SUPPLEMENT TO "IDENTIFICATION AND ESTIMATION OF TRIANGULAR SIMULTANEOUS EQUATIONS MODELS WITHOUT ADDITIVITY"

(*Econometrica*, Vol. 77, No. 5, September 2009, 1481–1512)

BY GUIDO W. IMBENS AND WHITNEY K. NEWEY

PROOFS OF LEMMAS 10 AND 11 AND THEOREMS 12 AND 13

Throughout this supplementary material, C will denote a generic positive constant that may be different in different uses. Also, we will abbreviate the phrases with probability approaching 1 as w.p.a.1, positive semidefinite as p.s.d., and positive definite as p.d.; $\lambda_{\min}(A)$, $\lambda_{\max}(A)$, and $A^{1/2}$ will denote the minimum eigenvalues, the maximum eigenvalues, and the square root, respectively, of a symmetric matrix A. Let $\sum_i$ denote $\sum_{i=1}^n$. Also, let CS, M, and T refer to the Cauchy–Schwarz, Markov, and triangle inequalities, respectively. Also, let CM refer to the following well known result: *If* $E[|Y_n||Z_n] = O_p(r_n)$, *then* $Y_n = O_p(r_n)$.

PROOF OF LEMMA 10: The joint PDF of (x, η) is $f_Z(x-\eta)f_\eta(\eta)$, where $f_Z(\cdot)$ is the PDF of Z and $f_\eta(\cdot)$ is the PDF of η. By a change of variable $v = F_\eta(\eta)$, the PDF of (x, v) is

$$f_Z(x - F_\eta^{-1}(v)),$$

where $F_\eta(\cdot)$ is the CDF of η_0. Consider $\alpha = \bar{\alpha} + \delta > (1-R^2)/R^2 = \sigma_\eta^2/\sigma_Z^2$. Then for $\eta = F_\eta^{-1}(v)$ and $0 < v < 1$,

$$\frac{f_Z(x - F_\eta^{-1}(v))}{v^\alpha(1-v)^\alpha} = C\exp\left(-\frac{1}{2}\left(\frac{x-\eta}{\sigma_Z^2}\right)^2\right)\Phi\left(\frac{\eta}{\sigma_\eta}\right)^{-\alpha}\Phi\left(-\frac{\eta}{\sigma_\eta}\right)^{-\alpha}.$$

It is well known that $\phi(u)/\Phi(u)$ is monotonically decreasing, so there is $C > 0$ such that $\Phi(u)^{-1} \geq C\phi(u)^{-1}$, $u \leq 0$, and similarly $\Phi(u)^{-1} \geq C\phi(u)^{-1}$, $u \geq 0$. Then by $\Phi(u)^{-1} \geq 1$ for all u,

$$\Phi(u)^{-1}\Phi(-u)^{-1} \geq C\phi(u)^{-1}.$$

Therefore, for $\eta = \sigma_\eta \Phi^{-1}(v)$,

$$\begin{aligned}\frac{f_Z(x - F_\eta^{-1}(v))}{v^\alpha(1-v)^\alpha} &\geq C\exp\left\{-\frac{1}{2}\left(\frac{x-\eta}{\sigma_Z}\right)^2\right\}\exp\left(\frac{1}{2}\frac{\alpha\eta^2}{\sigma_\eta^2}\right)\\ &= C\exp\left\{\frac{-x^2}{2\sigma_Z^2} + \frac{x\eta}{\sigma_Z^2} + \frac{\eta^2}{2\sigma_Z^2}\left(\frac{\alpha\sigma_Z^2}{\sigma_\eta^2} - 1\right)\right\}.\end{aligned}$$

 DOI: 10.3982/ECTA7108

The expression following the equality is bounded away from zero for $|x| \leq B$ and all $\eta \in \mathbb{R}$ by $\alpha > \sigma_\eta^2/\sigma_Z^2$.

The upper bound follows by a similar argument, using the fact that there is a C with $\phi(u)/\Phi(u) \leq |u| + C$ for all u. *Q.E.D.*

Before proving Lemma 11, we prove some preliminary results. Let $q_i = q^L(Z_i)$ and $\omega_{ij} = 1(X_{1j} \leq X_{1i}) - F_{X_1|Z}(X_{1i}|Z_j)$.

