g}}^g$.

Let $\mathcal{S}^g$ be the collection of all subsets of Θ^g that contain $\theta_{j^{*g}}^g$. For each such subset $\widetilde{\Theta}^g \in \mathcal{S}^g$, let $\pi^g[\widetilde{\Theta}^g]$ be some distribution in the corresponding one-good problem whose support is $\widetilde{\Theta}^g$ and for which the unique optimal mechanism is a posted price of $\theta_{j^{*g}}^g$. (This can be constructed, for example, by placing large enough probability mass on $\theta_{j^{*g}}^g$.) Now, by choosing a sufficiently small positive weight $\eta^g[\widetilde{\Theta}^g]$ for each $\widetilde{\Theta}^g \in \mathcal{S}^g$, we can write π^g as a convex combination of distributions

$$\pi^g = \sum_{\widetilde{\Theta}^g \in \mathcal{S}^g} \eta^g[\widetilde{\Theta}^g]\pi^g[\widetilde{\Theta}^g] + \eta^g[\emptyset]\overline{\pi}^g,$$

where $\overline{\pi}^g$ is some distribution that still has full support on Θ^g and still has the property that the unique optimal price is $\theta^g_{j^*g}$. For convenience, write $\overline{\mathcal{S}}^g = \mathcal{S}^g \cup \{\emptyset\}$ and $\pi^g[\emptyset] = \overline{\pi}^g$; this allows us to write more simply

$$\pi^g = \sum_{\widetilde{\Theta}^g \in \overline{\mathcal{S}}^g} \eta^g\big[\widetilde{\Theta}^g\big]\pi^g\big[\widetilde{\Theta}^g\big]. \tag{B.7}$$

Let $\overline{\mathcal{S}} = \times_g \overline{\mathcal{S}}^g$. Consider any choice of $\widetilde{\Theta} = (\widetilde{\Theta}^1, \dots, \widetilde{\Theta}^G) \in \overline{\mathcal{S}}$. We know that for each separate good g, setting a price of $\theta^g_{j^*g}$ for each item is optimal against each marginal distribution $\pi^g[\widetilde{\Theta}^g]$. Accordingly, let $\pi[\widetilde{\Theta}]$ be the joint distribution constructed in Section 4.2, so that its marginals are the distributions $\pi^g[\widetilde{\Theta}^g]$ and such that (x^*, t^*) is an optimal mechanism for distribution $\pi[\widetilde{\Theta}]$.

Then we can define a joint distribution π on Θ by

$$\pi = \sum_{\widetilde{\Theta} \in \mathcal{S}} \left(\prod_{g=1}^{G} \eta^g\big[\widetilde{\Theta}^g\big] \right) \pi[\widetilde{\Theta}].$$

It is straightforward to check, using (B.7), that π is a distribution on Θ whose marginal on each Θ^g equals π^g, and (x^*, t^*) is an optimal mechanism for distribution π. We claim that in fact (x^*, t^*) is the unique optimal mechanism for π.

So let (x, t) be any optimal mechanism for π; we wish to show that it fully coincides with (x^*, t^*). Since $E_{\pi[\widetilde{\Theta}]}[t(\theta)] \leq E_{\pi[\widetilde{\Theta}]}[t^*(\theta)]$ for each $\pi[\widetilde{\Theta}]$, the only way t can obtain the same expected profit as t^* against π is to have equality for every $\pi[\widetilde{\Theta}]$, that is, (x, t) must be an optimal mechanism for every distribution $\pi[\widetilde{\Theta}]$.

Consider in particular any $\widetilde{\Theta}$ where each $\widetilde{\Theta}^g$ consists of $\theta^g_{j^*g}$ and at most one other type. For these sets, the special case of the lemma we have already proven shows that we must have $(x(\theta), t(\theta)) = (x^*(\theta), t^*(\theta))$ for each $\theta \in \times_g \widetilde{\Theta}^g$.

