Econometrica, Vol. 91, No. 6 (November, 2023), 2005–2039

PRICE SETTING WITH STRATEGIC COMPLEMENTARITIES AS A MEAN FIELD GAME

FERNANDO ALVAREZ
The Kenneth C. Griffin Department of Economics, University of Chicago and NBER

FRANCESCO LIPPI
Department of Economics and Finance, LUISS University and EIEF

PANAGIOTIS SOUGANIDIS
Department of Mathematics, University of Chicago

We study the propagation of monetary shocks in a sticky-price general equilibrium economy where the firms' pricing strategy features a complementarity with the decisions of other firms. In a dynamic equilibrium, the firm's price-setting decisions depend on aggregates, which in turn depend on the firms' decisions. We cast this fixed-point problem as a Mean Field Game and prove several analytic results. We establish existence and uniqueness of the equilibrium and characterize the impulse response function (IRF) of output following an aggregate shock. We prove that strategic complementarities make the IRF larger at each horizon. We establish that complementarities may give rise to an IRF with a hump-shaped profile. As the complementarity becomes large enough, the IRF diverges, and at a critical point there is no equilibrium. Finally, we show that the amplification effect of the strategic interactions is similar across models: the Calvo model and the Golosov–Lucas model display a comparable amplification, in spite of the fact that the non-neutrality in Calvo is much larger.

KEYWORDS: Strategic complementarities, mean field games, menu costs, impulse response analysis, monetary shocks.

1. INTRODUCTION

IN SPITE OF SUBSTANTIVE PROGRESS IN THE THEORY AND EMPIRICS of general equilibrium models with sticky prices, the need for tractability leads most analyses to abstract from the interactions between firms' decisions in price setting. Yet such complementarities are appealing because they amplify the non-neutrality of nominal shocks, as argued by Nakamura and Steinsson (2010) and Klenow and Willis (2016), and because of their empirical relevance.[1] Existing general equilibrium analyses proceed by exploring these effects numerically, as in Nakamura and Steinsson (2010), Klenow and Willis (2016),

Fernando Alvarez: f-alvarez1@uchicago.edu
Francesco Lippi: francescolippi@gmail.com
Panagiotis Souganidis: souganidis@uchicago.edu

We thank seminar participants at the University of Chicago, EIEF, Luiss, Peking, Ca' Foscari, Bologna, Ohio State, Zurich, the EUI, the BSE Summer Forum, the ECB, the 2021 NBER Summer Institute, the 5th Santiago Macro Workshop, the Richmond Fed, the Chicago Fed, the Theory Tea UofC workshop, and the workshop on Literacy in Partial Differential Equations at the University of Chicago. We thank Adrien Auclert, David Baquee, Andres Blanco, Paco Buera, Russel Cooper, Luca Dedola, Jennifer La'O, Rody Manuelli, and Rob Shimer for their comments. Ken Miyahara provided excellent research assistance. Souganidis was partially supported by National Science Foundation Grants DMS-1900599 and DMS-2153822, the Office for Naval Research Grant N000141712095, and the Air Force Office for Scientific Research Grant FA9550-18-1-0494. Lippi acknowledges financial support from the ERC Grant 101054421-DCS.

[1]Several empirical studies suggest the presence of non-negligible complementaries, for example, Cooper and Haltiwanger (1996), Gopinath and Itskhoki (2011), Amiti, Itskhoki, and Konings (2019).

and Mongey (2021), or abstracting from the decision about the timing of adjustments, as in Wang and Werning (2022), or abstracting from idiosyncratic shocks, as in Caplin and Leahy (1997). In this paper, we develop a new analytic approach to study a general equilibrium where the dynamic path of aggregates influences individual decisions, and vice versa. The results provide a thorough characterization of a sticky-price equilibrium in a *state-dependent* model featuring both idiosyncratic shocks and strategic complementarities, or substitutabilities, in pricing decisions. The approach is amenable to applications to other fields of macroeconomics.

A rigorous treatment of strategic complementarities in a general equilibrium model is involved: decisions depend on aggregate variables, which in turn depend on individual decisions. This fixed-point problem is especially difficult in models with lumpy behavior, where the optimal decisions are non-linear (Ss rules) and time-varying. A recent analysis by Wang and Werning (2022) presents analytic results for a dynamic oligopoly model. In this insightful paper, tractability is obtained by assuming that the timing of the firm's price adjustments is exogenous, à la Calvo. Our approach shares with Caplin and Leahy (1997) and Wang and Werning (2022) a quest for analytic results. A main difference with respect to these papers is that we consider a problem where the firm's decisions are state-dependent and where idiosyncratic shocks feature prominently at the firm level.[2]

We present several analytic results that characterize the firm's optimal policy and the general equilibrium in a dynamic environment featuring strategic complementarities (or substitutabilities) and state-dependent decisions. The key breakthrough is obtained by casting the problem using the mathematical structure of Mean Field Games (MFG), as laid out by Lasry and Lions (2007). The problem takes the form of a system of two coupled partial differential equations: one Bellman equation describing individual decisions, and one Kolmogorov equation describing aggregation. The usefulness of employing the MFG framework to study the dynamic behavior of high-dimensional cross-sections was highlighted by Achdou, Han, Lasry, Lions, and Moll (2022), Ahn, Kaplan, Moll, Winberry, and Wolf (2018) where numerical methods were discussed. Relative to the MFG literature, and its applications to economics, this paper innovates in two dimensions. First, we focus on an analytic characterization of the dynamics that ensue following a perturbation of the stationary equilibrium.[3] The presence of strategic complementarities can create, even in simple static models, lack of equilibrium or multiplicity, which makes analytical, as opposed to purely numerical methods, necessary.[4] Second, we consider an impulse control problem, instead of one with drift control, that is, we deal with the case of lumpy adjustments. This case, appearing in several economic contexts, motivates our interest and is mathematically more delicate since it requires to solve a problem with time-varying boundaries.

We consider an economy with random menu costs of the Calvo-plus type considered in Nakamura and Steinsson (2010). This model spans price-setting models in between the

[2]Absent idiosyncratic shocks, all price changes are either increases or decreases at a point in time. Instead, idiosyncratic shocks allow us to relate to the micro-data on price changes, which have been shown to encode information about shock propagation.

[3]We often refer to this one-off perturbation as to an "MIT shock". This requires solving for the equilibrium dynamics triggered by a small unexpected perturbation of the stationary distribution. See Boppart, Krusell, and Mitman (2018) for numerical techniques to solve for a similar type of perturbation, and for the same interpretation of the resulting equilibrium as an impulse response.

[4]Note that the well-known "monotonicity condition" for uniqueness, developed by Lasry–Lions and used in almost all papers in this area, corresponds to the case of strategic substitutability and thus is not useful for the solution of the economically more interesting case of strategic complementarities.

pure Ss model of Golosov and Lucas (2007) and the pure time-dependent model of Calvo (1983). The dynamic equilibrium for an economy *without strategic interactions* was solved analytically in Alvarez and Lippi (2022). We follow Klenow and Willis (2016) and extend that model to capture both micro and macro complementarities in the decision problem of the firm. These originate from the fact that the firm's flow profit depends on its own markup and the markup (or price) of the average firm, with a non-zero cross derivative. Our framework allows us to study analytically the effect of such interactions on the firm's optimal Ss rules after the shock as well as its effect on the aggregate dynamics.

Main Results. First, we establish conditions for the existence and uniqueness of the equilibrium and analytically characterize the impulse response function (IRF) of output to a once and for all nominal shock. If an equilibrium exists, it is unique. We show, moreover, that as the strategic complementarity becomes larger, the output's IRF increases at each horizon, in a convex fashion, that is, increasing more as the complementarity increases. Indeed, the IRF becomes arbitrarily large as the strategic complementarity approaches a critically high value. At that value, the equilibrium does not exist, and for strictly larger values the equilibrium is not well-posed; for example, it is not continuous as a function of the parameters. Around the critical values at which the equilibrium does not exist, the equilibrium outcomes change dramatically (i.e., they "have a pole"). On the other hand, the equilibrium always exists when the interactions involve substitutability (as opposed to complementarity) and the IRF converges to zero as substitutability becomes arbitrarily large, that is, the economy behaves as one with flexible prices. It may seem surprising that, even with a large degree of strategic complementarities, there are no multiple equilibria. Instead, the way that "excessively large" complementarities manifest is by the equilibrium being "ill-posed." The underlying reason is that the best response functions are linear, in a sense properly defined in Section 3 for the Calvo model and in Section 6.2 for the general case.

Second, we show that the presence of a sufficiently large strategic complementarity makes the IRF hump-shaped as a function of time elapsed since the shock, while if there is no complementarity, the IRF is monotone decreasing. This is a novel result that illustrates the substantive economic consequences of strategic interactions.

Third, we note that while most of the analysis focuses on a small monetary shock, our results can be used to study the impulse response following *any* perturbation of the initial distribution. For instance, we can study the response to a markup shock or to a volatility shock or, in general, to a one-off perturbation that affects the economy's steady-state distribution.

Fourth, while the core of the analysis focuses on the effect of a single shock and the associated impulse response, we also characterize the unconditional variance of output if monetary shocks are i.i.d., an experiment similar to the one in the classic articles by Caplin and Leahy (1997) and by Nakamura and Steinsson (2010). We show that, in this case, the unconditional variance of output is an increasing function of the strength of strategic complementarity.

Fifth, we show that for the models in the Calvo-plus class, the strategic complementarities amplify the Cumulative Impulse Response (CIR) by a measure that is approximately the same for all models within this class. For instance, the Calvo model and the Golosov–Lucas model display a comparable amplification, in spite of the fact that the level of the CIR in these models differs by a factor of 6. A simple analytic expression for the general case is given in equation (61).

Sixth, all the results described above also hold for the pure time-dependent Calvo model, which we study in Section 3 to introduce the tools used for the general case.

Related Literature. Our modeling of strategic complementarities shares with the classic article by Caplin and Leahy (1997) that the firm's profit function depends on both its own markup as well as on the average markup. One difference is that our economy features idiosyncratic shocks, while theirs does not. While they studied an equilibrium where the aggregate nominal shocks follow a drift-less Brownian motion, we mostly focus on an impulse response after a once and for all shock, which makes it easy for us to connect to, for example, the VAR evidence.[5]

Our work is closely related to Nakamura and Steinsson (2010) and Klenow and Willis (2016). The DSGE models in both papers consider an input-output structure that makes the (sticky) price of other industries part of the cost of each industry (i.e., "macro strategic complementarities"). Both papers, as well as ours, consider a frictionless labor market, idiosyncratic shocks at the firm level, and menu cost paid by firms to adjust prices. Nakamura and Steinsson (2010) allow, as we do, for a random menu cost. Klenow and Willis (2016) allow, as we do, for a non-constant demand elasticity at the firm level, which yields what they called "micro-strategic complementarities." We show that, up to second order, micro and macro complementarities are additive, so we capture both of them through a single parameter. Both papers use numerical techniques to characterize the effect of monetary shocks on aggregate output, while we provide analytic results.

Our analysis also relates to Wang and Werning (2022), who analyze the propagation of shocks in a sticky-price economy with strategic complementarities. They present an analytic solution assuming firms follow a time-dependent rule à la Calvo. Some features of the underlying environment are similar: the forces creating complementarities (variable demand elasticity, decreasing returns, non-zero Frisch elasticity) are fully summarized by a single parameter. Other modeling aspects are different: first, they consider a dynamic oligopoly without idiosyncratic shocks, while we focus on oligopolistically competitive markets with idiosyncratic shocks, a useful feature to connect to the distribution of price changes in the data. Second, the timing of adjustment is exogenous in their paper, while the firms in our setup choose both the timing as well as the size of the price adjustments. The simplification of the exogenous-timing and no-idiosyncratic shocks allows them to connect with the New Keynesian Phillips curve and to study the importance of strategic complementarities. Third, their setup features a finite number of firms (per sector), allowing them to analyze the role of concentration within an industry, a feature that we cannot address.

Another related contribution is Auclert, Rigato, Rognlie, and Straub (2022), who solve a discrete-time model with strategic complementarities for a time-dependent and for a state-dependent pricing model. A main difference is that they restrict the strength of the strategic complementarity to a specific value (corresponding to the case where $\theta = -1$ in our model). Their main novel theoretical result is to represent the outcomes of the state-dependent model as the sum of two time-dependent models. They obtain the equations that the system must obey, whose parameters have to be solved numerically. They use this decomposition, among other things, to evaluate how closely the Calvo model approximates standard versions of the state-dependent model.

A related contribution in the Mean Field Game literature is Bertucci (2018), who analyzed a problem with impulse control. His problem is simpler in that the decision maker considers only one adjustment, and that the target to which it adjusts is pre-specified. Additionally, he focuses on existence and uniqueness, using a slightly different notion of the solution.

[5]An interesting feature of Caplin and Leahy (1991, 1997) is to produce a state-dependent reaction to monetary shocks, perhaps the only model where a clear notion of "overheating" due to monetary policy appears.

Organization of the Paper. The next section lays out the general equilibrium environment of the problem and the microfoundations of strategic interactions in price setting. Section 3 uses a Calvo model to introduce our approach and to display some new results. Section 4 sets up the dynamic equilibrium as a MFG. Section 5 studies a linearized version of the MFG and derives analytic results that are key for the equilibrium analysis. Section 6 characterizes the dynamic equilibrium and discusses the economic implications of strategic interactions. Section 7 concludes and discusses future work. The Appendix may be found in the online Supplemental Material (Alvarez, Lippi, and Souganidis (2023)).

2. GENERAL EQUILIBRIUM SETUP AND COMPLEMENTARITIES

This section presents an economy where households maximize the present value of lifetime utility and firms maximize profits subject to costly price adjustments. We show that non-negligible complementarities between the price-setting strategies of firms can arise through two channels, possibly coexisting: first, from consumers' preferences that yield a demand system with a non-constant price elasticity, a phenomenon the literature dubbed *micro-complementarities*, as in Kimball (1995); second, a production structure with sticky-price intermediate goods, as in Klenow and Willis (2016) and Nakamura and Steinsson (2010), referred to as *macro-complementarities*. We establish that the effects of both channels on the firm's pricing strategy are summarized by a single parameter and that, at a symmetric equilibrium, the firm's problem is approximated by a quadratic return function that depends on the own price and the aggregate price, as in the classic work of Caplin and Leahy (1997).

Households. We consider a continuum of households with time discount ρ and utility $\int_0^\infty e^{-\rho t}(U(\mathcal{C}(t)) - \mathfrak{a}L(t) + \log\frac{M(t)}{P(t)})\,dt$, where U denotes a CRRA utility function over the consumption composite $\mathcal{C}$, the labor supply is L, M is the money stock, P is the consumption deflator, and $\mathfrak{a} > 0$ is a parameter. The linearity of the labor supply and the log specification for real balances are convenient simplifications also used in Golosov and Lucas (2007) and many other papers. We follow Kimball (1995) in modeling the consumption composite $\mathcal{C}$ using an implicit aggregator over a continuum of varieties k as follows: $1 = (\int_0^1 \Upsilon(\frac{c_k(t)}{\mathcal{C}(t)}A_k(t))\,dk)$, where A_k denotes a preference shock for variety k, and $\Upsilon(1) = 1$, $\Upsilon' > 0$, and $\Upsilon'' < 0$. The Kimball aggregator defines $\mathcal{C}$ implicitly, yielding an elasticity of substitution that varies with the relative demand $c_k/\mathcal{C}$. The standard CES demand is obtained as a special case when Υ is a power function.

The representative household chooses c_k, money demand, and labor supply to maximize lifetime utility subject to the budget constraint

$$M(0) + \int_0^\infty e^{-\int_0^t R(s)\,ds}\left[\tilde{\Pi}(t) + (1+\tau_L)W(t)L(t) - R(t)M(t) - \int_0^1 \tilde{p}_k(t)c_k(t)\,dk\right]dt = 0,$$

where $R(t)$ is the nominal interest rates, $W(t)$ the nominal wage, τ_L a constant labor subsidy, $\tilde{\Pi}(t)$ is the sum of the aggregate (net) nominal profits of firms and the lump sum nominal transfers from the government, and $\tilde{p}_k$ the price of each variety.