LEMMA S.1: *For $Z = (Z_1, \dots, Z_n)$ and $L \times 1$ vectors of functions $b_i(Z)$ ($i = 1, \dots, n$), if $\sum_{i=1}^n b_i(Z)'\hat{Q}b_i(Z)/n = O_p(r_n)$, then*

$$\sum_{i=1}^n \left\{ b_i(Z)' \sum_{j=1}^n q_j \omega_{ij}/\sqrt{n} \right\}^2 \Big/ n = O_p(r_n).$$

PROOF: Note that $|\omega_{ij}| \leq 1$. Consider $j \neq k$ and suppose without loss of generality that $j \neq i$ (otherwise reverse the role of j and k because we cannot have $i = j$ and $i = k$). By independence of the observations,

$$\begin{aligned} E\big[\omega_{ij}\omega_{ik}|Z\big] &= E\big[E[\omega_{ij}\omega_{ik}|Z, X_i, X_k]|Z\big] \\ &= E\big[\omega_{ik}E[\omega_{ij}|Z, X_i, X_k]|Z\big] \\ &= E\big[\omega_{ik}E[\omega_{ij}|Z_j, Z_i, X_i]|Z\big] \\ &= E\big[\omega_{ik}\big\{E[1(X_{1j} \leq X_{1i})|Z_j, Z_i, X_i] \\ &\qquad - F_{X_1|Z}(X_{1i}|Z_j)\big\}|Z\big] = 0. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} &E\left[\sum_{i=1}^n \left\{ b_i(Z)' \sum_{j=1}^n q_j \omega_{ij}/\sqrt{n} \right\}^2 \Big/ n \Big| Z \right] \\ &\quad \leq \sum_{i=1}^n b_i(Z)' \left\{ \sum_{j,k=1}^n q_j E[\omega_{ij}\omega_{ik}|Z] q_k'/n \right\} b_i(Z)/n \\ &\quad = \sum_{i=1}^n b_i(Z)' \left\{ \sum_{j=1}^n q_j E[\omega_{ij}^2|Z] q_j'/n \right\} b_i(Z)/n \leq \sum_{i=1}^n b_i(Z)'\hat{Q}b_i(Z)/n, \end{aligned}$$

so the conclusion follows by CM. *Q.E.D.*

LEMMA S.2—Lorentz (1986, p. 90, Theorem 8): *If Assumption 3 is satisfied, then there exists C such that for each x there is $\gamma(x)$ with $\sup_{z \in Z} |F_{X_1|Z}(x|z) - p^{K_1}(z)'\gamma(x)| \leq CK_1^{-d_1/r_1}$.*

LEMMA S.3: *If Assumption* 4 *is satisfied*, *then for each* K *there exists a nonsingular constant matrix* B *such that* $\tilde{p}^{K_2}(w) = Bp^{K_2}(w)$ *satisfies* $E[\tilde{p}^{K_2}(w_i) \times \tilde{p}^{K_2}(w_i)'] = I_{K_2}$, $\sup_{w\in\mathcal{W}} \|\tilde{p}^{K_2}(w)\| \leq CK_V^{\alpha}K_2$, $\sup_{w\in\mathcal{W}} \|\partial\tilde{p}^{K_2}(w)/\partial V\| \leq CK_V^{\alpha+2}K_2$, *and* $\sup_{t\in[0,1]} \|\tilde{p}^{K_V}(t)\| \leq CK_V^{1+\alpha}$.

PROOF: For $u \in [0, 1]$, let $P_j^{\alpha}(u)$ be the jth orthonormal polynomial with respect to the weight $u^{\alpha}(1-u)^{\alpha}$. Denote $\mathcal{X} = \prod_{\ell=1}^{r_2-1}[\underline{x}_{\ell}, \bar{x}_{\ell}]$. By the fact that the order of the power series is increasing and that all terms of a given order are included before a term of higher order, for each k and $\lambda(k,\ell)$ with $p_k(w) = \prod_{\ell=1}^{s} w_{\ell}^{\lambda(k,\ell)}$, there exists b_{kj} $(j \leq k)$ such that

$$\sum_{j=1}^{k} b_{kj}p_j(w) = \prod_{\ell=1}^{r_2-1} P^0_{\lambda(k,\ell)}([x_{\ell} - \underline{x}_{\ell}]/[\bar{x}_{\ell} - \underline{x}_{\ell}])P^{\alpha}_{\lambda(k,s)}(t).$$