But every type $\theta \in \Theta$ belongs to some such subspace of types, for appropriate $\widetilde{\Theta}$. Therefore, the optimal mechanism (x, t) must coincide with (x^*, t^*) everywhere. *Q.E.D.*

PROOF OF COROLLARY 5.3: As argued in the proof of Lemma 4.2 in the main paper, when looking for optimal mechanisms, we can restrict to those whose payments are all in $[-\bar{t}, \bar{t}]$ for some sufficiently high constant $\bar{t}$. Then the effective space of mechanisms $\mathcal{M}'$ becomes a convex polytope, since it is a compact set of $|\Theta| \cdot (G+1)$-dimensional vectors defined by certain linear constraints. Therefore, it is the convex hull of its vertices (see, e.g., Ziegler (1995, Theorem 1.1)), that is, there exist some mechanisms $M_1, \dots, M_K$ such that every mechanism in $\mathcal{M}'$ equals some convex combination of them.

By Lemma 5.2, there exists some particular $\pi^* \in \Pi$ for which the separate-sales mechanism earns *strictly* higher expected profit than any other mechanism. Since expected profit is a linear function on $\mathcal{M}'$, it is maximized at one of the corners, so the separate-sales mechanism must be one of these corners, say M_1. By continuity, for any sufficiently nearby π, M_1 still gives strictly higher expected profit than $M_2, \dots, M_K$, and so remains higher than any convex combination, that is, no mechanism attains higher profit than R^*. *Q.E.D.*

APPENDIX C: GENERALIZED VIRTUAL VALUES IN THE MONOPOLY PROBLEM

In this appendix we demonstrate in detail how the generalized virtual values we have defined in Section 4.1 reduce to the traditional ironed virtual value in the one-dimensional case. We focus on the benchmark monopoly problem with a single good (and a finite set of types). We could allow for a convex cost of production and the calculations would be virtually identical, but for simplicity we do not do so here.

Suppose that the set of possible values for the good is $\{\theta_1, \ldots, \theta_J\}$, with $0 \leq \theta_1 < \theta_2 < \cdots < \theta_J$, and that π is the distribution. Recall the notation for allocations: $X = \{0, 1\}$ and $u(x, \theta) = x\theta$. Also write $R_j = \theta_j \times \sum_{j'=j}^{J} \pi(\theta_{j'})$, the profit from setting a price of θ_j; write $R^* = \max_j R_j$, with j^* as the index attaining the maximum (if there are several, pick the lowest). It will also be convenient to put $R_{J+1} = 0$.

The traditional analysis of the problem begins by defining the virtual value of type θ_j as

$$\widetilde{\theta}_j = \theta_j - \frac{\sum_{j'>j} \pi(\theta_{j'})}{\pi(\theta_j)}(\theta_{j+1} - \theta_j), \quad \text{or} \quad \widetilde{\theta}_j = \theta_j \quad \text{if } j = J.$$

(This is the discrete-type analogue of the classical formula from Myerson (1981); see, e.g., Vohra (2011, p. 118).)

Consider first the no-ironing case, where $\widetilde{\theta}_1 \leq \cdots \leq \widetilde{\theta}_J$. In this case, the traditional solution uses just the IR constraint of the lowest type θ_1 and the adjacent downward IC constraints $\theta_j \to \theta_{j-1}$ to show that no mechanism can earn more than R^*, which is achieved by allocating the goods to the types θ_j with nonnegative virtual value.

This method corresponds to a solution to the dual problem that puts positive weights only on these constraints. To fully illustrate the connection, we will explicitly write out what this dual solution is and then check that our generalized virtual values defined by (4.9) correspond to the virtual values $\widetilde{\theta}_j$. Recall that in a typical screening problem, there may be many possible generalized virtual values, depending on the choice of dual solution. However, once we have decided to use a dual solution that puts weight only on the lowest IR and the adjacent downward IC constraints, an easy induction using (4.7) shows that these weights are uniquely determined. Thus it makes sense to talk about *the* generalized virtual values representing this approach to the screening problem.