Firms. There is a continuum of firms, indexed by $k \in [0, 1]$, that use a labor (L_k) and intermediate-good inputs (I_k) to produce the final good y_k with a constant returns to scale technology (omit time index) as follows: $y_k = c_k + q_k = (L_k/Z_k)^\alpha I_k^{1-\alpha}$. Note that final goods are used by consumers, c_k, and also as an input in the production of the intermediate good $Q = \int_0^1 I_k \, dk$ through the production function $1 = \int_0^1 \Upsilon(\frac{q_k}{Q} A_k)\, dk$. The aggregates Q and $\mathcal{C}$ have the same unit price, P, since they are produced with identical inputs and the same function Υ. The labor productivity of firm k is $1/Z_k$ and we assume that $Z_k = \exp(\sigma \mathcal{W}_k)$ where $\mathcal{W}_k$ are standard Brownian motions, independent and identically distributed across firms, so that the log of Z_k follows a diffusion with variance σ^2. The households' labor supply L is used to produce each of the k goods and the price-adjustment services L_p, so $L = \int_0^1 L_k \, dk + L_p$.

The Demand for Final Goods. The first-order conditions of consumers and intermediate-good producers yield the demand system, whose form depends on the function Υ. Given a total expenditure E, the demand for variety k, evaluated at a symmetric equilibrium, is

$$y_k = \frac{1}{\Upsilon^{-1}(1)} \frac{E}{PA_k} D\left(\frac{p}{P}\right) \quad \text{where } D\left(\frac{p}{P}\right) \equiv (\Upsilon')^{-1}\left(\frac{p}{P}\Upsilon'(\Upsilon^{-1}(1))\right) \text{ and } p \equiv \tilde{p}/A.$$

The Firm's Profit Function. Let the nominal wage W be the numeraire, and $\tilde{p}_k = pA_k$ be the firm's price. Notice that the firm's marginal (and average) cost is $\chi \equiv (Z_k W)^\alpha P^{1-\alpha}$, where P is the price of intermediate inputs. We can write the firm's (nominal) profit as $y_k \cdot (pA_k - (Z_k W)^\alpha P^{1-\alpha})$. Assuming that $Z_k^\alpha = A_k$, that is, that preference shocks are proportional to marginal cost shocks, then we have that each firm maximizes $\Pi(p, P) = y_k A_k W(\frac{p}{W} - (\frac{P}{W})^{1-\alpha})$ so the profits of the individual firm do not depend on A_k since $y_k A_k = \frac{E}{P\Upsilon^{-1}(1)} D(\frac{p}{P})$.[6] The notation emphasizes that the firm's decision depends on both the own price, p, and the aggregate price P, and that prices are homogeneous in W.

Let us write the firm's profit in terms of the demand $D(p/P)$ and the cost function $\chi = \chi(P)$, giving the marginal cost. We have $\frac{\Pi(p,P)}{W} = \frac{E}{P\Upsilon^{-1}(1)} D(p/P)(p - \chi(P))$. The first-order condition for optimality implicitly defines the pricing function: $p^*(P) = \frac{\eta(p/P)}{\eta(p/P)-1}\chi(P)$, where $\eta(p/P) \equiv -\frac{p}{D(p/P)} \frac{\partial D(p/P)}{\partial p}$ so η is the elasticity of the demand with respect to the own price p. We have the following:

PROPOSITION 1: *Consider a value for P such that $p^*(\bar{P}) = \bar{P}$. Assume that D is decreasing and that $\Pi(p, P)$ is strictly concave at $(p^*(\bar{P}), \bar{P}) = (\bar{P}, \bar{P})$. We have*

$$\frac{\bar{P}}{p^*(\bar{P})} \frac{\partial p^*(\bar{P})}{\partial P} = \frac{1}{1 + \frac{\eta'(1)}{\eta(1)(\eta(1)-1)}} \Bigg[\underbrace{\frac{\eta'(1)}{\eta(1)(\eta(1)-1)}}_{\text{micro elasticity}} + \underbrace{\frac{P}{\chi(P)} \frac{\partial \chi(P)}{\partial P}}_{\text{macro elasticity}} \Bigg], \tag{1}$$

where $\eta(1) > 1$ and $1 + \frac{\eta'(1)}{\eta(1)(\eta(1)-1)} > 0$. Expanding the profit function around $(\bar{P}, \bar{P})$:

$$\frac{\Pi(p, P)}{\Pi(\bar{P}, \bar{P})} = 1 - B\left(\frac{p - \bar{P}}{\bar{P}} + \theta \frac{P - \bar{P}}{\bar{P}}\right)^2 + \iota(P) + o\left(\left\| \frac{p - \bar{P}}{\bar{P}}, \frac{P - \bar{P}}{\bar{P}} \right\|^2\right), \tag{2}$$

[6]The assumption $Z_k^\alpha = A_k$ allows the problem to be described by a scalar stationary state variable, the price gap x. This is used to write the dynamic programming problem of the firm as well as to keep the expenditure shares stationary across goods in the presence of permanent idiosyncratic shocks. This is the "price to pay" to formulate the problem using Caballero and Engel's "price gaps." This assumption is also used in Woodford (2009), Bonomo, Carvalho, and Garcia (2010), Midrigan (2011).

where $\iota(\cdot)$ is a function that does not depend on p, *and where*

$$B \equiv -\frac{1}{2}\frac{\Pi_{11}(\bar{P},\bar{P})}{\Pi(\bar{P},\bar{P})}\bar{P}^2 = \frac{\left[\eta'(1)+\eta(1)\big(\eta(1)-1\big)\right]}{2} > 0 \quad \textit{and}$$

$$\theta \equiv \frac{\Pi_{12}(\bar{P},\bar{P})}{\Pi_{11}(\bar{P},\bar{P})} = -\frac{\bar{P}}{p^*}\frac{\partial p^*}{\partial P}\bigg|_{p^*=\bar{P}}.$$

A few remarks are in order. First, equation (2) shows that the profit maximization problem of the firm is approximated by the minimization of the quadratic period return $B(x+\theta X)^2$, where $x \equiv \frac{p-\bar{P}}{\bar{P}}$ and $X \equiv \frac{P-\bar{P}}{\bar{P}}$ denote the percent deviation from the symmetric equilibrium of the own and the aggregate price, respectively.

Second, the extent of strategic interactions between the own price and the aggregate price is captured by the parameter θ. Notice that static profits are maximized by setting $x=-\theta X$. The firm's strategy exhibits strategic complementarity if $\theta<0$, and it exhibits strategic substitutability if $\theta>0$. Clearly, if $\theta \neq -1$, the only static equilibrium is $X=0$.

Third, in the absence of macro complementarity, for example, if $\frac{\partial \chi}{\partial P}=0$, we have $\theta = -\frac{\eta'}{\eta(\eta-1)+\eta'}$ so that $\theta<0$ occurs if $\eta'>0$. This condition has a clear economic interpretation: if $\eta'>0$, a higher P lowers the demand elasticity, which induces the firm to raise its markup. Thus, $\eta'>0$ implies that the own price and the aggregate price are strategic complements. Note, moreover, that if $\frac{\partial \chi}{\partial P}=0$, the strength of strategic complementarities is bounded, since $\theta>-1$. Instead, if $\frac{\partial \chi}{\partial P}>0$, we can have $\theta<-1$, a case of interest in the discussion of the equilibrium characterization and existence (see Section 3 and Section 6).

Impulse Response of Output to a Monetary Shock. Note that an increase in the aggregate nominal wage for all firms reduces the average deviation of markups from its optimal value, that is, it lowers X. One of the most interesting objects is the path of $X(t)$ after a small displacement of the stationary distribution, given by the initial condition $m_0(x)=\tilde{m}(x+\delta)$, where $\tilde{m}$ is the stationary density. The value of $X(t)$ is inversely proportional to the deviation from steady-state output t periods after the monetary shock δ. Below, we also consider a more general perturbation $m_0(x)=\tilde{m}(x)+\delta\nu(x)$.

3. A SIMPLE BENCHMARK: THE CALVO CASE

This section discusses a problem with strategic complementarities and Calvo's (1983) pricing. Due to its tractability, this is the most common case analyzed in the sticky-price literature. The model offers a simple setup to introduce the essential elements of the analysis and to discuss some key results, such as existence, uniqueness, and the non-monotone impulse response profiles, that will also appear in the state-dependent problem.[7]

The economy features a continuum of atomistic firms. Each firm takes as given the path of average deviation of markups $X(t)$ for all times $t\geq 0$. The firm can change its price *only* at random times $\{\bar{\tau}_k\}$, given by a Poisson process with parameter ζ. We refer to these times as *adjustment opportunities*, and to the state chosen at those times as the *optimal reset value*. After resetting its price at time t, the firm's markup gap $x(t)$ evolves as a driftless Brownian motion with variance σ^2. The markup jumps right after a price change at

[7]We are thankful to an anonymous referee for suggesting that we study the Calvo problem.

$t = \bar{\tau}_k$ by the amount $\bar{J}_k$; thus, the markup gap evolves as

$$x(s) = x(t) + \sigma\big[\mathcal{W}(s) - \mathcal{W}(t)\big] + \sum_{k:\bar{\tau}_k \le s} \bar{J}_k \quad \text{for all } s \in [t, T], \tag{3}$$

where $\mathcal{W}$ is a standard Brownian motion.

We assume that the strategic complementarities are at work only up to horizon T, and allow T to be finite or infinite. In particular, for $t < T$, the period flow cost is $B(x + \theta X)^2$ with $B > 0$, which feature strategic interactions, corresponding to the description in Proposition 1. Each firm minimizes the expected discounted value of the flow cost—with discount rate ρ—taking the path $X(t)$ for $t \in [0, T)$ as given. At time $t = T$, a firm with state x has a continuation $u_T(x)$, independent of θ and $X(t)$. We will assume that u_T equals the steady-state value function $\tilde{u}(x) = \frac{B}{(\rho+\zeta)}(\frac{\sigma^2}{\rho} + x^2)$ (see the proof of Lemma 1).

Optimal Price Setting. For $t \in [0, T)$, the state of the firm is (x, t), the value function is $u(x, t)$, and the optimal reset value at t is $x^*(t)$. The firm takes as given $u_T : \mathbb{R} \to \mathbb{R}$ and $X : [0, T) \to \mathbb{R}$, and its value function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ solves

$$u(x, t) = \min_{\{\bar{J}_k\}_{k=1}^{\infty}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} B\big(x(s) + \theta X(s)\big)^2 \, ds + e^{-\rho(T-t)} u_T\big(x(T)\big) \Big| x(t) = x\right], \tag{4}$$

where the state evolves subject to equation (3).

LEMMA 1: *The value function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ solves the PDE*:

$$\rho u(x, t) = B\big(x + \theta X(t)\big)^2 + \frac{\sigma^2}{2} u_{xx}(x, t) + u_t(x, t) + \zeta\Big(\min_z u(z, t) - u(x, t)\Big), \tag{5}$$

with terminal condition $u(x, T) = \tilde{u}(x)$ for all x, and the optimal reset $x^* : [0, T) \to \mathbb{R}$ solves $x^*(t) = \arg\min_z u(z, t)$, and it is given by

$$x^*(t) = -(\rho + \zeta)\theta \int_t^T e^{-(\rho+\zeta)(s-t)} X(s) \, ds \quad \textit{for all } t \in [0, T). \tag{6}$$

The value function $u(x, 0)$ is finite for all x if and only if

$$\int_0^T e^{-\rho t} B\big(x^*(t) + \theta X(t)\big)^2 \, dt < \infty. \tag{7}$$

The optimal policy at the times when $t = \bar{\tau}_k$ is for x to jump to $x^*(t)$, that is, $\bar{J}_k = x^*(t) - x(t^-)$. Three features will hold, with appropriate modifications, in the general case: the PDE in equation (5), the condition that $0 = u_x(x^*(t), t)$, and that the optimal decision rule $x^*(t)$ is a (linear) function of the path of future X's. One difference with the state-dependent model is that neither B nor σ^2 affects the optimal reset price in equation (5). If the condition in equation (7) is violated, the expected discounted profits of the firm diverge; this condition, given equation (6), restricts the path of $X(t)$. Furthermore, for future reference, note that the integral equation (6) is equivalent to the following ODE and boundary condition:

$$\dot{x}^*(t) = (\rho + \zeta)\big(x^*(t) + \theta X(t)\big) \quad \text{for all } t \in [0, T) \text{ and } e^{-(\rho+\zeta)T} x^*(T) = 0. \tag{8}$$

Aggregation. For the models of interest, $X(t)$ is the cross-sectional average of the x's. Consider a discrete-time version with a (short) interval of length dt. In this interval, a fraction of firm $\zeta\, dt$ change its price, so their markup becomes $x^*(t)$. The remaining firms keep their (expected) value since x evolves as a drift-less Brownian motion. Thus, $X(t + dt) = (1 - \zeta\, dt)X(t) + \zeta\, dt x^*(t)$. Taking the limit as $dt \to 0$, we obtain

$$\dot{X}(t) = \zeta\big(x^*(t) - X(t)\big) \quad \text{for all } \in [0, T), \text{ with } X(0) = -\delta, \quad \text{or equivalently} \tag{9}$$

$$X(t) = X(0)e^{-\zeta t} + \zeta \int_0^t e^{-\zeta(t-s)} x^*(s)\, ds \quad \text{for all } t \in [0, T). \tag{10}$$

Next, we define the equilibrium:

DEFINITION 1: An equilibrium for an initial condition $X(0) = -\delta$ is two paths $\{x^*(t), X(t)\}$ for $t \in [0, T)$ that solve the integral equation (6), encoding optimality, the integral equation (10), encoding aggregation, and satisfy the finite-value condition in equation (7). Alternatively, one could replace the two integral equations with the o.d.e's and boundary conditions in equation (8) and equation (9).

A few comments are in order. First, the initial condition $X(0) = -\delta$ has the interpretation of the impact effect of a once and for all shock to nominal wages, caused by an expansionary monetary shock. In what follows, we normalize $\delta = 1$ and focus on the impulse response with initial condition $X(0) = -1$. This is without loss of generality since the impulse response in Calvo is linear in the shock size.

Second, optimal decisions are "forward looking," and are solved backward from the terminal condition $x^*(T) = 0$. Aggregation is "backward looking," and is solved forward given the initial condition $X(0)$. Third, the (integral) equation (6) and equation (10) are both linear, so the equilibrium is the fixed point of a linear operator. Fourth, for the case of $T = \infty$, there is a constraint on the square discounted integral of the paths.

We define the effect on the steady-state deviation of output after a monetary shock of (normalized) size $\delta = 1$ to be $Y_\theta(t, T) \equiv -X(t)$, since the output deviation from steady state is, up to first order, the negative of the (average) markup deviation.

Next, we present a characterization of the equilibrium based on the solution of the system of ODE's. Later, we discuss the equivalent solution based on solving the integral equation and the characterization of its kernel. We note that defining the equilibrium as the solution of the integral equation is less conventional in economics than using the (equivalent) system of ODE's. Nevertheless, the analysis of the kernel previews ideas that will be used in the state-dependent case where the system of ODE's is not a viable equilibrium description.