Let B_k denote a $K_2 \times 1$ vector $B_k = (b_{k1}, \dots, b_{kk}, 0')'$, $b_{kk} \neq 0$, where 0 is a $(K - k)$-dimensional zero vector, and let $\bar{B}$ be the $K_2 \times K_2$ matrix with kth row B_k'. Then $\bar{B}$ is a lower triangular matrix with nonzero diagonal elements and so is nonsingular. As shown in Andrews (1991), there is C such that $|P_j^{\alpha}(u)| \leq C(j^{\alpha+1/2} + 1) \leq Cj^{\alpha+1/2}$ and $|dP_j^{\alpha}(u)/du| \leq Cj^{\alpha+5/2}$ for all $u \in [0, 1]$ and $j \in \{1, 2, \dots\}$. Then for $\bar{p}^{K_2}(w) = \bar{B}p^{K_2}(w)$, it follows that $|\bar{p}_k(w)| \leq C\lambda(k,s)^{\alpha+1/2}\prod_{\ell=1}^{s-1}\lambda(k,\ell)^{1/2}$, so that $\|\bar{p}^{K_2}(w)\| \leq CK_V^{\alpha}K_2$, and $\sup_{w\in\mathcal{W}} \|\partial\bar{p}^{K_2}(w)/\partial t\| \leq CK_V^{\alpha+2}K_2$. Then by Assumption 4, it follows that $\Omega_{K_2} = E[\bar{p}^{K_2}(w_i)\bar{p}^{K_2}(w_i)'] \geq CI_{K_2}$. Let $\tilde{B} = \Omega_{K_2}^{-1/2}$ and define $\tilde{p}^{K_2}(w) = \tilde{B}\bar{p}^{K_2}(w)$. Then $\|\tilde{p}^{K_2}(w)\| = \sqrt{\tilde{p}^{K_2}(w)'\tilde{p}^{K_2}(w)} \leq \sqrt{\bar{p}^{K_2}(w)'\Omega^{-1}\bar{p}^{K_2}(w)} \leq C\|\bar{p}^{K_2}(w)\|$ and an analogous inequality holds for $\|\partial\tilde{p}^{K_2}(w)/\partial t\|$, giving the conclusion. *Q.E.D.*

Henceforth define $\zeta = CK_V^{\alpha}K_2$ and $\zeta_1 = CK_V^{\alpha+2}K_2$. Also, since the estimator is invariant to nonsingular linear transformations of $p^{K_2}(w)$, we can assume that the conclusion of Lemma S.3 is satisfied with $p^{K_2}(w)$ replacing $\tilde{p}^{K_2}(w)$.

PROOF OF LEMMA 11: Let $\delta_{ij} = F_{X_1|Z}(X_{1i}|Z_j) - q_j'\gamma^{K_1}(X_{1i})$, with $|\delta_{ij}| \leq K_1^{-d_1/r_1}$ by Lemma S.2. Then for $\tilde{V}_i = \tilde{a}^{K_1}_{1(X_1\leq X_{1i})}(Z_i)$,

$$\tilde{V}_i - V_i = \Delta_i^{\mathrm{I}} + \Delta_i^{\mathrm{II}} + \Delta_i^{\mathrm{III}},$$

where

$$\Delta_i^{\mathrm{I}} = q_i'\hat{Q}^{-}\sum_{j=1}^{n} q_j\omega_{ij}/n, \quad \Delta_i^{\mathrm{II}} = q_i'\hat{Q}^{-}\sum_{j=1}^{n} q_j\delta_{ij}/n, \quad \Delta_i^{\mathrm{III}} = -\delta_{ii}.$$

Note that $|\Delta_i^{\mathrm{III}}| \leq CK_1^{-d_1/r}$ by Lemma S.2. Also, by $\hat{Q}$ p.s.d. and symmetric, there exists a diagonal matrix of eigenvalues Λ and an orthonormal matrix B

such that $\hat{Q} = B\Lambda B'$. Let Λ^- denote the diagonal matrix of inverse of nonzero eigenvalues and zeros, and let $\hat{Q}^- = B\Lambda^- B'$. Then $\sum_i q_i'\hat{Q}^- q_i = \operatorname{tr}(\hat{Q}^-\hat{Q}) \le CL$. By CS and Assumption 3,

$$\sum_{i=1}^n (\Delta_i^{\mathrm{II}})^2/n \le \sum_{i=1}^n \left(q_i'\hat{Q}^- q_i \sum_{j=1}^n \delta_{ij}^2/n \right) \Big/ n \le C \sum_{i=1}^n (q_i'\hat{Q}^- q_i) L^{-2d_1}/n$$
$$= CK_1^{-2d_1/r} \operatorname{tr}(\hat{Q}^-\hat{Q}) \le CK_1^{1-2d_1/r}.$$

