SUPPLEMENT TO "ROTTEN PARENTS AND DISCIPLINED CHILDREN: A POLITICO-ECONOMIC THEORY OF PUBLIC EXPENDITURE AND DEBT"
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By Zheng Song, Kjetil Storesletten, and Fabrizio Zilibotti

APPENDIX B

IN THIS APPENDIX, we present some supplementary material. Section B.1 provides the politico-economic microfoundations of the model. Section B.2 provides the derivation of Equation (9). Section B.3 provides the proof of Proposition 2. Section B.4 extends the analysis under CRRA utility. Section B.5 shows the result of numerical analysis using the alternative method of Krusell, Kuruscu, and Smith (2002). Finally, Section B.6 shows the details of the numerical analysis of a demographic transition such as the one discussed in the text.

B.1. *Probabilistic Voting Model*

The political equilibrium discussed in the paper has an explicit politico-economic microfoundation in terms of a politico-economic voting model, based on Lindbeck and Weibull (1987). In this appendix, we describe the voting model, which is an application of Persson and Tabellini (2000) to a dynamic voting setting.

The population has a unit measure and consists of two groups of voters, young and old, of equal size (we discuss below the extension to groups of different sizes). The electoral competition takes place between two office-seeking candidates, denoted by A and B. Each candidate announces a fiscal policy vector, b', τ, and g, subject to the government budget constraint, $b' = Rb + g - w(R)\tau H(\tau)$, and to $b' \leq \overline{b}$.[22] Since there are new elections every period, the candidates cannot make credible promises over future policies (i.e., there is lack of commitment beyond the current period). Voters choose either of the candidates based on their fiscal policy announcements and on their relative *appeal*, where the notion of appeal is explained below. In particular, a young voter prefers candidate A over B if, given the inherited debt level b, preference parameter θ, the world interest rate, and the equilibrium policy functions $\langle B, G, T \rangle$ which apply from tomorrow and onward,

$$U_Y\big(\tau_A, g_A, G(b'_A); b, \theta, R\big) > U_Y\big(\tau_B, g_B, G(b'_B); b, \theta, R\big).$$

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[22]Note that the announcement over the current fiscal policy raises no credibility issue, due to the assumption that the politicians are pure office seekers and have no independent preferences on fiscal policy.

Corruption Perception Index (CPI) vs. Debt-to-GDP Ratio
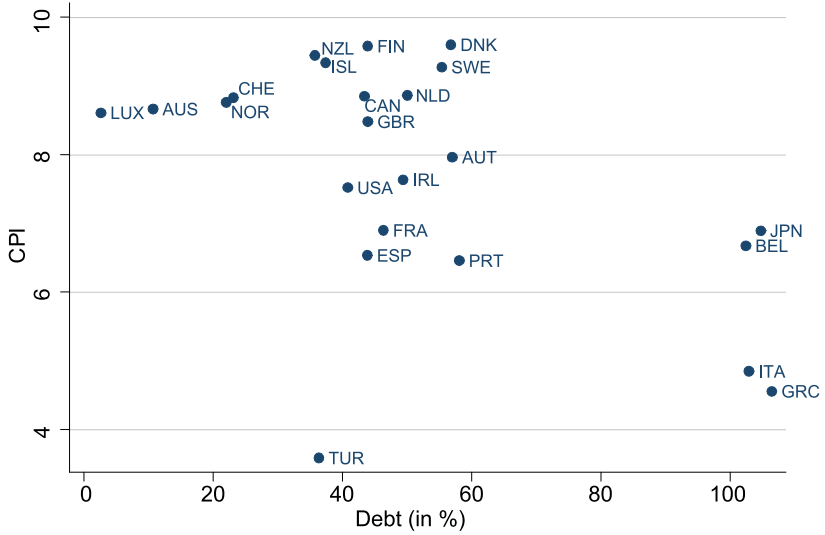
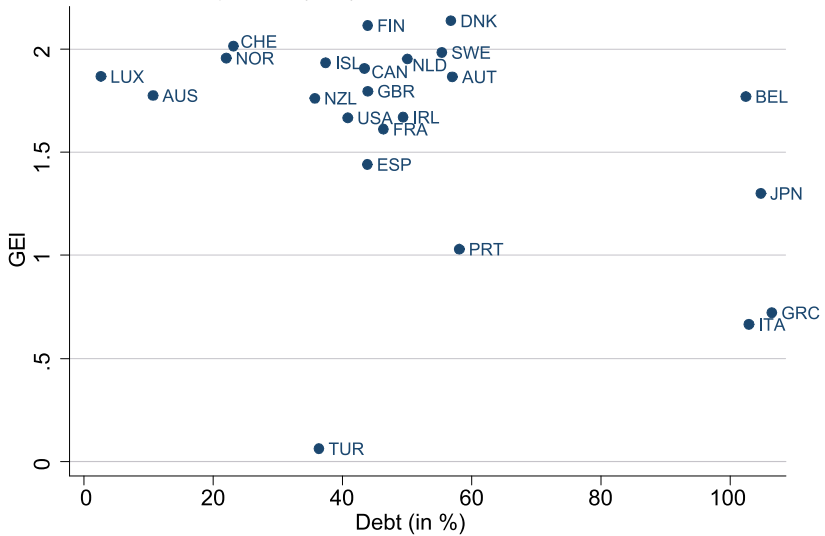Government Efficiency Index (GEI) vs. Debt-to-GDP Ratio

FIGURE 3.—Government efficiency and corruption versus government debt. The upper panel plots the index of corruption perception provided by Transparency International against the ratio of central government debt to GDP (source: OECD). The lower panel plots the index of effectiveness of governance from the World Bank's Worldwide Governance Indicators against the ratio of central government debt to GDP (source: OECD). The observations are averages over 1990–2008 and include all countries that were OECD members over the entire period.