In our proposed dual solution, the IR and IC multipliers are

$$\kappa[\theta_1] = 1, \qquad \lambda[\theta_j \to \theta_{j-1}] = \sum_{j'=j}^{J} \pi(\theta_{j'}),$$

and all other λ and κ variables are zero. Also, for each θ_j, we define

$$\mu_0[\theta_j] = \max\{\pi(\theta_j)\widetilde{\theta}_j, 0\}, \qquad \mu_1[\theta_j] = \max\{0, -\pi(\theta_j)\widetilde{\theta}_j\},$$
$$\nu[\theta_j] = -\max\{\pi(\theta_j)\widetilde{\theta}_j, 0\}.$$

Let us check that this is indeed an optimal solution. It is immediate that all the λ, κ, and μ variables are nonnegative. It is also immediate that (4.6) holds with $a = 0$ (this means not allocating the good) since all the $u(a, \theta)$ terms are zero. For $a = 1$ (allocating the good), the first three terms in (4.6) add up to $-\pi(\theta_j)\widetilde{\theta}_j$, while the last two add up to $-(\max\{0, -\pi(\theta_j)\widetilde{\theta}_j\} - \max\{\pi(\theta_j)\widetilde{\theta}_j, 0\}) = \pi(\theta_j)\widetilde{\theta}_j$. So (4.6) is satisfied. It is straightforward to check that (4.7) is satisfied as well.

And for (4.8), note that

$$\pi(\theta_j)\widetilde{\theta}_j = R_j - R_{j+1}$$

for each $j = 1, \ldots, J$. Since j^* is the (lowest) index for which R_j attains the maximum, this implies $\widetilde{\theta}_{j^*-1} < 0$ (if $j^* > 1$) and $\widetilde{\theta}_{j^*} \geq 0$. So since the $\widetilde{\theta}_j$ are increasing, we have $\widetilde{\theta}_j \geq 0$ precisely when $j \geq j^*$. Consequently, we have

$$\sum_{j=1}^{J} \nu[\theta_j] = -\sum_{j \geq j^*} \pi(\theta_j)\widetilde{\theta}_j = -\sum_{j \geq j^*}(R_j - R_{j+1}) = -R^*.$$

So (4.8) is satisfied, and we do indeed have an optimal solution to the dual program.

Now consider the generalized virtual value of type θ_j as defined in (4.9) with these dual variables. Certainly $\overline{u}(a, \theta_j) = 0$ when $a = 0$ (all the $u(a, \theta)$ terms are zero); the relevant case is $a = 1$. For each $\theta = \theta_j$ we only have one nonzero term $\lambda[\widehat{\theta} \to \theta]$, namely $\widehat{\theta} = \theta_{j+1}$ (if $j = J$ there are no such terms), and then it is clear that $\overline{u}(1, \theta_j)$ is indeed equal to the traditional virtual value $\widetilde{\theta}_j$.

Now we turn to the general case, where the $\widetilde{\theta}_j$ are not necessarily increasing so there will be ironing. We follow the ironing procedure in Myerson (1981) (with adjustments for discrete types). Consider the set of points in the plane,

$$S = \left\{ \left(\sum_{j'<j} \pi(\theta_{j'}), R_j \right) \,\middle|\, j = 1, \ldots, J+1 \right\},$$

and define the function $G : [0, 1] \to \mathbb{R}$ to be the upper boundary of the convex hull of this S, that is, the lowest concave function such that $G(x) \geq y$ for each point $(x, y) \in S$. Then define the ironed revenue

$$\overline{R}_j = G\left(\sum_{j'<j} \pi(\theta_{j'}) \right)$$

for each $j = 1, \ldots, J+1$. We immediately have $R_j \leq \overline{R}_j \leq \max_{(x,y)\in S} y = R^*$ for all j, with equality for $j = j^*$. Define also

$$\overline{\widetilde{\theta}}_j = \frac{\overline{R}_j - \overline{R}_{j+1}}{\pi(\theta_j)}$$

for each $j = 1, \ldots, J$. These are the ironed virtual values. Concavity of G implies they are increasing, $\overline{\widetilde{\theta}}_1 \leq \cdots \leq \overline{\widetilde{\theta}}_J$.