Note that for $b_i(Z) = q_i'\hat{Q}^-/\sqrt{n}$ we have

$$\sum_{i=1}^n b_i(Z)'\hat{Q} b_i(Z)/n = \operatorname{tr}(\hat{Q}\hat{Q}^-\hat{Q}\hat{Q}^-)/n = \operatorname{tr}(\hat{Q}\hat{Q}^-)/n$$
$$\le CK_1/n = O_p(K_1/n),$$

so it follows by Lemma S.1 that $\sum_{i=1}^n (\Delta_i^{\mathrm{I}})^2/n = O_p(L/n)$. The conclusion then follows by T and by $|\tau(\tilde{V}) - \tau(V)| \le |\tilde{V} - V|$, which gives $\sum_i (\hat{V}_i - V_i)^2/n \le \sum_i (\tilde{V}_i - V_i)^2/n$. *Q.E.D.*

Before proving other results, we give some useful lemmas. For these results let $p_i = p^{K_2}(w_i)$, $\hat{p}_i = p^{K_2}(\hat{w}_i)$, $p = [p_1, \ldots, p_n]$, $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_n]$, $\hat{P} = \hat{p}'\hat{p}/n$, and $\tilde{P} = p'p/n$, $P = E[p_i p_i']$. Also, as in Newey (1997), it can be shown that without loss of generality we can set $P = I_{K_2}$.

LEMMA S.4: *If the hypotheses of Theorem* 1 *are satisfied*, *then* $E[Y|X, Z] = m(X, V)$.

PROOF: By the proof of Theorem 1, $V = F_{X_1|Z}(X_1|Z)$ is a function of X_1 and Z that is invertible in X_1 with inverse $X_1 = \bar{h}(Z, V)$, where $\bar{h}(z, v)$ is the inverse of $F_{X_1|Z}(x|z)$ in its first argument. Therefore, (V, Z) is a one-to-one function of (X, Z). By independence of Z and (ε, η), ε is independent of Z conditional on V, so that by equation (4),

$$E[Y|X, Z] = E[Y|V, Z] = E\big[g(\bar{h}(Z, V), \varepsilon)|V, Z\big]$$
$$= \int g(\bar{h}(Z, V), e) F_{\varepsilon|Z,V}(de|Z, V)$$
$$= \int g(\bar{h}(Z, V), e) F_{\varepsilon|V}(de|V) = m(X, V). \qquad \textit{Q.E.D.}$$

Let $u_i = Y_i - m(X_i, V_i)$ and let $u = (u_1, \ldots, u_n)'$.

LEMMA S.5: *If* $\sum_i \|\hat{V}_i - V_i\|^2/n = O_p(\Delta_n^2)$ *and Assumptions* 3–6 *are satisfied, the following equalities hold*:

(i) $\|\tilde{P} - P\| = O_p(\zeta\sqrt{K_2/n})$,
(ii) $\|p'u/n\| = O_p(\sqrt{K_2/n})$,
(iii) $\|\hat{p} - p\|^2/n = O_p(\zeta_1^2 \Delta_n^2)$,
(iv) $\|\hat{P} - \tilde{P}\| = O_p(\zeta_1^2\Delta_n^2 + \sqrt{K_2}\zeta_1\Delta_n)$,
(v) $\|(\hat{p} - p)'u/n\| = O_p(\zeta_1\Delta_n/\sqrt{n})$.