LEMMA 2: *Fix $\rho \geq 0$, $\zeta > 0$, $T < \infty$, and let $\gamma \equiv \frac{\zeta}{\zeta+\rho}$. The solution of the two-dimensional system in* (8) *and* (9) *features two eigenvalues*, $\lambda_1(\theta)$, $\lambda_2(\theta)$, *given by*

$$\lambda_1(\theta) = \zeta \frac{\big(1 - \gamma - \Delta(\theta)\big)}{2\gamma},$$

$$\lambda_2(\theta) = \zeta \frac{\big(1 - \gamma + \Delta(\theta)\big)}{2\gamma} \quad \textit{where } \Delta(\theta) \equiv \sqrt{(1 + \gamma)^2 + 4\gamma\theta}.$$

The critical value θ^* *solves* $\Delta(\theta^*) = 0$; *it is given by* $-1 \geq \theta^* = -\frac{(1+\gamma)^2}{4\gamma}$. *If* $\theta \neq \theta^*$, *the solution for output*, $Y_\theta(t, T) \equiv -X(t)$, *is*

$$Y_\theta(t; T) = \big(1 + \mathfrak{c}(\theta, T)\big)e^{\lambda_2(\theta)t} - \mathfrak{c}(\theta, T)e^{\lambda_1(\theta)t} \quad \text{for } t \in (0, T), \tag{11}$$

$$\mathfrak{c}(\theta, T) \equiv \frac{\big(1 + \gamma + \Delta(\theta)\big)e^{\lambda_2(\theta)T}}{\big(1 + \gamma - \Delta(\theta)\big)e^{\lambda_1(\theta)T} - \big(1 + \gamma + \Delta(\theta)\big)e^{\lambda_2(\theta)T}}.$$

If $\theta = \theta^*$, *the solution for output*, $Y_\theta(t, T) \equiv -X(t)$, *is*

$$Y_{\theta^*}(t; T) = e^{\frac{\rho}{2}t}\left(1 - \frac{\zeta t}{\zeta T + \dfrac{2\gamma}{1+\gamma}}\right) \quad \text{for } t \in (0, T). \tag{12}$$

We then have:

1. *If* $\theta > \theta^*$, *then* $\Delta(\theta)$, $\lambda_1(\theta)$, $\lambda_2(\theta)$ *are real*, $\mathfrak{c}(\theta, T)$ *is finite*. *The solution* $Y_{\theta^*}(t; T)$ *is given by equation* (11).
2. *If* $\theta = \theta^*$, *then* $\Delta(\theta^*) = 0$, *and* $\lambda_1(\theta^*) = \lambda_2(\theta^*) = \rho/2$. *The solution* $Y_{\theta^*}(t; T)$ *is given by equation* (12).
3. *If* $\theta < \theta^*$, *then* $\Delta(\theta)$ *is purely complex*; *the roots* $\lambda_1(\theta)$, $\lambda_2(\theta)$ *are complex conjugates*. *The solution* $Y_{\theta^*}(t; T)$ *is given by equation* (11). *The coefficient* $\mathfrak{c}(\theta, T)$ *is finite if and only if* $\theta \neq \theta_j$, *where the sequence* $\{\theta_j\}_{j=1}^{\infty}$ *is given by*

$$\theta_j = \theta^* - \frac{(\Delta_j)^2}{4\gamma} \quad \text{where } \Delta_j \text{ solves } \Delta_j = -(1+\gamma)\tan\left(\Delta_j \frac{\zeta T}{2\gamma}\right) \quad \text{and}$$

$$\theta_1 \approx \theta^* - \gamma\left(\frac{\pi}{\zeta T + 2\dfrac{\gamma}{(1+\gamma)}}\right)^2 \quad \text{for large } \zeta T \text{ and } \theta_1 \to \theta^* \text{ as } \zeta T \to \infty.$$

4. *Define* $\theta_0 \in (\theta_1, \theta^*)$ *as that value of* θ *that solves* $0 = \lambda_2(\theta) + \mathfrak{c}(\theta, T)(\lambda_2(\theta) - \lambda_1(\theta))$. *Then*, *for* $\theta = \theta_0$, *we have* $\frac{\partial Y_\theta(t,T)}{\partial t}\big|_{t=0} = 0$, *so that when* $\theta = \theta_0$, *the output impulse response has a zero derivative at* $t = 0$.

The equilibrium impulse response is given in equation (11), provided that the equation is well defined. Note that all impulse responses start at $Y_\theta(0; T) = 1$. Inspection of the equation shows that the equilibrium exists provided the coefficient $\mathfrak{c}(\theta, T)$, defined in equation (11), is finite, either real or complex. The lemma defines the critical thresholds $-1 > \theta_1 > \theta_2 > \cdots$, where the equilibrium does not exist. It also defines the threshold $\{\theta_0\}$, satisfying $\theta_1 < \theta_0 < \theta^* \leq -1$, that is useful to determine the region for hump-shaped impulse responses.

The next proposition uses Lemma 2 to characterize the equilibrium as a function of the degree of strategic complementarities θ and of the time horizon T.

PROPOSITION 2: *Fix* $\rho \geq 0$ *and* $\zeta > 0$. *Let* $\lambda_1(\theta)$, $\lambda_2(\theta)$, γ, θ^*, *and* $\{\theta_j\}_{j=1}^{\infty}$ *be as defined in Lemma* 2. *The equilibrium output solving equation* (8) *and equation* (9), $Y_\theta(t; T) \equiv -X(t)$, *is given by equation* (11) *for* $\theta \neq \theta_j$; *it is given by equation* (12) *for* $\theta = \theta^*$. *We have*:

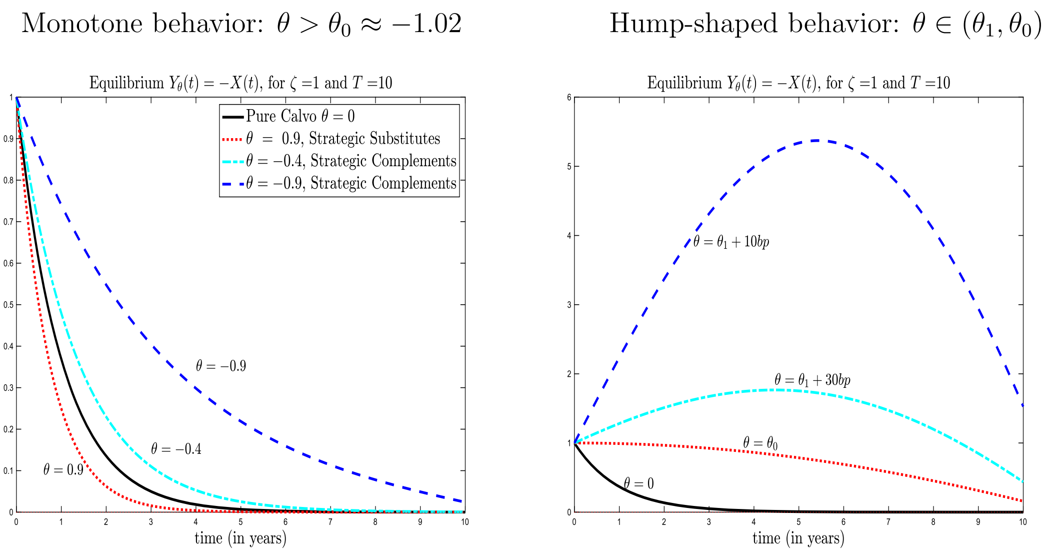

FIGURE 1.—IRF for the Calvo model in the well-posed region ($\theta > \theta_1$).

1. *Whenever an equilibrium exists, it is unique.*
2. *If $T < \infty$ and $\theta > \theta_1$, the equilibrium exists, it is a continuous function of θ, and it is well-posed, that is, $Y_\theta(t, T) > 0$. Moreover, we have (recall $\theta_1 < \theta_0 < \theta^*$):*
   - (2.a) *For $\theta > \theta_0$, then $Y_\theta(t, T)$ is decreasing in t. For $T \to \infty$, then $Y_\theta(t, T) \to e^{\lambda_1(\theta)t}$.*
   - (2.b) *For $\theta \in (\theta_1, \theta_0)$, then $Y_\theta(t, T)$ is "hump shaped" in t, that is, $\frac{\partial Y_\theta(t,T)}{\partial t}|_{t=0} > 0$.*
3. *If $T < \infty$ and $\theta \leq \theta_1$, the equilibrium exists provided $\theta \neq \theta_j$, for $j = 1, 2, \ldots$. The equilibrium is not well-posed over this region, that is, $Y_\theta(t, T)$ has countably many jump discontinuities at $\theta = \theta_j$. In particular:*
   - (3.a) *If $\theta \leq \theta_1$, the equilibrium $Y_\theta(t, T)$ oscillates with frequency $\frac{|\Delta(\theta)|(\rho+\zeta)}{4\pi}$, and amplitude $e^{\rho t/2}$. Moreover, for any $t > 0$, then $Y_\theta(t, T)$ does not converge as $T \to \infty$.*
   - (3.b) *If $\theta = \theta_j$ for some $j \geq 1$, there is no equilibrium. Fix $t > 0$; the function $Y_\theta(t, T)$ has a pole at $\theta = \theta_j$, so it changes sign and satisfies $\lim_{\theta \to \theta_j} Y_\theta(t, T) = \pm\infty$.*
4. *If $T = \infty$:*
   - (4.a) *If $\theta > \theta^*$, there is a unique equilibrium given by $Y_\theta(t, \infty) = e^{\lambda_1(\theta)t}$. Fix any $t > 0$; then $Y_\theta(t, \infty)$ is strictly increasing and convex in $(-\theta)$, converges to 1 for all t as $\theta \to \theta^*$, and converges to zero for all t as $\theta \to \infty$. As $\rho \to 0$, the impulse response coefficient is $\lambda_1 = -\zeta\sqrt{1 + \theta}$.*
   - (4.b) *If $\theta \leq \theta^*$, there is no equilibrium.*

A few remarks are in order, beginning with the case where T is finite. First, the proposition shows that if an equilibrium exists (i.e., if $\mathfrak{c}(\theta, T)$ is finite), then it is unique. Multiple equilibria cannot occur as the system of differential equations has at most one solution.

Second, over the region $\theta \in (\theta_1, \infty)$, the equilibrium exists, it is well-posed, that is, $Y_\theta(t; T) > 0$, and it is continuous as a function of the parameter θ. For T finite, the state space is divided in two regions. For $\theta > \theta_0$, the equilibrium output $Y_\theta(t; T)$ is monotone decreasing in t, while for $\theta \in (\theta_1, \theta_0)$, it is hump-shaped in t, as shown in Figure 1. In Appendix A of the Supplemental Material, we display how the interval (θ_1, θ_0) depends on the value of T. The economics of this result is that large strategic complementarities induce firms to over-react to the initial shock. If the aggregate markup falls by 1%, the firm now wants to react by lowering it by more than 1%. This amplifies the effect of the original shock and produces oscillations.

Third, as $\theta \to \theta_1^+$, the equilibrium values diverge to $+\infty$; at $\theta = \theta_1$, there is no equilibrium; and as $\theta \to \theta_1^-$, the equilibrium values diverge to $-\infty$; see the left panel of Figure 2. In other words, around the critical value θ_1, as well as all other critical values $\theta_2, \theta_3, \dots$, the equilibrium is not continuous as a function of the parameters and output displays an oscillatory behavior. In the region $\theta < \theta_1$, output crosses the steady-state value ($Y = 0$) with a frequency that increases with $-\theta$, and diverges as θ approaches θ_j; see the right panel of Figure 2. Additionally, for $\rho > 0$, the amplitude of these oscillations increases with time. As $T \to \infty$, there is no solution to these equations.

Finally, we note that when $T = \infty$, the equilibrium only exists for $\theta > \theta^* > \theta_1$. In this case, the IRF is a single exponential, as in Wang and Werning (2022). Indeed, as $\rho \to 0$, the impulse response is an exponential with coefficient $\lambda_1 = -\zeta\sqrt{1+\theta}$.

An Alternative Solution Approach. The next lemma analyzes the kernel of the system of integral equations. This alternative method will be used for the general model.

LEMMA 3: *The paths $\{x^*(t), X(t)\}$ are an equilibrium if and only if the path $\{X(t)\}$ satisfies $\int_0^T e^{-\rho t} X(t)^2\, ds < \infty$ and solves the integral equation*

$$X(t) = X(0)e^{-\zeta t} + \theta \int_0^T K(t,s)X(s)\, ds \quad \text{for all } t \in [0, T) \tag{13}$$

obtained by substituting equation (6) in equation (10). The kernel $K(t,s)$ is given by

$$K(t,s) \equiv \frac{\zeta(\rho+\zeta)}{2\zeta+\rho}\left(1 - e^{(2\zeta+\rho)\min\{t,s\}}\right)e^{-\zeta(t+s)-\rho s} \quad \text{for all } (t,s) \in [0,T)^2.$$

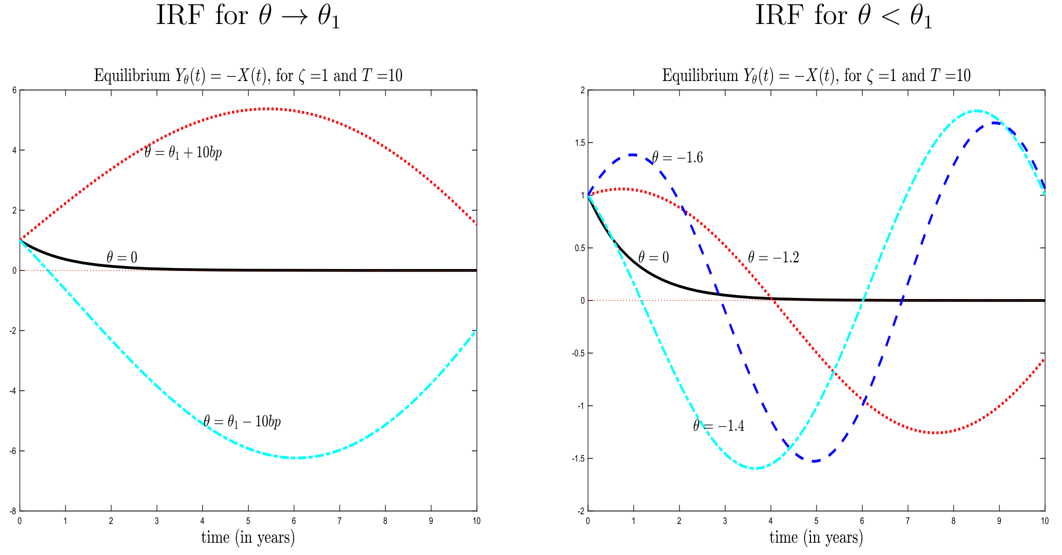

FIGURE 2.—IRF for the Calvo model in the ill-posed region.

The kernel satisfies: (*i*) $K(t,s) \leq 0$; (*ii*) $K(t,s)e^{-\rho t}$ *is symmetric in* (t,s); (*iii*) *for all* T: $\sup_t \int_0^T |K(t,s)|\,ds \leq 1$; (*iv*) $\int_0^T \int_0^T K(t,s)^2\,ds\,dt = \mathfrak{B}(T) < \infty$, *and* $\mathfrak{B}(T) \to \infty$ *as* $T \to \infty$; (*v*) *for* $T < \infty$, *the kernel has countably many eigenvalues* $\mu_j < 0$. *They are ordered as* $|\mu_1| > |\mu_2| > |\mu_3| \cdots$, *and* $|\mu_j| \to 0$ *as* $j \to \infty$. *The corresponding eigenfunctions*, ϕ_j, *form an orthonormal base*.

The lemma yields all the results obtained with the ODE system, as we briefly sketch next. In the case of $T = \infty$, existence and uniqueness of the solution follow by a contraction mapping argument, provided that $\theta\mu_1 < 1$; moreover, using that $K \leq 0$ and $\mu_j < 0$, one immediately gets the monotonicity of the solution. For $T < \infty$, and any θ, the "Fredholm alternative theorem" implies that if $\theta \neq \theta_j \equiv 1/\mu_j$ for all j, then there is a unique solution given by $X(t) = \sum_{j=1}^{\infty} \frac{\langle \phi_j, X_0 \rangle}{1-\theta\mu_j} \phi_j(t) = -Y_\theta(t)$, where ϕ_j are the eigenfunctions of K, where $X_0 \equiv X(0)e^{-\zeta t}$, and $\langle \cdot, \cdot \rangle$ is the corresponding inner product. Alternatively, if $\theta = \theta_j \equiv 1/\mu_j$, there is no solution. Since by the Perron–Frobenius theorem ϕ_1 is the only positive eigenfunction, one can see that, as θ converges to $1/\mu_1$, there must be a hump on $Y_\theta(t) = -X(t)$. Moreover, from the expansion in term of eigenfunctions, one can see that when θ crosses θ_j the solution goes through a pole, that is, it is discontinuous at that critical θ_j. Thus, if $\theta < \theta_1$, the solution is ill-posed. Notice the complete equivalence with the θ_j where the equilibrium fails to exist and the IRF behavior discussed in Proposition 2.

In the more general menu cost problem discussed below, where the firm chooses the optimal adjustment times, analyzing the linear operator defined by an integral equation similar to the one discussed here is the only available solution. Indeed, in the treatment of the model with fixed costs discussed below, Lemma 8 is the analogous result to Lemma 3. Moreover, the analysis of the $T = \infty$ and $T < \infty$ cases follows the same steps in Section 6.1 and Section 6.2 as the one sketched here, where we obtain a similar characterization of the solution in terms of the eigenfunctions in equation (59).

4. EQUILIBRIUM AS A MEAN FIELD GAME: GENERAL CASE

This section introduces the elements to set up the general equilibrium of the problem as a mean field game. We first describe the problem of a firm whose value function u depends on the state x and time t. The firm minimizes the discounted value of the sum of flow cost $B(x + \theta X)^2$ with $B > 0$ as derived in Proposition 1 and the fixed cost of adjustment ψ, where $\rho \geq 0$ is the discount rate. Additionally, with a Poisson probability rate $\zeta > 0$, a "free adjustment opportunity" arrives at time $\bar{\tau}_k$ and the firm can change its price without paying a cost. The possibility of adjusting at any time by paying the fixed cost ψ is the crucial difference with the Calvo model described above.