Also, as in the no-ironing case, $\overline{R}_{j^*} = \max_j \overline{R}_j$ implies that $\overline{\widetilde{\theta}}_{j^*-1} \leq 0$ (if $j^* > 1$) and $\overline{\widetilde{\theta}}_{j^*} \geq 0$.

Now we describe how to translate the ironing approach into a solution to the dual problem. The usual colloquial description of ironing is that it maximizes revenue subject only to the adjacent downward IC constraints (and IR of the lowest type) plus a monotonicity constraint, $x(\theta_j)$ increasing in j. However, monotonicity is itself obtained as a consequence of the adjacent downward and upward IC constraints. So in fact the corresponding dual solution will put weight on both the downward and upward adjacent ICs.

For the IR constraints, we put $\kappa[\theta_1] = 1$ and $\kappa[\theta_j] = 0$ otherwise, as before. For the IC constraints, we put the following weights on the adjacent incentive constraints:

$$\lambda[\theta_j \to \theta_{j-1}] = \sum_{j'=j}^{J} \pi(\theta_{j'}) + \frac{\overline{R}_j - R_j}{\theta_j - \theta_{j-1}}, \qquad \lambda[\theta_{j-1} \to \theta_j] = \frac{\overline{R}_j - R_j}{\theta_j - \theta_{j-1}}$$

(and all other $\lambda[\cdots]$ multipliers equal to zero). Notice that these weights are nonnegative, since $\overline{R}_j \geq R_j$. Put also

$$\mu_0[\theta_j] = \max\{\pi(\theta_j)\overline{\overline{\theta}}_j, 0\}, \qquad \mu_1[\theta_j] = \max\{0, -\pi(\theta_j)\overline{\overline{\theta}}_j\},$$
$$\nu[\theta_j] = -\max\{\pi(\theta_j)\overline{\overline{\theta}}_j, 0\}.$$

In particular, the μ's are nonnegative as well.

Let us check that this is an optimal dual solution. First let us check (4.6) (in the case $a = 1$, since $a = 0$ is easy). We will do the case $1 < j < J$ here. Then the first three terms on the left side of (4.6) are

$$\begin{aligned}
&\lambda[\theta_{j+1} \to \theta_j]\ \theta_{j+1} + \lambda[\theta_{j-1} \to \theta_j]\theta_{j-1} - \big(\lambda[\theta_j \to \theta_{j+1}] + \lambda[\theta_j \to \theta_{j-1}]\big)\theta_j \\
&= \sum_{j'=j+1}^{J} \pi(\theta_{j'})\theta_{j+1} - \sum_{j'=j}^{J} \pi(\theta_{j'})\theta_j + \frac{\overline{R}_{j+1} - R_{j+1}}{\theta_{j+1} - \theta_j}\theta_{j+1} + \frac{\overline{R}_j - R_j}{\theta_j - \theta_{j-1}}\theta_{j-1} \\
&\quad - \frac{\overline{R}_{j+1} - R_{j+1}}{\theta_{j+1} - \theta_j}\theta_j - \frac{\overline{R}_j - R_j}{\theta_j - \theta_{j-1}}\theta_j \\
&= (R_{j+1} - R_j) + \frac{\overline{R}_{j+1} - R_{j+1}}{\theta_{j+1} - \theta_j}(\theta_{j+1} - \theta_j) - \frac{\overline{R}_j - R_j}{\theta_j - \theta_{j-1}}(\theta_j - \theta_{j-1}) \\
&= (R_{j+1} - R_j) + (\overline{R}_{j+1} - R_{j+1}) - (\overline{R}_j - R_j) \\
&= \overline{R}_{j+1} - \overline{R}_j \\
&= -\pi(\theta_j)\overline{\overline{\theta}}_j.
\end{aligned}$$

The other two terms on the left side of (4.6) equal $-(\mu_1[\theta_j] + \nu[\theta_j]) = \pi(\theta_j)\overline{\overline{\theta}}_j$. Thus, (4.6) is satisfied. (This is the case $1 < j < J$, but the remaining cases are similar, using the identities $\overline{R}_1 = R_1$ and $\overline{R}_{J+1} = R_{J+1}$ to account for the missing terms.)