PROOF: The first two results follow as in equation (A.1) and page 162 of Newey (1997). For (iii), a mean value expansion gives $\hat{p}_i = p_i + [\partial p^{K_2}(\tilde{w}_i)/\partial V](\hat{V}_i - V_i)$, where $\tilde{w}_i = (x_i, \tilde{V}_i)$ and $\tilde{V}_i$ lies in between $\hat{V}_i$ and V_i. Since $\hat{V}_i$ and V_i lie in [0, 1], it follows that $\tilde{V}_i \in [0, 1]$, so that by Lemma S.3, $\|\partial p^{K_2}(\tilde{w}_i)/\partial V\| \le C\zeta_1$. Then by CS, $\|\hat{p}_i - p_i\| \le C\zeta_1|\hat{V}_i - V_i|$. Summing up gives

$$\|\hat{p} - p\|^2/n = \sum_{i=1}^n \|\hat{p}_i - p_i\|^2/n = O_p(\zeta_1^2\Delta_n^2). \tag{S.1}$$

For (iv), by Lemma S.3, $\sum_{i=1}^n \|p_i\|^2/n = O_p(E[\|p_i\|^2]) = \operatorname{tr}(I_{K_2}) = K_2$. Then by T, CS, and M,

$$\begin{aligned}\|\hat{P} - \tilde{P}\| &\le \sum_{i=1}^n \|\hat{p}_i\hat{p}_i' - p_ip_i'\|/n \le \sum_{i=1}^n \|\hat{p}_i - p_i\|^2/n \\ &\quad + 2\left(\sum_{i=1}^n \|\hat{p}_i - p_i\|^2/n\right)^{1/2}\left(\sum_{i=1}^n \|p_i\|^2/n\right)^{1/2} \\ &= O_p(\zeta_1^2\Delta_n^2 + \sqrt{K_2}\zeta_1\Delta_n).\end{aligned}$$

Finally, for (v), for $\overrightarrow{Z} = (Z_1, \dots, Z_n)$ and $\overrightarrow{X} = (X_1, \dots, X_n)$, it follows from Lemma S.4, Assumption 6, and independence of the observations that $E[uu'|\overrightarrow{X}, \overrightarrow{Z}] \le CI_n$, so that by p and $\hat{p}$ depending only on $\overrightarrow{Z}$ and $\overrightarrow{X}$,

$$\begin{aligned}E\big[\|(\hat{p} - p)'u/n\|^2|\overrightarrow{X}, \overrightarrow{Z}\big] &= \operatorname{tr}\big\{(\hat{p} - p)'E[uu'|\overrightarrow{X}, \overrightarrow{Z}](\hat{p} - p)/n^2\big\} \\ &\le C\|\hat{p} - p\|^2/n^2 = O_p(\zeta_1^2\Delta_n^2/n). \quad \textit{Q.E.D.}\end{aligned}$$

LEMMA S.6: *If Assumptions* 3–6 *are satisfied and* $K_2\zeta_1^2\Delta_n^2 \longrightarrow 0$, *then w.p.a.1*, $\lambda_{\min}(\hat{P}) \ge C$, $\lambda_{\min}(\tilde{P}) \ge C$.

PROOF: By Lemma S.3 and $\zeta_1^2 K_2/n \le CK_2\zeta_1^2K_1/n$, we have $\|\hat{P} - \tilde{P}\| \xrightarrow{p} 0$ and $\|\tilde{P} - P\| \xrightarrow{p} 0$, so the conclusion follows as on page 162 of Newey (1997). *Q.E.D.*

Let $m = (m(w_1), \ldots, m(w_n))'$, and $\hat{m} = (m(\hat{w}_1), \ldots, m(\hat{w}_n))'$.

LEMMA S.7: *If* $\sum_i \|\hat{V}_i - V_i\|^2/n = O_p(\Delta_n^2)$, *Assumptions* 3–6 *are satisfied*, $\sqrt{K_2}\zeta_1\Delta_n \to 0$, *and* $K_2\zeta^2/n \to 0$, *then for* $\tilde{\alpha} = \hat{P}^{-1}\hat{p}'\hat{m}/n$ *and* $\bar{\alpha} = \hat{P}^{-1}\hat{p}'m/n$, *the following equalities hold*:

(i) $\|\hat{\alpha} - \bar{\alpha}\| = O_p(\sqrt{K_2/n})$,
(ii) $\|\tilde{\alpha} - \bar{\alpha}\| = O_p(\Delta_n)$,
(iii) $\|\tilde{\alpha} - \alpha^{K_2}\| = O_p(K_2^{-d_2/r_2})$.