The firm takes as given a path for $\{X(t)\}$ for $t \in [0, T)$, and a terminal value function $u_T(x)$. As in the previous section, we will assume that u_T equals the steady-state value function $\tilde{u}$ described below. We study the cases when T is finite, and also the limit as $T \to \infty$. The firm chooses stopping times and adjustments $\{\tau_i, J_i\}$, as well as adjustments $\{\bar{J}_k\}$ at the exogenously given times $\{\bar{\tau}_k\}$. The firm value function solves

$$u(x,t) = \min_{\{\bar{J}_k, J_i, \tau_i\}} \mathbb{E}\Bigg[\int_t^T e^{-\rho(s-t)} B\big(x(s) + \theta X(s)\big)^2\,ds + \psi \sum_{i=1,\tau_i \leq T}^{\infty} e^{-\rho\tau_i} + e^{-\rho(T-t)} u_T\big(x(T)\big) \Big| x(t) = x\Bigg],$$

where the state evolves as

$$x(s) = x(t) + \sigma\big[\mathcal{W}(s) - \mathcal{W}(t)\big] + \sum_{i:\tau_i \le s} J_i + \sum_{k:\bar{\tau}_k \le s} \bar{J}_k \quad \text{for all } s \in [t, T],$$

where $\mathcal{W}$ is a standard Brownian motion. The reason why the state of the firm includes time t is the time dependence of $X(t)$. The value function solves the following variational inequalities for all x and $t \in [0, T)$:

$$\begin{aligned} \rho u(x, t) = \min\Big\{ &\rho\Big(\psi + \min_z u(z, t)\Big), \\ &B\big(x + \theta X(t)\big)^2 + \frac{\sigma^2}{2} u_{xx}(x, t) + u_t(x, t) + \zeta\Big(\min_z u(z, t) - u(x, t)\Big)\Big\}, \end{aligned} \tag{14}$$

with $u(x, T) = u_T(x)$ for all x. The first term in the RHS of equation (14) gives the possibility of paying the fixed cost and adjusting. The second has the flow cost plus the continuation, which consists of the effect of uncontrolled changes in x, changes in time t, and the expected change if there is a free adjustment opportunity. The optimal decision rule of the firm at each time t consists of dividing the state space in a region where control is not exercised, the inaction region, and a complementary region where control is exercised and the state is reset by an impulse. Three time paths describe the decision rule: $\underline{x}(t)$, $\bar{x}(t)$, and $x^*(t)$ for $t \in [0, T)$. At a given time t, the optimal rule is represented by the interval $[\underline{x}(t), \bar{x}(t)]$ so that if $x(t)$ is in this interval, the firm does *not* exercise control, that is, inaction is optimal, but if $x(t) \notin (\underline{x}(t), \bar{x}(t))$, then $t = \tau_i$, and the firm exercises control, immediately changing its price with a jump J_i from $x(t^-)$ to $x(t^+) = x^*(t)$. Additionally, the firm will reset its price to $x^*(t)$ if t is a time when a free adjustment opportunity occurs, that is, if $t = \bar{\tau}_k$. We refer to $\underline{x}(t)$ and $\bar{x}(t)$ as the boundaries of the range of inaction, to $x^*(t)$ as the optimal return point.

Mean Field Game (MFG) Definition. Given initial and terminal conditions m_0, u_T, a mean field game consists of the functions u, m, mapping $\mathbb{R} \times [0, T]$ to $\mathbb{R}$, and functions $\underline{x}$, $\bar{x}$, x^*, X mapping $[0, T]$ to $\mathbb{R}$. The equilibrium is given by the solution of the coupled system of partial differential equations: the HJB equation for the firm's value function u, and the KFE for the cross-sectional density m. For all $t \in [0, T]$ and for all $x \in [\underline{x}(t), \bar{x}(t)]$, the equations are

$$\begin{aligned} 0 = {} & u_t(x, t) - \rho u(x, t) + \frac{\sigma^2}{2} u_{xx}(x, t) + B\big(x + \theta X(t)\big)^2 \\ & + \zeta\big[u\big(x^*(t), t\big) - u(x, t)\big], \end{aligned} \tag{15}$$

$$0 = -m_t(x, t) + \frac{\sigma^2}{2} m_{xx}(x, t) - \zeta m(x, t) \quad \text{for } x \neq x^*(t), \tag{16}$$

where

$$X(t) = \int_{\underline{x}(t)}^{\bar{x}(t)} x m(x, t)\, dx \quad \text{and} \quad x^*(t) = \arg\min_x u(x, t). \tag{17}$$

The boundary and terminal conditions for u are

$$u_x(\bar{x}(t), t) = u_x(\underline{x}(t), t) = u_x(x^*(t), t) = 0 \quad \text{for all } t \in [0, T], \tag{18}$$

$$u(\bar{x}(t), t) = u(\underline{x}(t), t) = u(x^*(t), t) + \psi \quad \text{for all } t \in [0, T], \tag{19}$$

$$u(x, T) = u_T(x) \quad \text{for all } x. \tag{20}$$

The boundary and initial conditions for m are

$$0 = m(\bar{x}(t), t) = m(\underline{x}(t), t) \quad \text{for all } t \in [0, T], \tag{21}$$

$$\zeta = \frac{\sigma^2}{2}\left[m_x(x^{*-}(t), t) - m_x(x^{*+}(t), t) + m_x(\bar{x}(t), t) - m_x(\underline{x}(t), t)\right] \quad \text{for all } t \in [0, T], \tag{22}$$

$$m(x, 0) = m_0(x) \quad \text{for all } x. \tag{23}$$

We now comment on the assumptions used above. First, the boundary conditions for the HJB in equation (18) are typically referred to as "smooth pasting" and "optimal return point," and the ones in equation (19) are referred to as "value matching." They follow from optimality and are a consequence of our assumption that, for each t, the value function $u(\cdot, t)$ is once differentiable for all x, and twice differentiable in the range of inaction. In particular, for any x outside the range of inaction, the value function must satisfy $u(x, t) = u(x^*(t), t) + \psi$.

Second, we will assume throughout that the inaction region is connected, that is, given by a single interval $[\underline{x}(t), \bar{x}(t)]$, which is without loss of generality for our purposes. Third, the density outside the inaction region, that is, where $u(x, t) = \psi + u(x^*(t), t)$, is zero. In particular, $m(x, t) = 0$ for all $x \notin [\underline{x}(t), \bar{x}(t)]$. Then, assuming continuity of $m(\cdot, t)$ for all x, we obtain the boundary condition in equation (21). This is the condition to be expected at the boundaries of the range of inaction, since no density can accumulate at these "exit" points. Fourth, the Kolmogorov forward equation does not hold at $x = x^*(t)$ since this is an "entry" point, at which $m_x(\cdot, t)$ is discontinuous. The condition in equation (22) equates the flow that probability entering at $x^*(t)$ with the sum of the probability per unit of time that comes from the exit points $\bar{x}(t)$ and $\underline{x}(t)$ plus the one that comes from free adjustments from everywhere. As a consequence of these boundary conditions, the probability is preserved, that is,

$$1 = \int_{\underline{x}(t)}^{\bar{x}(t)} m(x, t)\, dx \quad \text{for all } t \in [0, T]. \tag{24}$$

See Bertucci (2020) for a derivation of the boundary conditions for m for related problems.

Fifth, the equilibrium features a fixed point. The value function and its optimal policy $\{u, \underline{x}, x^*, \bar{x}\}$ are solved for a given path X, and likewise the density m is solved for a given policy $\{\underline{x}, x^*, \bar{x}\}$. They are coupled by requiring that the average value is consistent with both: $X(t) = \int_{\underline{x}}^{\bar{x}} x m(x, t)\, dx$ in equation (17).

Sixth, recall that the condition $\theta < 0$ corresponds to the case of strategic complementarities, and $\theta > 0$ to the case of strategic substitutability. We are particularly interested in $\theta < 0$, but we will cover both cases. The standard case treated in the MFG literature considers $\theta > 0$, which corresponds to the "monotonicity" condition that is at the center of the argument for uniqueness.[8]

Steady State: Initial and Terminal Conditions. We describe the stationary version of the MFG. Let $\bar{x}_{ss}$, $\underline{x}_{ss}$, and x^*_{ss} be three time-invariant thresholds, and let $\tilde{u}$ and $\tilde{m}$ be two time-invariant functions with domain in $[\underline{x}_{ss}, \bar{x}_{ss}]$ solving

$$\begin{aligned} 0 &= -\rho\tilde{u}(x) + \frac{\sigma^2}{2}\tilde{u}_{xx}(x) + B(x+\theta X_{ss})^2 + \zeta\big(\tilde{u}(x^*_{ss}) - \tilde{u}(x)\big) \\ &\quad \text{for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}], \\ 0 &= \frac{\sigma^2}{2}\tilde{m}_{xx}(x) - \zeta\tilde{m}(x) \quad \text{for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}], x \neq x^*_{ss}, \end{aligned} \tag{25}$$

where $X_{ss} = \int_{\underline{x}_{ss}}^{\bar{x}_{ss}} x\tilde{m}(x)\,dx$, with boundary conditions: $\tilde{u}_x(\bar{x}_{ss}) = \tilde{u}_x(\underline{x}_{ss}) = \tilde{u}_x(x^*_{ss}) = 0$, $\tilde{u}(\bar{x}_{ss}) = \tilde{u}(\underline{x}_{ss}) = \tilde{u}(x^*_{ss}) + \psi$, and $0 = \tilde{m}(\underline{x}_{ss}) = \tilde{m}(\bar{x}_{ss})$.

When $\zeta > 0$, we have the symmetric stationary distribution $\tilde{m}$ given by

$$\tilde{m}(x) = \frac{\ell}{2}\frac{e^{\ell(2\bar{x}_{ss}-x)} - e^{\ell x}}{\big(1-e^{\ell\bar{x}_{ss}}\big)^2} \quad \text{for } x \in [0, \bar{x}_{ss}], \tag{26}$$

where $\tilde{m}(x) = \tilde{m}(-x)$ for $x \in [-\bar{x}_{ss}, 0]$, and $\ell \equiv \sqrt{\frac{2\zeta}{\sigma^2}}$. We have the following:

PROPOSITION 3: *If $\theta \neq -1$, then $X_{ss} = 0$ is the only stationary state and it is independent of θ. If $\theta = -1$, then any X_{ss} is a steady state.*

5. SOLVING THE MFG AFTER A "SMALL SHOCK"

In this section, we develop results to analyze the dynamic response to a monetary shock in the presence of strategic interactions. We analyze the effect of a *small* shock by solving an equilibrium starting with an initial condition different from the steady state, what is sometimes referred to as an MIT-shock. In our case, the state is given by an infinite-dimensional object, that is, a cross-sectional distribution. To preserve analytic clarity and tractability, we analyze the equilibrium that follows a perturbation of the economy at the steady state.

The section is organized in three parts. In Section 5.1, we linearize the HJB equation for the firm's problem and solve it analytically. In Section 5.2, we linearize the KFE for the dynamics of the cross-sectional distribution and solve it analytically. In Section 5.3, we derive the fixed point implied by the HJB and the KFE equations and provide a characterization of the resulting kernel that will be central in the analysis of the equilibrium.

[8]In terms of the notions used in the MFG literature, letting m_i be an arbitrary measure and $X_i \equiv \int x\,dm_i$, the definition of monotonicity applied to the period return $B(x+\theta X)^2$ is that, for any two $m_1 \neq m_2$, the following inequality must hold: $0 < \int(B(x+\theta X_1)^2 - B(x+\theta X_2)^2)(dm_1(x) - dm_2(x)) = 2B\theta(X_1 - X_2)^2$. Hence, the monotonicity condition in MFGs corresponds to $\theta > 0$, or strategic substitutability.

Terminal and Initial Conditions for MFG. We use the stationary solution to define the initial density m_0 and the terminal value function u_T. For the initial condition, we consider a perturbation ν of the stationary density $\tilde{m}$, where we use the parameter δ to index the size of the perturbation, so

$$m_0(x) = \tilde{m}(x) + \delta\nu(x), \quad \text{where} \int_{\underline{x}_{ss}}^{\bar{x}_{ss}} \nu(x)\,dx = 0, \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}]. \tag{27}$$

A particular perturbation that we will consider, because of its prominence in the monetary economics literature, is one corresponding to an unanticipated aggregate nominal shock that changes the nominal costs of all firms by an amount δ, so that the density before any decision is taken is $m_0(x) = \tilde{m}(x+\delta)$. In this case, $\nu(x) = \tilde{m}_x(x)$.[9]

For the terminal condition, we set $u_T(x) = \tilde{u}(x)$ for all $x \in [\underline{x}_{ss}, \bar{x}_{ss}]$ and $u_T(x) = \tilde{u}(x^*_{ss}) + \psi$ for all $x \notin [\underline{x}_{ss}, \bar{x}_{ss}]$ so that at time $t = T$, the continuation corresponds to the steady-state value function. The interpretation of the terminal condition $u_T(x) = \tilde{u}(t)$ is that the problem of the firm can be regarded as an infinite-horizon problem. In this case, T measures the horizon over which the strategic interactions apply.

Normalization. To simplify the exposition, we normalize the parameters of the problem so that, at steady state, $\bar{x}_{ss} = 1$. In particular, given $\{\sigma^2, B, \rho, \zeta\}$, we set the fixed cost ψ so that $\bar{x}_{ss} = 1$. This amounts to measuring the shock δ in units of standard deviation of steady-state price changes, that is, in units of $\sqrt{\mathrm{Var}(\Delta p)}$. Moreover, we also define

$$k \equiv \frac{\sigma^2}{2}, \qquad \eta \equiv \sqrt{\frac{\rho+\zeta}{k}}, \qquad \ell \equiv \sqrt{\frac{2\zeta}{\sigma^2}}. \tag{28}$$

For future reference, the average number of price changes in steady state is given by $N = \zeta(\frac{\cosh(\ell)}{\cosh(\ell)-1})$ for $\ell > 0$ and $N = 2k$ for $\ell = 0$.

The Benchmark Initial Condition. In general, $m_0 : [-1, 1] \to \mathbb{R}$ given by equation (27) for some $\nu(x)$. As mentioned, in most of the analysis, we focus on $\nu(x) = \tilde{m}_x(x)$ to relate to the effects of a permanent monetary shock. Direct computation on equation (26) gives

$$\begin{aligned} \tilde{m}_x(x) &= -\frac{\ell^2}{2}\frac{e^{\ell(2-x)} + e^{\ell x}}{\left(1-e^{\ell}\right)^2} \quad \text{if } \ell > 0, \quad \text{or} \\ \tilde{m}_x(x) &= -1 \quad \text{if } \ell = 0, \quad \text{for all } x \in (0, 1], \end{aligned} \tag{29}$$

where, for $x \in [-1, 0)$, we use that $\tilde{m}_x$ is antisymmetric, that is, $\tilde{m}_x(x) = -\tilde{m}_x(-x)$.

Equilibrium for Symmetric Initial Conditions. Next, we establish that if the initial displaced distribution m_0 is symmetric, that is, if $m_0(x) = m_0(-x)$, then the equilibrium cross-section average has no dynamics $X(t) = X_{ss} = 0$, that is, a flat impulse response. This result is important because it will allow us to ignore the symmetric component of the initial perturbation $\nu(x)$, and to focus on the antisymmetric part. We have the following:

[9]The interpretation of this initial condition is that, after the monetary shock δ, the nominal cost jumps immediately and hence the value of the state x for each firm jumps from x to $x - \delta$.

PROPOSITION 4: *Let $m_0(x)$ be a symmetric distribution with support on $[-1, 1]$, that is, $m_0(x) = m_0(-x)$ and $\int_{-1}^{1} m_0(x)\,dx = 1$. Then there exists an equilibrium with $X(t) = X_{ss} = 0$, $\bar{x}(t) = \bar{x}_{ss} = 1$, $\underline{x}(t) = \underline{x}_{ss} = -1$, and $x^*(t) = x^*_{ss} = 0$ for all $t \in [0, T]$ and where $m(x, t)$ is symmetric in x for all $t \in [0, T]$. This equilibrium is unique in the class of symmetric m.*

A few comments are in order. First, while $X(t) = X_{ss} = 0$, the distribution $m(\cdot, t)$ evolves through time. Second, the proposition establishes uniqueness of the equilibrium only among those in which m is symmetric. A symmetric displacement can be generated by shocking, for example, the variance of the fundamental shocks (σ^2), or the market power of firms (B).