It is straightforward to see that (4.7) is satisfied as well: the only difference from the no-ironing case is the addition of the $(\overline{R}_{j+1} - R_{j+1})/(\theta_{j+1} - \theta_j)$ and $(\overline{R}_j - R_j)/(\theta_j - \theta_{j-1})$ terms, which each appear once with a $+$ sign and once with a $-$ sign on the left side, and so they cancel out.

And because $\overline{\overline{\theta}}_1 \leq \cdots \leq \overline{\overline{\theta}}_{j^*-1} \leq 0 \leq \overline{\overline{\theta}}_{j^*} \leq \cdots \leq \overline{\overline{\theta}}_J$, we have $\nu[\theta_j] = 0$ for $j < j^*$ and $= -\pi(\theta_j)\overline{\overline{\theta}}_j$ for $j \geq j^*$. Therefore,

$$\sum_{j=1}^{J} \nu[\theta_j] = -\sum_{j \geq j^*} \pi(\theta_j)\overline{\overline{\theta}}_j = -\sum_{j \geq j^*}(\overline{R}_j - \overline{R}_{j+1}) = -\overline{R}_{j^*} = -R^*.$$

Thus (4.8) holds as well, and we indeed have an optimal solution to the dual problem.

With this choice of dual variables, the generalized virtual value for allocating the object to type θ_j, as defined in (4.9), equals

$$\begin{aligned}
\overline{u}(1,\theta_j) &= \theta_j - \left(\frac{\lambda[\theta_{j+1}\to\theta_j]}{\pi(\theta_j)}(\theta_{j+1}-\theta_j) + \frac{\lambda[\theta_{j-1}\to\theta_j]}{\pi(\theta_j)}(\theta_{j-1}-\theta_j)\right) \\
&= \theta_j - \frac{\sum_{j'=j+1}^{J}\pi(\theta_{j'})}{\pi(\theta_j)}(\theta_{j+1}-\theta_j) - \frac{\left(\dfrac{\overline{R}_{j+1}-R_{j+1}}{\theta_{j+1}-\theta_j}\right)}{\pi(\theta_j)}(\theta_{j+1}-\theta_j) \\
&\quad - \frac{\left(\dfrac{\overline{R}_j-R_j}{\theta_j-\theta_{j-1}}\right)}{\pi(\theta_j)}(\theta_{j-1}-\theta_j) \\
&= \widetilde{\theta}_j - \frac{\overline{R}_{j+1}-R_{j+1}}{\pi(\theta_j)} + \frac{\overline{R}_j-R_j}{\pi(\theta_j)} \\
&= \frac{(R_j-R_{j+1})-(\overline{R}_{j+1}-R_{j+1})+(\overline{R}_j-R_j)}{\pi(\theta_j)} \\
&= \frac{\overline{R}_j-\overline{R}_{j+1}}{\pi(\theta_j)} \\
&= \overline{\widetilde{\theta}}_j.
\end{aligned}$$

(Again, this is for $1<j<J$, but the calculation for $j=1,J$ is almost identical.)

Thus, as promised, the generalized virtual values are equal to the ironed virtual values as traditionally defined.

REFERENCES

MADARÁSZ, K., AND A. PRAT (2012): "Sellers With Misspecified Models," *Review of Economic Studies* (forthcoming). [1]

MCAFEE, R. P., J. MCMILLAN, AND M. D. WHINSTON (1989): "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values," *Quarterly Journal of Economics*, 104 (2), 371–383. [4]

MYERSON, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6 (1), 58–73. [7,8]

VOHRA, R. (2011): *Mechanism Design: A Linear Programming Approach*. New York: Cambridge University Press. [7]

ZIEGLER, G. M. (1995): *Lectures on Polytopes*. Graduate Texts in Mathematics, Vol. 152. New York: Springer-Verlag. [6]

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