PROOF: For (i),

$$\begin{aligned}
&E\big[\big\|\hat{P}^{1/2}(\hat{\alpha} - \bar{\alpha})\big\|^2 | \overrightarrow{X}, \overrightarrow{Z}\,\big] \\
&\quad = E[u'\hat{p}\hat{P}^{-1}\hat{p}'u/n^2 | \overrightarrow{X}, \overrightarrow{Z}\,] \\
&\quad = \operatorname{tr}\big\{\hat{P}^{-1/2}\hat{p}'E[uu'|\overrightarrow{X}, \overrightarrow{Z}\,]\hat{p}\hat{P}^{-1/2}\big\}/n^2 \\
&\quad \le C\operatorname{tr}\{\hat{p}\hat{P}^{-1}\hat{p}'\}/n^2 \le C\operatorname{tr}(I_{K_2})/n \\
&\quad = CK_2/n.
\end{aligned}$$

Since by Lemma S.6, $\lambda_{\min}(\hat{P}) \ge C$ w.p.a.1, this implies that $E[\|\hat{\alpha} - \bar{\alpha}\|^2 | \overrightarrow{X}, \overrightarrow{Z}\,] \le CK_2/n$. Similarly, for (ii),

$$\begin{aligned}
\big\|\hat{P}^{1/2}(\tilde{\alpha} - \bar{\alpha})\big\|^2 &\le C(\hat{m} - m)'\hat{p}\hat{P}^{-1}\hat{p}'(\hat{m} - m)/n^2 \le C\|\hat{m} - m\|^2/n \\
&= O_p(\Delta_n^2),
\end{aligned}$$

which follows from $m(w)$ being Lipschitz in V, so that also $\|\tilde{\alpha} - \bar{\alpha}\|^2 = O_p(\Delta_n^2)$. Finally for (iii),

$$\begin{aligned}
\big\|\hat{P}^{1/2}(\tilde{\alpha} - \alpha^{K_2})\big\|^2 &= \|\tilde{\alpha} - \hat{P}^{-1}\hat{p}'\hat{p}\alpha^{K_2}/n\|^2 \\
&\le C(\hat{m} - \hat{p}'\alpha^{K_2})'\hat{p}\hat{P}^{-1}\hat{p}'(\hat{m} - \hat{p}'\alpha^{K_2})/n^2 \\
&\le \|\hat{m} - \hat{p}\alpha^{K_2}\|^2/n \le C\sup_{w \in \mathcal{W}} |m_0(w) - p^K(w)'\alpha^{K_2}|^2 \\
&= O_p\big(K_2^{-2d_2/r_2}\big),
\end{aligned}$$

so that $\|\hat{P}^{1/2}(\tilde{\alpha} - \alpha^{K_2})\|^2 = O_p(K_2^{-2d_2/r_2})$. *Q.E.D.*

PROOF OF THEOREM 12: Note that by Lemma 11, for $\Delta_n^2 = K_1/n + K_1^{1-2d_1/r_1}$, we have $\sum_i \|\hat{V}_i - V_i\|^2/n = O_p(\Delta_n^2)$, so by $K_2\zeta^2/n \le CK_2\zeta_1^2K_1/n$, the hypotheses of Lemma S.7 are satisfied. Also by Lemma S.7 and T, $\|\hat{\alpha} - \alpha^{K_2}\|^2 =$

$O_p(K_2/n + K_2^{-2d_2/r_2} + \Delta_n^2)$. Then

$$\begin{aligned}
&\int [\hat{m}(w) - m(w)]^2 F_w(dw) \\
&\quad = \int \big[p^{K_2}(w)'(\hat{\alpha} - \alpha^{K_2}) + p^{K_2}(w)'\alpha^{K_2} - m(w)\big]^2 F_w(dw) \\
&\quad \le C\|\hat{\alpha} - \alpha^{K_2}\|^2 + CK_2^{-2d_2/r_2} = O_p\big(K_2/n + K_2^{-2d_2/r_2} + \Delta_n^2\big).
\end{aligned}$$

For the second part of Theorem 12,

$$\begin{aligned}
&\sup_{w\in\mathcal{W}} |\hat{m}(w) - m(w)| \\
&\quad = \sup_{w\in\mathcal{W}} |p^{K_2}(w)'(\hat{\alpha} - \alpha^{K_2}) + p^{K_2}(w)'\alpha^{K_2} - \beta(w)| \\
&\quad = O_p\big(\zeta\big(K_2/n + K_2^{-2d_2/r_2} + \Delta_n^2\big)^{1/2}\big) + O_p\big(K_2^{-d_2/r_2}\big) \\
&\quad = O_p\big(\zeta\big(K_2/n + K_2^{-2d_2/r_2} + \Delta_n^2\big)^{1/2}\big). \qquad \textit{Q.E.D.}
\end{aligned}$$