5.1. *Linearization and Solution of the HJB Equation*

This section derives a linearization of the HJB with respect to the shock δ. We consider an equilibrium with $\{\bar{x}(t, \delta), \underline{x}(t, \delta), x^*(t, \delta), X(t, \delta), u(x, t, \delta), m(x, t, \delta)\}$, where δ indexes the perturbation of the initial condition for a given ν. We differentiate all the equilibrium objects with respect to δ and evaluate them at $\delta = 0$. For all $t \in [0, T]$, we denote these derivatives as follows:

$$v(x, t) \equiv \frac{\partial}{\partial \delta} u(x, t, \delta)\bigg|_{\delta=0} \quad \text{for all } x \in [-1, 1],$$

$$n(x, t) \equiv \frac{\partial}{\partial \delta} m(x, t, \delta)\bigg|_{\delta=0} \quad \text{for all } x \in [-1, 1], x \neq 0,$$

$$\bar{z}(t) \equiv \frac{\partial}{\partial \delta} \bar{x}(t, \delta)\bigg|_{\delta=0}, \qquad \underline{z}(t) \equiv \frac{\partial}{\partial \delta} \underline{x}(t, \delta)\bigg|_{\delta=0}, \qquad z^*(t) \equiv \frac{\partial}{\partial \delta} x^*(t, \delta)\bigg|_{\delta=0} \quad \text{and}$$

$$Z(t) \equiv \frac{\partial}{\partial \delta} X(t, \delta)\bigg|_{\delta=0}.$$

Once these derivatives are solved for, all objects of interest can be computed as, for example, $u(x, t, \delta) \approx \tilde{u}(x) + \delta v(x, t)$, $m(x, t, \delta) \approx \tilde{m}(x) + \delta n(x, t)$, or $X(t, \delta) \approx \delta Z(t)$ since we consider a perturbation around the steady state, where the approximation error is of order smaller than δ.

We study the evolution of the derivative of the value function, $v(x, t)$, as function of the path of the average price gap $\{Z(t)\}$. To do so, we first obtain the PDE and boundary conditions that $v(\cdot, t)$ satisfies. We then look for an explicit solution of $v(\cdot, t)$, which we use to compute the thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$ as a function of the path of $\{Z(t)\}$.

Linearization of the HJB and Its Boundary Conditions. We differentiate $u(x, t, \delta)$, $x^*(t)$, and $X(t)$ in the HJB equation (15) with respect to δ for $x \in [-1, 1]$, $t \in (0, T)$, and evaluate the derivatives at $\delta = 0$, to obtain

$$0 = v_t(x, t) - (\rho + \zeta) v(x, t) + k v_{xx}(x, t) + 2B\theta x Z(t) + \zeta v(0, t) \quad \text{for } x \in [-1, 1], t \in (0, T), \tag{30}$$

where $2B\theta x Z(t)$ is the derivative with respect to δ of the flow cost, and where we use that $u_x(x^*, t) = 0$ for all t.

Differentiating the value matching conditions in equation (19) with respect to δ, for example, $u(\bar{x}(t,\delta),t,\delta)=\psi+u(x^*(t,\delta),t,\delta)$, and evaluating them at $\delta=0$, we get[10]

$$v(-1,t)=v(0,t), \qquad v(1,t)=v(0,t) \quad \text{for all } t\in(0,T), \tag{31}$$

where we use that $u(x,t,\delta)|_{\delta=0}=\tilde{u}(x)$ and that $\tilde{u}_x(-1)=\tilde{u}_x(0)=\tilde{u}_x(1)=0$. Using the boundary condition at $t=T$, that is, that the firm's value function is independent of δ, gives

$$0=v(x,T) \quad \text{all } x\in[-1,1]. \tag{32}$$

Solution of the HJB Equation. We prove two intermediate results before characterizing the dynamics of the optimal thresholds.

LEMMA 4: *The function $v(x,t)$ is antisymmetric in x for each t, that is, $v(x,t)=-v(-x,t)$ for all $x\in[-1,1]$ and $t\in[0,T]$, and hence it satisfies the boundary condition*:

$$0=v(-1,t)=v(1,t)=v(0,t) \quad \textit{all } t\in(0,T). \tag{33}$$

We can solve the PDE for v in equation (30) for all t, x, which is the heat equation with source $2B\theta xZ(t)$, with a zero space boundary at $t=T$ given by equation (32), and the boundary conditions given in equation (33). We summarize this in the following lemma.

LEMMA 5: *Given the source $Z(t)$ for all $t\in[0,T]$, then the unique solution of the heat equation* (30) *with the Dirichlet boundary conditions in equation* (33) *for all $t\in[0,T]$, and with terminal space condition $v(x,T)=0$ for all $x\in[0,1]$, is*

$$v(x,t)=-4B\theta\int_t^T\sum_{j=1}^{\infty}e^{(\eta^2+(j\pi)^2)k(t-\tau)}Z(\tau)\frac{(-1)^j}{j\pi}\sin(j\pi x)\,d\tau, \tag{34}$$

where the constants η and k are defined in equation (28).

Differentiating the smooth pasting conditions in equation (18) with respect to δ, for example, $u_x(\bar{x}(t,\delta),t,\delta)=0$, and evaluating it at $\delta=0$ gives $v_x(1,t)+\tilde{u}_{xx}(1)\bar{z}(t)=0$. This equation, together with Lemma 5, allows us to characterize the dynamics of the optimal thresholds $\{\underline{z}(t),\bar{z}(t)\}$. The next proposition summarizes the nature of the optimal decision rules for a firm facing a path of future values for the cross-sectional average price gap or markup.

PROPOSITION 5: *Taking as given a path $Z(t)$ for $t\in[0,T]$, the solution to the firm's problem implies the following path for its optimal thresholds $\{\underline{z}(t),z^*(t),\bar{z}(t)\}$*:

$$\bar{z}(t)=\bar{T}(Z)(t)\equiv\theta\bar{A}\int_t^T\bar{H}(\tau-t)Z(\tau)\,d\tau \quad \textit{for all } t\in[0,T), \tag{35}$$

$$z^*(t)=T^*(Z)(t)\equiv\theta A^*\int_t^T H^*(\tau-t)Z(\tau)\,d\tau \quad \textit{for all } t\in[0,T), \tag{36}$$

[10]For example, the derivative at the high threshold gives $v(1,t)+\tilde{u}_x(1)\bar{z}(t)=v(0,t)+\tilde{u}_x(0)z^*(t)$.

where $\underline{z}(t) = \bar{z}(t)$ *and where* $\bar{H}$ *and* H^* *are defined as*

$$\begin{aligned} \bar{H}(s) &\equiv \sum_{j=1}^{\infty} e^{-(\eta^2+(j\pi)^2)ks} > 0, \\ H^*(s) &\equiv \sum_{j=1}^{\infty} e^{-(\eta^2+(j\pi)^2)ks}(-1)^j < 0 \quad \text{for all } s > 0, \end{aligned} \tag{37}$$

and

$$\begin{aligned} \bar{A} &\equiv \frac{4B}{\tilde{u}_{xx}(1)} = k\frac{2\eta^2}{[1-\eta\coth(\eta)]} < 0, \\ A^* &\equiv \frac{4B}{\tilde{u}_{xx}(0)} = k\frac{2\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} > 0. \end{aligned} \tag{38}$$

The ratio $A^*/|\bar{A}|$ *is strictly increasing in* η, *with* $\frac{\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} \to 6$, $|\frac{\eta^2}{[1-\eta\coth(\eta)]}| \to 3$ *as* $\eta \to 0$.

A few comments are in order. First, the current value of the thresholds $z^*(t)$ and $\bar{z}(t)$ depends on future values of the average price gap $Z(\tau)$ with $\tau \in (t, T)$. In this sense, this mapping is forward-looking.

Second, the result that $\bar{z}(t) = \underline{z}(t)$ means that the width of the inaction region, but not its position, is constant through time. The economics of this result is that the width of the inaction region reflects the option value of waiting, which is mainly affected by σ^2, the curvature of the payoff function, and the fixed costs. Since none of these objects is affected by the monetary shock, the width of the inaction region stays constant. While the width is constant, its position and the location of the optimal return point within it change through time.

Third, θ only appears multiplicatively in the expressions for z^* and $\bar{z}$, since neither $\bar{A}$, A^* nor $\bar{H}$, H^* depend on it. Thus, in the special case without strategic interactions, $\theta = 0$, the thresholds are kept at the steady-state values, that is, $z^* = \bar{z} = 0$.

Fourth, given the sign of the expressions above, if there is strategic complementarity ($\theta < 0$), a firm facing higher values of $Z(\tau)$ for $\tau \geq t$ sets a higher value of the optimal return $z^*(t)$, and a larger value of both the upper and lower thresholds of the inaction band, $\bar{z}(t)$, $\underline{z}(t)$. If $\theta > 0$, the result is the opposite. The strength of the result depends on θ as well as on $\eta \equiv \sqrt{2(\rho + \zeta)/\sigma^2}$. Also, as expected, values of $Z(\tau)$ closer to t receive higher weight in the firm's optimal decisions $\{\bar{z}, z^*\}$. The parameter η also enters into the expressions for $\bar{A}$ and A^*, which reflect how the curvature of the value function changes as η changes. The reason that $\tilde{u}_{xx}$ appears in the expressions is because we are perturbing the economy around the steady state. Equation (38) shows that the curvature of the steady-state value function $\tilde{u}_{xx}$, characterized in Lemma 10, affects the speed of convergence.

5.2. *Linearization and Solution of the KF Equation*

In this subsection, we study the evolution of $n(x, t)$ as function of the path of thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$. To do so, we first obtain the PDE and boundary conditions that $n(\cdot, t)$ satisfies. We then look for an explicit solution of $n(\cdot, t)$, which we use to compute $Z(t)$ as a function of the path of thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$.

Linearization of the KFE and Its Boundary Conditions. We differentiate the KFE for $m(x, t, \delta)$ given in equation (16) with respect to δ at each (x, t) to obtain

$$0 = -n_t(x, t) + kn_{xx}(x, t) - \zeta n(x, t) \quad \text{in } x \in [-1, 1], t \in (0, T), x \neq 0. \tag{39}$$

Differentiating the boundary condition $m(\bar{x}(t, \delta), t, \delta) = 0$ in equation (21) with respect to δ for each t, we get $0 = n(1, t) + \tilde{m}_x(1)\bar{z}(t)$. Likewise, differentiating the boundary condition $m(\underline{x}(t, \delta), t, \delta) = 0$ with respect to δ, we get $0 = n(-1, t) + \tilde{m}_x(-1)\underline{z}(t)$. Then the boundary conditions are

$$n(1, t) = -\tilde{m}_x(1)\bar{z}(t) \quad \text{and} \quad n(-1, t) = -n(1, t) \quad \text{all } t \in (0, T), \tag{40}$$

where we used that $\bar{z}(t) = \underline{z}(t)$ from Proposition 5. The reason why $\tilde{m}_x$ appears is because we are perturbing the economy around the steady state.

Differentiating the mass preservation equation (24) with respect to δ, we obtain: $0 = \int_{-1}^{1} n(x, t)\, dx$ for all $t \in (0, T)$. Differentiating this equation with respect to time and using the KFE in equation (39), we have

$$0 = n_x(1, t) - n_x(0^+, t) + n_x(0^-, t) - n_x(-1, t) \quad \text{all } t \in (0, T). \tag{41}$$

The initial condition for n comes from differentiating $m_0(x)$ with respect to δ; this gives

$$n(x, 0) = \nu(x) \quad \text{for } x \in (-1, 1), \tag{42}$$

which in the benchmark case of the small monetary shock is $n(x, 0) = \tilde{m}_x(x)$, whose expression was given by equation (29). Given n, we can compute $Z(t)$ as

$$Z(t) = \int_{-1}^{1} xn(x, t)\, dx \quad \text{all } t \in (0, T). \tag{43}$$

Equilibrium of the Perturbed Mean Field Game. The equilibrium of the MFG with initial condition given by the perturbation ν is described by functions $\{Z, \bar{z}, z^*, n\}$ that solve equations (35), (36), (39), (40), (41), (42), and (43).

Irrelevance of the Symmetric Component of the Perturbation ν. Any perturbation ν can be written as the sum of a symmetric component and an antisymmetric component. Given the linearity of the system, the equilibrium for a given ν is obtained as the sum of the equilibrium that corresponds to each of the components. The next corollary highlights a straightforward consequence of Proposition 4.

COROLLARY 1: *Let $\nu(x)$ be symmetric around $x = 0$. Then there is an equilibrium for this initial condition with $Z(t) = 0$ for all $t \in [0, T]$. This equilibrium is unique in the class of symmetric $n(x, t)$.*

Proposition 4 established the result for an equilibrium with an arbitrary symmetric initial condition, not just a perturbation. The perturbation can be obtained using $n(x, t) = (m(x, t) - \tilde{m}(x))/\delta$, including $\nu(x) = (m_0(x, t) - \tilde{m}(x))/\delta$. Intuitively, a symmetric displacement of the steady-state distribution has no effect on the mean of the distribution, Z. Given the symmetric law of motion for x, the mean remains at the steady-state value.

Solution of the KFE for an Antisymmetric ν. We will look for a solution of n that satisfies the PDE given in equation (39), its boundary condition in equation (40), mass preservation as given by equation (41), and the initial condition for $n(\cdot, 0)$.

First, we define the left and right limits of $n(\cdot, t)$ as $a(t)$ and $b(t)$, respectively:

$$n(0^+, t) = b(t) \quad \text{all } t \geq 0 \quad \text{and} \quad n(0^-, t) = a(t) \quad \text{all } t \geq 0.$$

Given the boundary behavior and the initial conditions, it is natural to look for antisymmetric solutions. Indeed, the next lemma shows that this has to be the case.

LEMMA 6: *If the initial condition is antisymmetric, that is, $\nu(x) = -\nu(-x)$, and $a(t) + b(t)$ is continuous on $(0, T]$, and n satisfies equation (39), equation (40), equation (41), then $n(x, t)$ is antisymmetric in x for all t, and thus $a(t) = -b(t)$ for all $t \in [0, T]$.*

Next, we use the antisymmetric nature of n to find an expression for $b(t) - a(t)$ in terms of the threshold $z^*(t)$.

LEMMA 7: *Assume that $m(x^*(t, \delta), t, \delta)$ is continuous, and right- and left-differentiable at $\delta = 0$. Then $z^*(t) = \frac{a(t)-b(t)}{2\tilde{m}_x(0^+)}$.*

The antisymmetric nature of n and Lemma 7 have the important implication that

$$b(t) = n(0^+, t) = -\tilde{m}_x(0^+) z^*(t) = -n(0^-, t) = -a(t) \quad \text{for all } t \geq 0.$$

Next, we present a PDE that $n(x, t)$ must satisfy. The key simplification is that, due to the antisymmetric nature of $n(x, t)$, it suffices to define it for $x \in (0, 1]$, for every t. Moreover, being antisymmetric, the mass preservation is satisfied. Finally, the characterization in Lemma 7 gives us a boundary condition at $x = 0$ for all t. Hence, the system given by equation (39), (40), (41), and (42) becomes the following system:

$$n_t(x, t) = k n_{xx}(x, t) - \zeta n(x, t) \quad \text{for } x \in [0, 1] \text{ and } t > 0, \tag{44}$$

$$n(1, t) = -\tilde{m}_x(1)\bar{z}(t) \quad \text{and} \quad n(0, t) = -\tilde{m}_x(0^+) z^*(t) \quad \text{for all } t > 0, \tag{45}$$

$$n(x, 0) = \nu(x) \quad \text{for } x \in [0, 1]. \tag{46}$$

The above system is well understood. It corresponds to a one-dimensional heat equation with a bounded spatial domain, an initial spatial condition, and a specification of time-varying values on the boundaries of the domain (see Chapter 6 in Cannon (1984)). The initial condition is given by ν and the time-varying boundaries are given by z^* and $\bar{z}$. This equation has a unique solution that can be written in terms of these three functions. The solution is a linear functional of z^*, $\bar{z}$, and ν, it is algebraic intensive, and explicit expressions are given in Lemma 11 in Appendix B. We use this explicit solution to write the impulse response of the mean $Z(t)$ for given path of the thresholds $\{\bar{z}(t), z^*(t)\}$, using the expression for $Z(t)$ in equation (43). We have the following:

PROPOSITION 6: *Taking as given the paths of $\{z^*(t), \bar{z}(t)\}$, and an initial condition given by an antisymmetric perturbation $\nu(x)$, the solution of the KFE gives the following path for the average value $\{Z(t)\}$:*

$$Z(t) = T_Z(z^*, \bar{z})(t) \equiv Z_0(t) + 4k \int_0^t G^*(t - \tau) z^*(\tau)\, d\tau + 4k \int_0^t \bar{G}(t - \tau)\bar{z}(\tau)\, d\tau \tag{47}$$

for all $t \in [0, T]$ and where $\bar{G}$, G^*, and Z_0, are defined as

$$\bar{G}(s) \equiv -\tilde{m}_x(1) \sum_{j=1}^{\infty} e^{-(\ell^2+(j\pi)^2)ks} > 0 \quad \text{and} \quad G^*(s) \equiv -\tilde{m}_x(0^+) \sum_{j=1}^{\infty} (-1)^{j+1} e^{-(\ell^2+(j\pi)^2)ks} > 0$$

for all $s \geq 0$, $\tilde{m}_x(1)$ and $\tilde{m}_x(0^+)$ are given in equation (29), and

$$Z_0(t) \equiv -4 \sum_{j=1}^{\infty} (-1)^j \frac{e^{-(\ell+(j\pi)^2)kt}}{j\pi} \int_0^1 \sin(j\pi x)\nu(x)\,dx, \tag{48}$$

where the constants k and ℓ are defined in equation (28).

This proposition gives the evolution of the average price gap or markup, $Z(t)$, as a function of the path of decisions up to time t. The current value of $Z(t)$ depends on past values of the thresholds $\{z^*(\tau), \bar{z}(\tau)\}$ with $\tau \in (0, t)$. In this sense, the mapping is backward-looking. Given our normalization, the mapping T_Z depends only on $k \equiv \sigma^2/2$ and ℓ.

A few remarks are due. We note that the expression for $Z(t)$ is made of two parts. The first one, $Z_0(t)$, gives the dynamics of the average price gap due to the displacement ν of the initial distribution when the thresholds are constant, that is, $\bar{z} = z^* = 0$. It corresponds to the impulse response of the average price gap in an economy where there are no strategic interactions, that is, $\theta = 0$. The other part, given by the two integrals, describes the effect on $Z(t)$ caused by past changes of the thresholds.

Second, the mapping is monotone, as larger values of past thresholds lead to larger values of the average markup $Z(t)$, that is, $G^*(s) > 0$ and $\bar{G}(s) > 0$ for all $s > 0$. Also note that the pairs $\{z^*(\tau), \bar{z}(\tau)\}$ for τ closer to t receive a higher weight than those further away in time.

Third, for the benchmark initial condition for a monetary shock, where $\nu = \tilde{m}_x$, as in equation (29), and without strategic interactions $\theta = 0$, Alvarez and Lippi (2022) showed that

$$Z_0(t) = \begin{cases} 2 \displaystyle\sum_{j=1}^{\infty} \frac{\ell^2}{\ell^2 + (j\pi)^2} \left(\frac{(-1)^j (1 + e^{2\ell}) - 2e^{\ell}}{(1 - e^{\ell})^2} \right) e^{-(\ell^2+(j\pi)^2)kt} & \text{for } \ell > 0, \\ 4 \displaystyle\sum_{j=1}^{\infty} \frac{[(-1)^j - 1]}{(j\pi)^2} e^{-(j\pi)^2 kt} & \text{for } \ell = 0. \end{cases} \tag{49}$$

We note that $Z_0(0) = -1$, and that $Z(t)$ is increasing and converges to zero as $t \to \infty$.

5.3. *Deriving the Fixed Point*

In this section, we put together the solution for the HJB equation and the KFE derived in Proposition 5 and in Proposition 6, respectively, to arrive to a single linear equation that $\{Z(t)\}$ must solve. We denote the fixed point by $Z = \mathcal{T}(Z)$. The mapping $\mathcal{T}$ is the composition of T_Z with $\bar{T}$ and T^* described above, that is, $\mathcal{T}(Z) = T_Z(T^*(Z), \bar{T}(Z))$. Direct computation gives the following:

PROPOSITION 7: *Let ν be an arbitrary perturbation. The equilibrium of a MFG must solve $Z = \mathcal{T}(Z)$ given by*

$$Z(t) = \mathcal{T}(Z)(t) \equiv Z_0(t) + \theta \int_0^T K(t,s)Z(s)\,ds \quad \textit{all } t \in [0,T], \tag{50}$$

where Z_0 is given by

$$Z_0(t) \equiv -2\sum_{j=1}^{\infty}(-1)^j \frac{e^{-(\ell+(j\pi)^2)kt}}{j\pi}\int_{-1}^{1}\sin(j\pi x)\nu(x)\,dx, \tag{51}$$

and where the kernel K is

$$K(t,s) = 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\big[\bar{A}_\ell - A_\ell^*(-1)^{j+i}\big]\frac{\big[e^{[(j\pi)^2+(i\pi)^2+\eta^2+\ell^2]k(t\wedge s)} - 1\big]e^{-(j\pi)^2kt-\ell^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2+\ell^2}, \tag{52}$$

where the constants η, k, ℓ are defined in equation (28), $\bar{A}_\ell \equiv -\tilde{m}_x(1)\bar{A} < 0$ and $A_\ell^* \equiv -\tilde{m}_x(0^+)A^* > 0$; $\tilde{m}_x$ is given in equation (29), and $\bar{A} < 0$ and $A^* > 0$ in equation (38).

Equation (50) is a non-homogeneous Fredholm integral equation of the second kind, where the parameter is given by θ. The path $\{Z_0\}$ is the solution of the MFG when there are no strategic interactions, that is, when $\theta = 0$, and the perturbation is given by ν. In our benchmark case of a monetary shock, $\nu = \tilde{m}_x$, and then Z_0 is given by equation (49).

An important result is that the kernel K, given in equation (52), is independent of θ, as well as of the initial perturbation ν. This means that we can analyze the impulse responses for different values of θ, or different values of initial shocks (as embedded in Z_0 through the perturbation ν), using a single kernel.[11] We will exploit this property in the characterization of the equilibrium in Proposition 12. This result reveals a surprisingly simple structure of the workings of strategic complementarities. It is also extremely convenient in exploring the results numerically.[12]

We define three objects that will be used below. The first is a notion of inner product between vectors, which we apply to functions of time. For any two functions V, W, we define the inner product $\langle\cdot,\cdot\rangle$ using weights given by the time discount as follows:

$$\langle V, W\rangle \equiv \frac{\rho}{1-e^{-\rho T}}\int_0^T V(t)W(t)e^{-\rho t}\,dt. \tag{53}$$

The second is a linear operator, $\mathcal{K}$, akin to a matrix multiplication:

$$\mathcal{K}(V)(t) \equiv \int_0^T K(t,s)V(s)\,ds \quad \text{for all } t \in [0,T] \tag{54}$$

[11]Notice that since the kernel K arises from the composition of a backward and a forward operator, then $K(t,s)$ is different from zero for all t, s.

[12]While the equilibrium condition for Z in equation (50) uses functional analysis techniques, in Appendix C we use a finite-dimensional approximation, based on standard linear algebra, that has the same structure. This gives an easy to implement numerical method to compute the equilibrium.

for any function $V : [0, T] \to \mathbb{R}$. The third is a bound on the kernel K. This comes in two types that are used for different analyses of the fixed point. One is a Lipschitz bound and the other is a form of L_2 bound:

$$\begin{aligned} \mathrm{Lip}_K &\equiv \sup_{t\in[0,T]} \int_0^T \left|K(t,s)\right| ds \quad \text{and} \\ \|K\|_2^2 &\equiv \frac{\rho^2}{\left(1-e^{-\rho T}\right)^2} \int_0^T \int_0^T K^2(t,s) e^{-\rho(t+s)}\, dt\, ds. \end{aligned} \tag{55}$$

The next lemma gathers important properties of the kernel K that will be used to characterize the equilibrium, as discussed below. The lemma considers the case where $\ell = 0$, which corresponds to the pure Ss model, as well as the case where $\ell > 0$, which typically regularizes the kernel.

LEMMA 8: *Consider the kernel in* (52) *and the inner product in* (53), *and the constants* η, ℓ *defined in equation* (28). *We have*:

1. *K is symmetric if $\rho = 0$, that is, $K(t,s) = K(s,t)$ for all (t,s). For $\rho \geq 0$, the operator $\mathcal{K}$ is self-adjoint, that is, for any V, W, we have $\langle \mathcal{K}V, W\rangle = \langle V, \mathcal{K}W\rangle$*:

$$\int_0^T \int_0^T K(t,s)V(s)W(t)e^{-\rho t}\, ds\, dt = \int_0^T \int_0^T K(t,s)W(s)V(t)e^{-\rho t}\, ds\, dt.$$

2. *All elements of K are negative, that is, $K(t,s) < 0$ for all $(t,s) \in (0,T)^2$.*
3. *$\mathcal{K}$ is negative semidefinite, $\langle \mathcal{K}V, V\rangle \leq 0$, that is, $\int_0^T \int_0^T K(t,s)V(t)V(s)e^{-\rho t}\, dt\, ds \leq 0$.*
4. *If $\ell = 0$, then $\mathrm{Lip}_K < \frac{\eta^2}{18}\left(\frac{1}{1-\eta\,\mathrm{csch}(\eta)} - \frac{4}{1-\eta\coth(\eta)}\right)$. Moreover, for small ρ, we have $\mathrm{Lip}_K < 1 - \frac{7}{180}\eta^2 + o(\eta^2)$.*
5. *Let $K(t,s;\eta,\ell)$ be the kernel as a function of η, ℓ. Then $|K(t,s;\eta,\ell)| \leq |\tilde{m}_x(0^+)| \times |K(t,s;\eta,0)|$ for all $t, s \in [0,T]$.*
6. *If $\ell = 0$, and $\rho \geq 0$, then $\|K\|_2^2 < c_0 \frac{\rho^2 T}{(1-e^{-\rho T})^2}\left(\frac{\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} - \frac{\eta^2}{[1-\eta\coth(\eta)]}\right)$ for a constant $c_0 > 0$ independent of any other parameters.*
7. *If $\ell \geq 0$ and $\rho > 0$, then $\|K\|_2^2 < \rho\left[\frac{1-e^{-2\rho T}+6\rho}{(1-e^{-\rho T})^2}\right]c_1$ for a constant $c_1 > 0$ independent of ρ and T.*

A few remarks are in order. The lemma establishes that the operator $\mathcal{K}$ is self-adjoint (point 1). This property is key for the existence of an orthonormal basis for the operator, and to represent the impulse response using standard eigenvalue-eigenfunction projection methods. The negative-definiteness of $\mathcal{K}$ (point 2) implies that all the eigenvalues are negative. Second, the fact that K is negative for all t, s implies the monotonicity of the equilibrium for $\theta < 0$. Third, the Lipschitz bound (points 4 and 5) gives values of θ for which the right-hand side of equation (50) is a contraction for T unbounded (in Proposition 10). Likewise, the bound for the norm $\|K\|_2^2$ (points 6 and 7) establishes the compactness of the operator $\mathcal{K}$, which together with the self-adjointness, allows us to give conditions for existence, uniqueness, and a characterization of the solution for a finite, arbitrarily large, T (in Proposition 12).

6. EQUILIBRIUM CHARACTERIZATION FOR THE MONETARY SHOCK

In this section, we characterize the dynamic equilibrium. As initial condition, we consider a perturbation ν to the stationary density, focusing on the monetary shock described in equation (29). We cover both the pure Ss model ($\zeta/k \equiv \ell^2 = 0$) as in Golosov and Lucas (2007) and Klenow and Willis (2016), as well as the Calvo-plus model ($\zeta/k \equiv \ell^2 > 0$) as in Nakamura and Steinsson (2010) and Alvarez, Le Bihan, and Lippi (2016). In these models, output is negatively proportional to price gaps, so that, denoting by $Y_\theta(t)$ the impulse response of output to a small monetary shock, we have $Y_\theta(t) = -Z(t)$ where we index the impulse response by the parameter θ and $Y_0(t) \equiv -Z_0(t)$. The impulse response function solves $Y_\theta = \mathcal{T} Y_\theta$ as follows:

$$Y_\theta(t) = (\mathcal{T} Y_\theta)(t) \equiv Y_0(t) + \theta \int_0^T K(t,s) Y_\theta(s)\, ds \quad \text{all } t \in [0,T]. \tag{56}$$

Section Contents. We study the existence and uniqueness of Y_θ, solving the integral equation (56), for different cases. Each of these cases provides new insights on the nature of the solution. In Section 6.1, we restrict $|\theta|$ to be bounded and allow T to be infinite provided that $\rho > 0$. A key result in Proposition 10 shows that the equilibrium exists, it is unique, and it is well posed if $|\theta|$ is bounded. We give a characterization of the impulse response as a function of θ, showing that the size of the response to a monetary shock at any given time t is bigger the larger the strength of strategic complementarity (smaller θ). In Section 6.2, we restrict $T < \infty$ and consider θ arbitrary and $\rho \geq 0$: the finite T allows us to use a spectral theorem and to represent the equilibrium impulse response $Y_\theta(t)$ analytically using the eigenvalues and eigenfunctions of $\mathcal{K}$; see Proposition 12. We show that the impulse response is hump-shaped if the complementarity is sufficiently large. In Section 6.3, we show that larger strategic complementarities increase the variance of output due to monetary shocks. In Section 6.4, we show that the amplification effect of strategic complementarities is similar across the models of the Calvo-plus type.

Our motivation to explore both an infinite horizon, $T \to \infty$ in Section 6.1, as well as a finite arbitrary large T in Section 6.2, is to provide a thorough characterization of the results. The case with $T \to \infty$ allows us to use a contraction theorem to prove the equilibrium existence in Proposition 9 and characterize some properties of the IRF (decreasing and convex in θ) that cannot be proven otherwise. The finite T assumption is used to establish that the operator $\mathcal{K}$ is compact, and is thus key to obtain the spectral representation of the impulse response used in Proposition 11 (see the discussion after Lemma 8).

6.1. *Equilibrium With Bounded* $|\theta|$

In this section, we analyze the case where the strength of the strategic interactions θ is bounded. For future reference, we define the series

$$S_\theta(t) = \sum_{r=0}^{\infty} \theta^r (\mathcal{K})^r (Y_0)(t) \quad \text{for all } t \in [0,T], \tag{57}$$

where $(\mathcal{K})^{r+1}(V)(t) \equiv \int_0^T K(t,s)(\mathcal{K})^r(V)(s)\, ds$ is the rth iteration of $\mathcal{K}$ defined in equation (54). Our first simple result shows that all impulse responses start at the same point, independent of θ:

PROPOSITION 8: *Let* $Y_\theta(t)$ *be the solution of equation* (57). *Then*, *at* $t = 0$, *we have* $Y_\theta(0) = Y_0(0) = 1$.

The next proposition establishes a bound for $|\theta|$, in terms of the fundamental model parameters, that gives a sufficient condition for existence and uniqueness. In particular, we use Lemma 8 to verify the conditions for the Banach contraction fixed-point theorem. This establishes existence and uniqueness of the solution of equation (56) for a range of θ including both positive (strategic substitution) and negative (strategic complementarity) values. Additionally, the proposition allows for any arbitrary initial perturbation ν.

PROPOSITION 9: *Assume that* $T < \infty$ *if* $\rho = 0$, *but otherwise these parameters take arbitrary values*. *Consider any perturbation* ν. *A sufficient condition for the existence and uniqueness of the equilibrium IRF*, *that is*, *of the uniqueness and existence of a solution to equation* (56) *in* $L^1([0, T])$, *is that* $|\theta| \operatorname{Lip}_K < 1$. *In this case*, $Y_\theta(t) = S_\theta(t)$ *as in equation* (57). *A sufficient condition* $|\theta| \operatorname{Lip}_K < 1$ *is*

$$|\theta| \frac{\ell^2}{2} \frac{e^{2\ell}}{(1 - e^{\ell})^2} \frac{\eta^2}{18} \left(\frac{1}{1 - \eta \operatorname{csch}(\eta)} - \frac{4}{1 - \eta \coth(\eta)} \right) < 1.$$

For the special case of $\ell = 0$, *this gives* $|\theta| \frac{\eta^2}{18} (\frac{1}{1-\eta \operatorname{csch}(\eta)} - \frac{4}{1-\eta \coth(\eta)}) < 1$.

The proof of this proposition is an immediate application of the contraction theorem. The modulus of the contraction is given by the $\theta \operatorname{Lip}_K$ bound characterized in Lemma 8 (points 4 and 5). For the pure Ss case, that is, when $\ell = 0$, we can use the approximation for small ρ in part 4 of Lemma 8 to obtain an expression for small η: $|\theta|(1 - \frac{7}{180}\eta^2) < 1$. Thus, for practical purposes in the pure Ss case, we can take the sufficient condition for a contraction to be $|\theta| \leq 1$.[13] We note that the proposition gives a sufficient condition for the infinite sum in equation (57) to converge. When the sufficient condition is violated, there may still be solutions to equation (56), although they cannot be represented by the infinite series in equation (57).

The next proposition gives a characterization of the equilibrium for the case of strategic complementarity ($\theta < 0$) and for an initial perturbation such that $Y_0(t) > 0$.