PROOF OF THEOREM 13: Let $\bar{p} = \int_0^1 p^{K_V}(t)\,dt$ and note that by Lemma S.3, $\bar{p}'\bar{p} \le CK_V^{2+2\alpha}$. Also,

$$\bar{p}(x) \overset{\text{def}}{=} \int_0^1 p^K(w)\,dt = p^{K_x}(x) \otimes \bar{p}. \tag{S.2}$$

As above, $E[uu'|\overrightarrow{X}, \overrightarrow{Z}] \le CI_n$, so that by Fubini's theorem,

$$\begin{aligned}
&E\bigg[\int \{\bar{p}(x)'(\hat{\alpha} - \bar{\alpha})\}^2 F_X(dx)|\overrightarrow{X}, \overrightarrow{Z}\bigg] \\
&\quad = \int \big\{\bar{p}(x)'\hat{P}^{-1}\hat{p}'E[uu'|\overrightarrow{X}, \overrightarrow{Z}]\hat{p}\hat{P}^{-1}\bar{p}(x)\big\} F_X(dx)/n^2 \\
&\quad \le C\int \bar{p}(x)'\hat{P}^{-1}\bar{p}(x)F_X(dx)/n \le CE[\bar{p}(X)'\bar{p}(X)]/n \\
&\quad = C\big\{E[p^{K_x}(X)'p^{K_x}(X)](\bar{p}'\bar{p})\big\}/n = K_x K_V^{2+2\alpha}/n.
\end{aligned}$$

It then follows by CM that $\int \{\bar{p}(x)'(\hat{\alpha} - \bar{\alpha})\}^2 F_X(dx) = O_p(K_x K_V^{2+2\alpha}/n)$. Also,

$$\int \bar{p}(x)\bar{p}(x)'F_X(dx) = I_{K_x} \otimes \bar{p}\bar{p}' \le CI_{K_2}\bar{p}'\bar{p} \le CI_{K_2}K_V^{2+2a},$$

so that by Lemma S.7 and T,

$$\begin{aligned}
&\int \{\bar{p}(x)'(\bar{\alpha}-\alpha^K)\}^2 F_X(dx) \\
&\quad \leq (\bar{\alpha}-\alpha^K)' \int \bar{p}(x)\bar{p}(x)' F_X(dx)(\bar{\alpha}-\alpha^K) \\
&\quad \leq C K_V^{2+2a} \|\bar{\alpha}-\alpha^K\|^2 = O_p\big(K_V^{2+2a}\big(K_2^{-2d_2/s}+\Delta_n^2\big)\big).
\end{aligned}$$

Also, by CS,

$$\begin{aligned}
&\int \{\bar{p}(x)'\alpha^K - \mu(x)\}^2 F_X(dx) \\
&\quad \leq \int \int_0^1 \{p^K(w)'\alpha - \beta(w)\}^2 \, dV \, F_X(dx) = O\big(K_2^{-2d_2/s}\big).
\end{aligned}$$

Then the conclusion follows by T and

$$\begin{aligned}
&\int [\hat{\mu}(x) - \mu(x)]^2 F_0(dx) \\
&\quad = \int \{\bar{p}(x)'(\hat{\alpha}-\alpha^K) + \bar{p}(x)'\alpha^K - \mu(x)\}^2 F_X(dx) \\
&\quad = O_p\big(K_V^{2+2\alpha}\big(K_x/n + K_2^{-2d_2/r_2} + \Delta_n^2\big)\big). \qquad \textit{Q.E.D.}
\end{aligned}$$

REFERENCES

ANDREWS, D. W. K. (1991): "Asymptotic Normality of Series Estimators for Nonparametric and Semiparametric Regression Models," *Econometrica*, 59, 307–345.

LORENTZ, G. (1986): *Approximation of Functions*. New York: Chelsea Publishing Company.

NEWEY, W. K. (1997): "Convergence Rates and Asymptotic Normality for Series Estimators," *Journal of Econometrics*, 79, 147–168.

Dept. of Economics, Littauer Center, Harvard University, 1805 Cambridge Street, Cambridge, MA 02138, U.S.A.; imbens@fas.harvard.edu

and

Dept. of Economics, Massachusetts Institute of Technology, Cambridge, MA 02142-1347, U.S.A.; wnewey@mit.edu.

Manuscript received April, 2007; final revision received January, 2009.