PROPOSITION 10: *Assume that* $T < \infty$ *if* $\rho = 0$, *but otherwise these parameters take arbitrary values*. *Let* ν *be any perturbation such that* $Y_0(t) > 0$, *and* $\|Y_0\|_\infty < \infty$ *and* $Y_0(t)$ *is continuous*. *Let* $\theta \in (\underline{\theta}, 0]$, *where* $\underline{\theta}$ *is such that the series* S_θ *in equation* (57) *converges*. *The unique solution of equation* (56) *has the following properties*:
1. *For each* $t \in (0, T)$, *the fixed point is positive*, *that is*, $Y_\theta(t) > 0$.
2. *For each* $t \in (0, T)$, *the fixed point* $Y_\theta(t)$ *is* (*strictly*) *monotone decreasing in* θ.
3. *For each* $t \in (0, T)$, *the fixed point* $Y_\theta(t)$ *is* (*strictly*) *convex in* θ.

The proof of this proposition uses that $K \leq 0$ (Lemma 8), and thus for $\theta < 0$ we have that $\theta\mathcal{K}$ is monotone, it has a Lipschitz bound, and preserves the sign of Y_0. The positivity, the monotonicity, and convexity with respect to $\theta < 0$, follow since each term of the series

[13]For this case, $2k \equiv \sigma^2 = N \operatorname{Var}(\Delta p) = N$, where N is the expected number of price changes per unit of time in steady state, and where we use the normalization $\bar{x}_{ss} = 1$ and the definition of k. Thus, when $\eta^2 = \rho/k$, we can write the bound as $\frac{1}{|\theta|} > 1 - \frac{7}{90}\frac{\rho}{N}$.

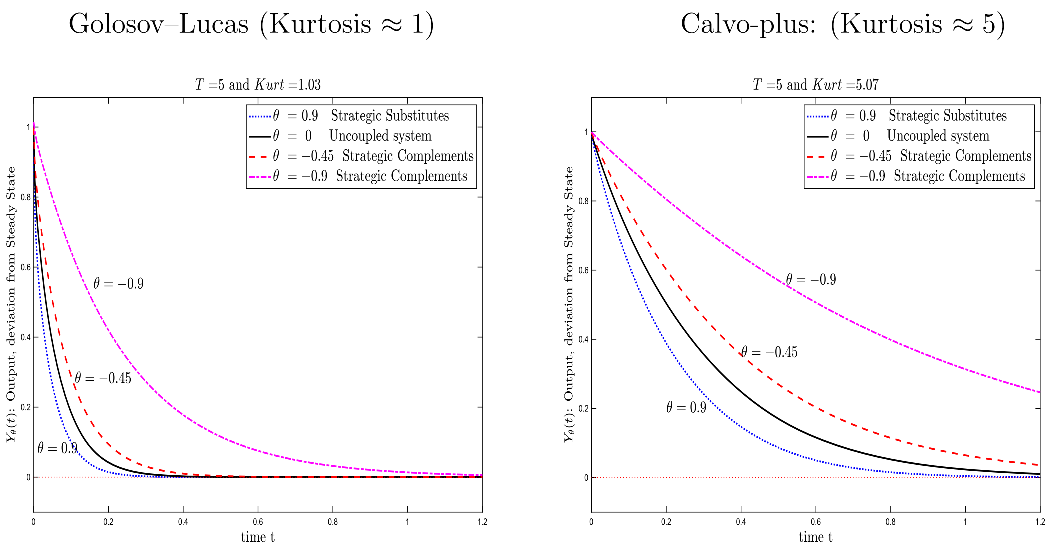

FIGURE 3.—Output response to a monetary shock.

for S_θ satisfies these properties. A few comments are in order. First, if $\nu = \tilde{m}_x(x)$, then $Y_0 > 0$ as assumed in the proposition (see equation (49)). Second, and most importantly, this proposition shows that as the strategic complementarity gets larger (more negative θ), then the aggregate response to the shock is larger, that is, $Y_\theta(t)$ is decreasing in θ at each date t. The proposition also shows that $Y_\theta(t)$ is a convex function of θ at each t. The monotonicity and convexity properties yield the following important corollary:

COROLLARY 2: *The assumptions of Proposition 10 imply that there is a* $-\infty < \underline{\theta} < 0$ *such that* $S_\theta(t) = +\infty$.

Thus, for sufficiently strong strategic complementarity, the series S_θ does not converge. This, in itself, does not imply that there is no equilibrium. We return to this question in the next section, where we show that, indeed, for values of θ sufficiently large (in absolute value), the model is not well posed: it may fail to have an equilibrium or, even when it has one, the equilibrium may not change continuously as a function of the parameters.

In Figure 3, we present the impulse responses produced by different values of θ, while keeping the average frequency of price adjustment constant across models. In the left panel, we display the IRF Y_θ for four values of θ and for $\ell \approx 0$, so it is essentially the pure Ss model, with unit kurtosis as in the Golosov–Lucas model. The figure illustrates Proposition 10: it can be seen that $Y_\theta(t)$ decreases in θ at each t, in a convex fashion. Also, all IRFs start at the same value, that is, $Y_\theta(0) = 1$, and it is evident that for larger strategic complementarity, the IRF is more persistent. The right panel displays the IRF for a version of the Calvo-plus model with a kurtosis of about 5 (given by a large value of ℓ). This model is thus quite close to Calvo, where kurtosis is 6. As in the pure Ss case, the IRFs are decreasing and convex in θ for each t. Comparing the two IRFs for the same θ across the two figures, it can be seen that the Calvo-plus model has a larger IRF than the one for the pure Ss model.

6.2. *Equilibrium Characterization With a Finite T*

In this section, we focus on a finite horizon $T < \infty$ and analyze how the equilibria vary as a function of θ. A main result is to provide an expression for the IRF Y_θ in terms of the projections onto an orthonormal base, and the associated eigenvalues, implied by the kernel of K. We begin by introducing a norm for linear operators, the HS norm, which is equivalent to the sum of the squared eigenvalues:

$$\|\mathcal{K}\|_{\mathrm{HS}}^2 \equiv \sum_{i,j} \big|\langle \mathcal{K} f_i, f_j\rangle\big|^2 = \sum_{i,j} \left(\frac{\rho}{1-e^{-\rho T}} \int_0^T \int_0^T K(t,s) f_i(s) f_j(t) e^{-\rho t}\, ds\, dt \right)^2, \tag{58}$$

where $\{f_j\}$ is any orthonormal base for the linear separable Hilbert space $\mathcal{H}$ of functions $V:[0,T] \to \mathbb{R}$ with $\langle V, V\rangle < \infty$. Recall that $\{\mu_j, \phi_j\}$ is an eigenvalue-eigenfunction pair of the operator $\mathcal{K}$ if the function $\phi_j \in \mathcal{H}$ solves the linear equation $\mu_j \phi_j = \mathcal{K}\phi_j$ for the scalar μ_j. The next proposition, which uses the results of Lemma 8, gives the necessary preliminary results.

PROPOSITION 11: *Assume that $T < \infty$. The HS norm is bounded by $\|\mathcal{K}\|_{\mathrm{HS}}^2 \le T^2 \|K\|_2^2$. In this case, the operator $\mathcal{K}$ is self-adjoint and compact, and thus it has countably many eigenvalues and eigenfunctions that we denote by $\{\mu_j, \phi_j\}_{j=1}^{\infty}$. The eigenvalues μ_j are real, negative, and ordered as $|\mu_1| > |\mu_2| > |\mu_3| > \cdots$, and they converge to zero $|\mu_j| \to 0$ as $j \to \infty$. There are at most finitely many eigenfunctions associated with each non-zero eigenvalue. The eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ form an orthonormal base for $\mathcal{H}$.*

The proposition is an instance of the spectral theorem for compact self-adjoint operators, a basic result in functional analysis; see Section 5 of Chapter II in Conway (2007). That the operator is self-adjoint was shown in part 1 of Lemma 8. That the operator is compact follows from finite Hilbert–Schmidt norm, which, as stated in equation (58), is equal to the L^2 norm of the kernel found in part 7 of Lemma 8. That the eigenvalues are negative follows from part 3 of Lemma 8. In Appendix C of the Supplemental Material, we develop a finite-dimensional approximation, which converts the equilibrium into a simple linear algebra problem where $\mathcal{K}$ is a matrix, and hence it is easily computed. There, we also show the convergence rate of the approximation.

The next proposition provides a main result of this paper: it identifies a range of θ where the equilibrium exists and is unique, and it presents a partial characterization of the impulse response function written in terms of the eigenvalues and eigenfunctions of $\mathcal{K}$.

PROPOSITION 12: *Assume that $T < \infty$. Then*

1. *If $\theta\mu_1 < 1$, there exists a unique equilibrium solving equation (56) given by*

$$Y_\theta(t) = \sum_{j=1}^{\infty} \frac{\langle \phi_j, Y_0\rangle}{1-\theta\mu_j} \phi_j(t) \quad \textit{for all } t \in (0,T). \tag{59}$$

2. *If $\theta \to +\infty$, then $Y_\theta(t) \to 0$ for all $t \in (0,T)$.*
3. *If $\theta = 1/\mu_1$, and ν is such that $Y_0 \ge 0$, then there is no solution to equation (56), that is, there is no equilibrium. Moreover, there is a pole at $\theta = 1/\mu_1$, that is, for all $t \in (0,T)$,*

$$\lim_{\theta \downarrow 1/\mu_1} Y_\theta(t) = +\infty \quad \textit{and} \quad \lim_{\theta \uparrow 1/\mu_1} Y_\theta(t) = -\infty. \tag{60}$$

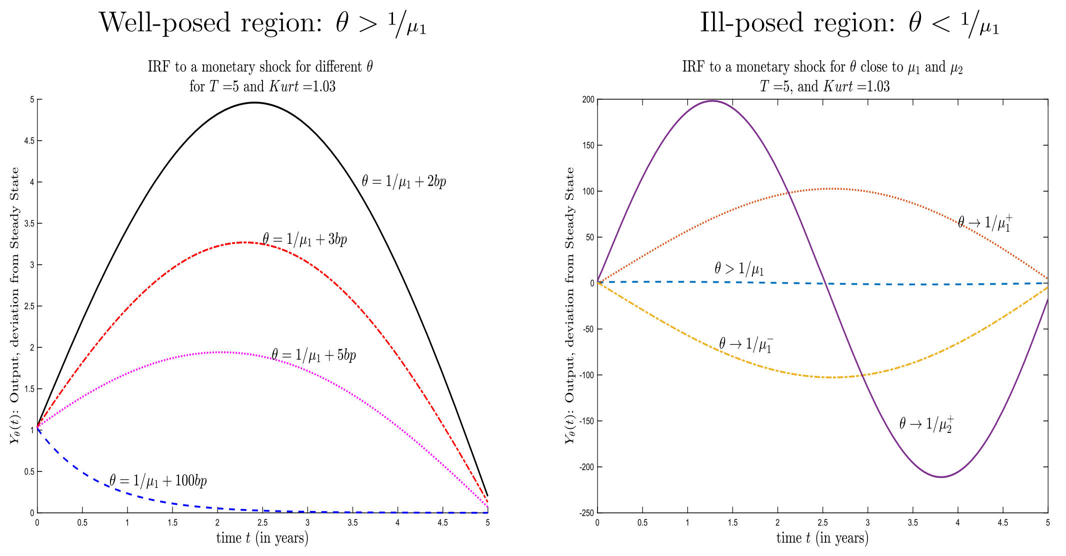

FIGURE 4.—IRF for θ near the boundary of the ill-posed region.

4. *There are at most countably many values of* $\theta \leq 1/\mu_1$ *for which the equilibrium does not exist, and where* Y_θ *has a pole at that value. Let* $j^* = \min\{j : \langle Y_0, \phi_j \rangle \neq 0\}$; *the equilibrium is well posed only for* $\theta \in (\frac{1}{\mu_{j^*}}, \infty)$.

A few comments are in order. This proposition shows that an equilibrium exists and is unique if $\theta > 1/\mu_1$, or equivalently that the equilibrium exists if $\theta \in (1/\mu_1, \infty)$. This result extends Proposition 9, since the existence result also covers $\theta > 0$, that is, all values of strategic substitutability. Second, the result complements Proposition 10, showing that as θ gets large, the IRF converges to the flexible price case.

Third, we note that the strategic complementarity parameter affects the problem exclusively through the $1 - \theta\mu_j$ terms, in equation (59). This leads to some interesting insights: as $\theta \to 1/\mu_1^+$, the IRF $Y_\theta(t)$ gets arbitrarily large as the weight term $1/(1 - \theta\mu_1)$ in equation (59) diverges to $+\infty$. Moreover, the left panel of Figure 4 shows that in this case, the IRF becomes hump-shaped, as the IRF is essentially an inflated version of the eigenfunction ϕ_1 (this happens since, as $\theta \to 1/\mu_1^+$, the weight assigned to ϕ_1 in equation (59) diverges).[14] Notice the scale of the vertical axis in the right panel of Figure 4: the original impulse responses (starting at $Y_\theta(0) = 1$) now appear like a horizontal dashed line.

Fourth, the right panel of the figure illustrates the lack of continuity established by point 3 of the proposition. Note how the impulse response changes its sign as θ approaches the critical value $1/\mu_1$ from above versus from below (compare the dotted vs the dash-dotted line). The impulse response for $\theta \to 1/\mu_1^-$ features large negative values of output (the dash-dotted yellow line). To gauge some economic intuition about the existence, the size, and the change in the sign of the equilibrium output response, consider a game where the best response of an agent, y^* to the actions of the other agents, y, is $y^* = y_0 + \theta\mu y$ where $y_0 > 0$, $\mu < 0$, and θ are all scalars. The equilibrium condition $y^* = y$ yields $y_\theta = \frac{y_0}{1-\theta\mu}$. This example shows that y_θ is positive as long as $\theta\mu < 1$, does not exist if $\theta\mu = 1$, and is

[14]It can be shown that the shape of the eigenfunction ϕ_j is akin to a sine function, with $j - 1$ zeros.

negative if $\theta\mu > 1$. As the value of θ approaches the critical point $1/\mu$, the size of the equilibrium outcome diverges. As θ crosses the critical point $1/\mu$, the slope of the best response flips, changing the sign of the equilibrium output from positive to negative.

Note that these patterns are recurrent: as θ approaches $1/\mu_2$, the IRF is given by a blown-up version of the second eigenfunction ϕ_2, shown in the right panel of Figure 4. The impulse response has countably many points where its values diverge, as $\mu \to 1/\mu_j$, and the sign of the impulse response jumps from being positive to negative as θ crosses these critical points. To further illustrate this feature, in Figure 5 we show that the cumulated IRF is a continuous function of θ for $\theta > 1/\mu_1$. We also show that, for $\theta < 1/\mu_1$, the cumulated value of output (denoted by a diamond) becomes a discontinuous function of θ, displaying huge swings every time that θ crosses the poles corresponding to the eigenvalues $1/\mu_j$ (marked by thin vertical lines). The diverging values and the swings in sign of the impulse response lead us to refer to the region where $\theta < 1/\mu_1$ as one where the problem is ill-posed.

Finally, we note that all the properties of the solution Y_θ hold in the finite-dimensional approximation developed in Appendix C, where the infinite sum in equation (59) is replaced by a finite one.

6.3. *Unconditional Output's Variance due to Monetary Shocks*

Starting with the seminal analysis of Caplin and Leahy (1997), several papers have used the output variance induced by monetary shocks as a summary measure of monetary non-neutrality, as in, for example, Nakamura and Steinsson (2010) and Midrigan (2011).

The linear expression for the impulse response given in equation (59) can be used to define a stochastic process for the deviation of output outside of the steady state. In particular, assume that the *random* monetary shocks are given by $\{d\epsilon(\tau)\}$, where $\epsilon(\tau)$ is a continuous-time process with independent changes and $\mathbb{E}[d\epsilon] = 0$ and $\mathbb{E}[d\epsilon] = \sigma_\delta^2\, dt$ for some parameter $\sigma_\delta > 0$. Our preferred example is a composite Poisson process for $\{\epsilon(\tau)\}$, where with probability $\varrho > 0$ per unit of time, $\epsilon(\tau)$ has a jump of size $\pm\delta$, each jump with probability $1/2$. In this case, $\sigma_\delta^2 = \varrho\delta^2$. The process for $\{\epsilon(\tau)\}$ generates the stationary stochastic process $\{y\}$ by $y(t) = \int_{-T}^{t} Y_\theta(t-\tau)\, d\epsilon(\tau)$ for all $t \ge 0$, using the impulse response $Y_\theta(t)$. The unconditional variance of this process is given by $\mathrm{Var}_\theta(y) = \sigma_\delta^2 \int_0^T Y_\theta^2(s)\, ds$.

PROPOSITION 13: *Assume that $\rho = 0$, $T < \infty$, and that $\theta > 1/\mu_1$. Assume the monetary shocks are i.i.d. and bounded. Then the unconditional variance of output* $\mathrm{Var}_\theta(y)$ *decreases with θ, that is,* $\mathrm{Var}_\theta(y) = \sum_{j=1}^{\infty} \frac{\langle \phi_j, Y_0\rangle^2}{(1-\theta\mu_j)^2}$ *and* $0 > \frac{1}{\mathrm{Var}_\theta(y)} \frac{\partial \mathrm{Var}_\theta(y)}{\partial\theta} = 2\sum_{j=1}^{\infty} \omega_j(\theta) \frac{\mu_j}{1-\theta\mu_j} > 2\frac{\mu_1}{1-\theta\mu_1}$, *where* $\omega_j(\theta) \equiv \frac{\langle \phi_j, Y_0\rangle^2}{(1-\theta\mu_j)^2\, \mathrm{Var}_\theta(y)}$ *are the weights.*

This proposition shows that the strength of strategic complementarities increases the unconditional variance of output—recall that $\theta < 0$ for strategic complementarities, and $\theta > 0$ for substitutability. This proposition complements the result in Proposition 10 that, at each t, the impulse response increases with the strength of strategic complementarity. Note that in the expression for $\mathrm{Var}_\theta(y)$, the parameter θ only enters in the factors $1/(1-\theta\mu_j)^2$, since Y_0, ϕ_j, μ_j do not depend on it. The functions Y_0, ϕ_j, μ_j depend on the particular price-setting model, that is, Golosov–Lucas, Calvo, or any variant of Calvo-plus.

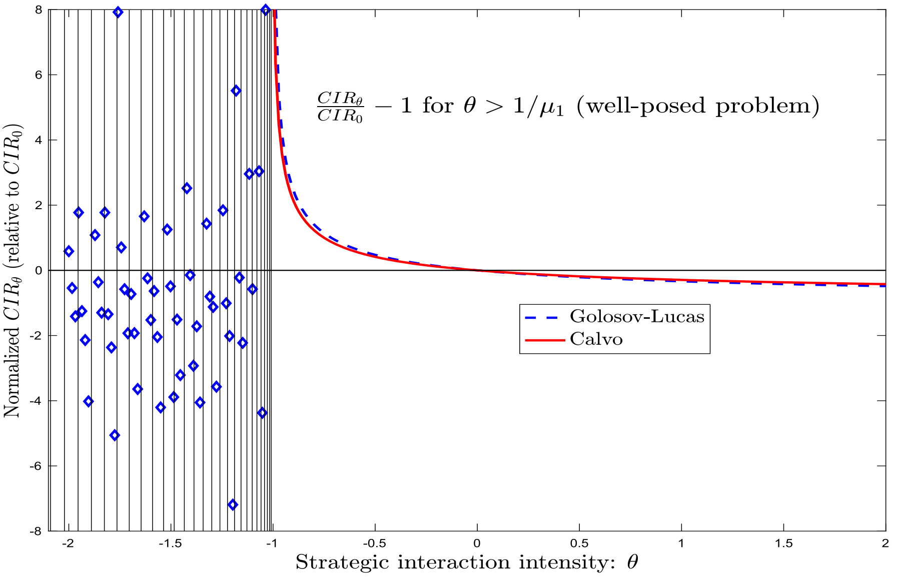

FIGURE 5.—The CIR over the ill-posed and the well-posed region.

6.4. *Strategic Complementarity and Selection Effects*

In this section, we return to the analysis of the Calvo-plus, that is, the model where we let $\ell > 0$. The Golosov–Lucas model is obtained as $\ell \to 0$, and the pure Calvo model as $\ell \to \infty$. We are interested in the relationship between strategic interactions, as measured by θ, and the selection effect in the price-setting behavior, measured by ℓ. We focus on the cumulative impulse function CIR_θ as a summary measure of the effect of a monetary shock.

Recall that, absent strategic interactions, that is, when $\theta = 0$, Alvarez, Le Bihan, and Lippi (2016) showed that the cumulative response function $\mathrm{CIR}_0 \equiv \int_0^\infty Y_0(t)\,dt$ depends only on ℓ and the frequency of price adjustment N. Indeed, in that paper, it is shown that $\mathrm{CIR}_0(\ell, N) \approx \mathrm{Kurt}(\ell)/(6N)$, where $\mathrm{Kurt}(\ell)$ is the steady-state kurtosis of the price changes, a statistic that depends only on ℓ. Motivated by these observations, we analyze the cumulative impulse response for different values of ℓ while keeping the steady-state number of price changes N constant. This implies that the difference between the CIR_0 in Calvo versus Golosov–Lucas is large, proportional to the different kurtosis of these models, equal to 6 in Calvo and 1 in Golosov–Lucas. In spite of this large difference, we will show that the effect of strategic interactions on aggregate output is approximately multiplicative, so that the percentage effect relative to the case of no interactions is similar across models.

Figure 5 plots the normalized cumulative impulse response $\mathrm{CIR}_\theta(\ell, N)/\mathrm{CIR}_0(\ell, N) - 1$, so that the value is larger than zero if the CIR_θ is larger than CIR_0, and equals zero in the baseline case of no strategic complementarities ($\theta = 0$). The notation emphasizes that the CIR depends on θ, on the frequency of price changes, N, and on the model type ℓ. We fix $N = 1$ and focus on the effect of θ across different ℓ. The region where the problem is well-posed, namely, $\theta > 1/\mu_1$, reports the normalized CIR for the Golosov–Lucas model (dashed line) as well as for the Calvo model (thick line). The range of θ considered includes both strategic substitutes ($\theta > 0$) and complements ($\theta < 0$).

Comparing the CIR for the Golosov–Lucas model ($\ell \approx 0$) with the one for the Calvo model (large ℓ), over a range of values for θ, it appears that the CIR_θ is decreasing and convex in θ, diverges towards $+\infty$ as θ approaches the reciprocal of the dominant eigenvalue ($1/\mu_1$), and converges to zero as $\theta \to \infty$.[15] What is remarkable is that the effect of θ in both models is similar over the whole domain. The figure shows that across several models, from the pure Ss to the Calvo model, the effect of strategic interactions is approximately multiplicative across a large range of values of θ. This means that in spite of the large *level* differences of the CIR in these models, as in, for example, Calvo being approximately six times larger than the Ss model when $\theta \approx 0$, the introduction of strategic interactions affects these models in a quantitatively similar way. The similar amplification effect between the models can also be given an analytic characterization. The next proposition shows the effect of a small change of θ on the cumulative response function CIR_θ.[16]

PROPOSITION 14: *Assume that* $\ell = 0$. *Consider the* CIR_θ *for the undiscounted case in a long horizon*. *Then*

$$\lim_{\rho \downarrow 0} \lim_{T \to \infty} \frac{1}{\text{CIR}_\theta} \frac{d\text{CIR}_\theta}{d\theta}\bigg|_{\theta=0} = 192 \sum_{m=1,3,5,\ldots} \left(\frac{1}{m\pi}\right)^5 \big[\text{csch}(m\pi) - \coth(m\pi)\big] \approx -0.578.$$

Likewise, using the characterization of Proposition 2, we compute the CIR_θ for the pure Calvo model, where $\zeta = N$, obtaining

$$\lim_{\rho \downarrow 0} \lim_{T \to \infty} \text{CIR}_\theta^{\text{Calvo}} = \frac{1}{N\sqrt{1+\theta}}, \quad \text{and} \quad \lim_{\rho \downarrow 0} \lim_{T \to \infty} \frac{1}{\text{CIR}_\theta^{\text{Calvo}}} \frac{d\text{CIR}_\theta^{\text{Calvo}}}{d\theta}\bigg|_{\theta=0} = -\frac{1}{2}.$$

Note that in the Calvo model, the proportional effect of θ on the cumulative impulse response CIR_θ at $\theta \approx 0$ is slightly smaller but overall very close to the value obtained for the pure Ss model. In the Calvo model, this elasticity is -0.5, while in the baseline Ss model, the elasticity is about -0.6, as shown above.

We use three results to derive the CIR for the general case: first, that $\text{CIR}_0(\ell, N) = \text{Kurt}(\ell)/(6N)$ in models without strategic complementarities, that is, $\theta = 0$; second, that $\text{CIR}_\theta^{\text{Calvo}} = \frac{1}{N\sqrt{1+\theta}}$; third, that $\text{CIR}_\theta(\ell, N)/\text{CIR}_0(\ell, N)$, as a function of θ, is approximately the same function across models (see Figure 5). Given these results, we write an approximation for the CIR for the general case, namely,

$$\text{CIR}_\theta(\ell, N) \approx \frac{\text{Kurt}(\ell)}{6N\sqrt{1+\theta}}. \tag{61}$$

Hence, the CIR depends on three determinants, $\{N, \ell, \theta\}$, in a simple multiplicative way.

7. CONCLUDING REMARKS

We analyzed the propagation of monetary shocks in a sticky-price general equilibrium model where firms' pricing decisions feature strategic interactions with the decision of

[15] Since $\text{CIR}_\theta \to 0$, then $(\text{CIR}_\theta - \text{CIR}_0)/\text{CIR}_0 \to -1$, as in the figure.

[16] The approximation is obtained by differentiating $Y_\theta(t) = Y_0(t) + \theta \int_0^T K(t,s) Y_\theta(s)\, ds$ with respect to θ and evaluating it at $\theta = 0$, obtaining $\frac{\partial}{\partial\theta} Y_\theta(t)|_{\theta=0} = \int_0^T K(t,s) Y_0(s)\, ds$.

other firms. This problem is involved and no encompassing analytic characterization of the determinants of the resulting equilibrium dynamics exists. We cast the fixed-point problem defining the equilibrium as a Mean Field Game (MFG) and establish several analytic results on equilibrium existence and on the analytic characterization of an impulse response.

The framework developed in this paper is useful to study the dynamics of equilibrium in related problems. In Alvarez, Ferrara, Gautier, LeBihan, and Lippi (2021), we used the equilibrium characterization developed in this paper to analyze the impulse response to shocks with a transitory component, as opposed to the once and for all shocks typically considered in the literature. Such an extension is important to map the model to the data, where nominal interest rate shocks typically feature a mean-reverting component.

We are also extending the framework to study higher-order perturbations. This should be key in the comparison between time- and state-dependent models, since these models react differently to large versus small shocks.

REFERENCES

ACHDOU, YVES, JIEQUN HAN, JEAN-MICHEL LASRY, PIERRE-LOUIS LIONS, AND BENJAMIN MOLL (2022): "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," *The Review of Economic Studies*, 89 (1), 45–86. [2006]

AHN, SEHYOUN, GREG KAPLAN, BENJAMIN MOLL, THOMAS WINBERRY, AND CHRISTIAN WOLF (2018): "When Inequality Matters for Macro and Macro Matters for Inequality," *NBER Macroeconomics Annual*, 32, 1–75. [2006]

ALVAREZ, FERNANDO, AND FRANCESCO LIPPI (2022): "The Analytic Theory of a Monetary Shock," *Econometrica*, 90 (4), 1655–1680. [2007,2027]

ALVAREZ, FERNANDO, FRANCESCO LIPPI, AND PANAGIOTIS SOUGANIDIS (2023): "Supplement to 'Price Setting With Strategic Complementarities as a Mean Field Game'," *Econometrica Supplemental Material*, 91, https://doi.org/10.3982/ECTA20797. [2009]

ALVAREZ, FERNANDO E., ANDREA FERRARA, ERWAN GAUTIER, HERVE LEBIHAN, AND FRANCESCO LIPPI (2021): "Empirical Investigation of a Sufficient Statistic for Monetary Shocks," Working Paper 29490, NBER. [2038]

ALVAREZ, FERNANDO E., HERVE LE BIHAN, AND FRANCESCO LIPPI (2016): "The Real Effects of Monetary Shocks in Sticky Price Models: A Sufficient Statistic Approach," *The American Economic Review*, 106 (10), 2817–2851. [2030,2036]

AMITI, MARY, OLEG ITSKHOKI, AND JOZEF KONINGS (2019): "International Shocks, Variable Markups, and Domestic Prices," *The Review of Economic Studies*, 86 (6), 2356–2402. [2005]

AUCLERT, ADRIEN, RODOLFO RIGATO, MATTHEW ROGNLIE, AND LUDWIG STRAUB (2022): "New Pricing Models, Same Old Phillips Curves?" Report, Stanford University. [2008]

BERTUCCI, CHARLES (2018): "Optimal Stopping in Mean Field Games, an Obstacle Problem Approach," *Journal de Mathématiques Pures et Appliquées*, 120, 165–194. [2008]

——— (2020): "Fokker–Planck Equations of Jumping Particles and Mean Field Games of Impulse Control," *Annales de l'Institut Henri Poincare, Analyse non lineaire*, 37 (5), 1211–1244. [2019]

BONOMO, MARCO, CARLOS CARVALHO, AND RENE GARCIA (2010): "State-Dependent Pricing Under Infrequent Information: A Unified Framework," Staff Reports 455, Federal Reserve Bank of New, York. [2010]

BOPPART, TIMO, PER KRUSELL, AND KURT MITMAN (2018): "Exploiting MIT Shocks in Heterogeneous-Agent Economies: The Impulse Response as a Numerical Derivative," *Journal of Economic Dynamics and Control*, 89 (C), 68–92. [2006]

CALVO, GUILLERMO A. (1983): "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12 (3), 383–398. [2007,2011]

CANNON, JOHN ROZIER (1984): *The One-Dimensional Heat Equation*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press. [2026]

CAPLIN, ANDREW, AND JOHN LEAHY (1991): "State-Dependent Pricing and the Dynamics of Money and Output," *The Quarterly Journal of Economics*, 106 (3), 683–708. [2008]

——— (1997): "Aggregation and Optimization With State-Dependent Pricing," *Econometrica*, 65 (3), 601–626. [2006-2009,2035]

CONWAY, JOHN (2007): *A Course in Functional Analysis* (Second Ed.). New York: Springer-Verlag. [2033]

COOPER, RUSSELL, AND JOHN HALTIWANGER (1996): "Evidence on Macroeconomic Complementarities," *The Review of Economics and Statistics*, 78 (1), 78–93. [2005]

GOLOSOV, MIKHAIL, AND ROBERT E. JR. LUCAS (2007): "Menu Costs and Phillips Curves," *Journal of Political Economy*, 115, 171–199. [2007,2009,2030]

GOPINATH, GITA, AND OLEG ITSKHOKI (2011): "In Search of Real Rigidities," *NBER Macroeconomics Annual*, 25, 261–310. [2005]

KIMBALL, MILES S. (1995): "The Quantitative Analytics of the Basic Neomonetarist Model," *Journal of Money, Credit and Banking*, 27 (4), 1241–1277. [2009]

KLENOW, PETER J., AND JONATHAN L. WILLIS (2016): "Real Rigidities and Nominal Price Changes," *Economica*, 83 (331), 443–472. [2005,2007-2009,2030]

LASRY, JEAN-MICHEL, AND PIERRE-LOUIS LIONS (2007): "Mean Field Games," *Japanese Journal of Mathematics*, 2, 229–260. [2006]

MIDRIGAN, VIRGILIU (2011): "Menu Costs, Multi-Product Firms, and Aggregate Fluctuations," *Econometrica*, 79 (4), 1139–1180. [2010,2035]

MONGEY, SIMON (2021): "Market Structure and Monetary Non-Neutrality," Working Paper 29233, National Bureau of Economic Research. [2006]

NAKAMURA, EMI, AND JON STEINSSON (2010): "Monetary Non-Neutrality in a Multisector Menu Cost Model," *The Quarterly Journal of Economics*, 125 (3), 961–1013. [2005-2009,2030,2035]

WANG, OLIVIER, AND IVAN WERNING (2022): "Dynamic Oligopoly and Price Stickiness," *American Economic Review*, 112 (8), 2815–2849. [2006,2008,2016]

WOODFORD, MICHAEL (2009): "Information-Constrained State-Dependent Pricing," *Journal of Monetary Economics*, 56, s100–s124. [2010]

Co-editor Charles I. Jones handled this manuscript.

Manuscript received 9 May, 2022; final version accepted 1 April, 2023; available online 7 August, 2023.

The replication package for this paper is available at https://doi.org/10.5281/zenodo.8200032. The Journal checked the data and codes included in the package for their ability to reproduce the results in the paper and approved online appendices.