Likewise, an old voter prefers candidate A over B if

$$U_O(g_A; b, \theta, R) > U_O(g_B; b, \theta, R).$$

σ^{iJ} (where $J \in \{Y, O\}$) is an individual-specific parameter drawn from a symmetric group-specific distribution that is assumed to be uniform in the support $[-1/(2\phi^J), 1/(2\phi^J)]$. Intuitively, a positive (negative) σ^{iJ} implies that voter i has a bias in favor of candidate B (candidate A). Note that the distributions have density ϕ^J and that neither group is, on average, biased toward either candidate. The parameter δ is an aggregate shock capturing the ex post average success of candidate B, whose realization becomes known after the policy platforms have been announced. δ is drawn from a uniform i.i.d. distribution on $[-1/(2\psi), 1/(2\psi)]$.[23] The sum of the terms $\sigma^{iJ} + \delta$ captures the relative appeal of candidate B: it is the inherent bias of individual i in group J for candidate B irrespective of the policy that the candidates propose. The assumption of uniform distributions is for simplicity (see Persson and Tabellini (2000), for a generalization).

Note that voters are rational and forward looking. They take into full account the effects of today's choice on future private- and public-good consumption. Because of repeated elections, they cannot decide directly over future fiscal policy. However, they can affect it through their choice of next-period debt level (b'), which affects future policy choices through the equilibrium policy functions B, T, and G.

It is at this point useful to identify the "swing voter" of each group, that is, the voter who is ex post indifferent between the two candidates:

$$\sigma^Y(b'_A, \tau_A, g_A, b'_B, \tau_B, g_B; b, \theta, R)$$
$$= U_Y(\tau_A, g_A, G(b'_A); b, \theta, R) - U_Y(\tau_B, g_B, G(b'_B); b, \theta, R) - \delta,$$
$$\sigma^O(g_A, g_B; b, \theta, R) = U_O(g_A; b, \theta, R) - U_O(g_B; b, \theta, R) - \delta.$$

Conditional on δ, the vote share of candidate A is

$$\pi_A(b'_A, \tau_A, g_A, b'_B, \tau_B, g_B; b, \theta, R)$$
$$= 1 - \pi_B(b'_A, \tau_A, g_A, b'_B, \tau_B, g_B; b, \theta, R)$$
$$= \frac{1}{2}\phi^Y\left(\sigma^Y(b'_A, \tau_A, g_A, b'_B, \tau_B, g_B; b, \theta, R) + \frac{1}{2\phi^Y}\right)$$
$$+ \frac{1}{2}\phi^O\left(\sigma^O(g_A, g_B; b, \theta, R) + \frac{1}{2\phi^O}\right)$$

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[23]The realization of δ can be viewed as the outcome of the campaign strategies to boost the candidates' popularity. Such an outcome is unknown ex ante.

$$= \frac{1}{2} + \frac{1}{2}\big(\phi^Y \times \big(U_Y\big(\tau_A, g_A, G\big(b'_A\big); b, \theta, R\big)$$
$$- U_Y\big(\tau_B, g_B, G\big(b'_B\big); b, \theta, R\big)\big) - \delta\big)$$
$$+ \frac{1}{2}\big(\phi^O \times \big(U_O(g_A; b, \theta, R) - U_O(g_B; b, \theta, R)\big) - \delta\big).$$

Note that π_A and π_B are stochastic variables, since δ is stochastic. The probability that candidate A wins is then given by

$$p_A = \mathrm{Prob}_\delta\left[\pi_A\big(b'_A, \tau_A, g_A, b'_B, \tau_B, g_B; b, \theta, R\big) \geq \frac{1}{2}\right]$$

$$= \mathrm{Prob}\left[\delta < \frac{\phi^Y}{\phi^Y + \phi^O}\big(U_Y\big(\tau_A, g_A, G\big(b'_A\big); b, \theta, R\big)\right.$$
$$- U_Y\big(\tau_B, g_B, G\big(b'_B\big); b, \theta, R\big)\big)$$
$$\left. + \frac{\phi^O}{\phi^Y + \phi^O}\big(U_O(g_A; b, \theta, R) - U_O(g_B; b, \theta, R)\big)\right]$$

$$= \frac{1}{2} + \psi(1 - \omega)\big(U_Y\big(\tau_A, g_A, G\big(b'_A\big); b, \theta, R\big)$$
$$- U_Y\big(\tau_B, g_B, G\big(b'_B\big); b, \theta, R\big)\big)$$
$$+ \psi\omega\big(U_O(g_A; b, \theta, R) - U_O(g_B; b, \theta, R)\big),$$

where $\omega \equiv \phi^O/(\phi^Y + \phi^O)$.

Since both candidates seek to maximize the probability of winning the election, the Nash equilibrium is characterized by the following equations:

$$\big(b'^*_A, \tau^*_A, g^*_A\big) = \max_{b'_A, \tau_A, g_A} (1 - \omega)\big(U_Y\big(\tau_A, g_A, G\big(b'_A\big); b, \theta, R\big)$$
$$- U_Y\big(\tau_B, g_B, G\big(b'_B\big); b, \theta, R\big)\big)$$
$$+ \omega\big(U_O(g_A; b, \theta, R) - U_O(g_B; b, \theta, R)\big),$$
$$\big(b'^*_B, \tau^*_B, g^*_B\big) = \max_{b'_B, \tau_B, g_B} (1 - \omega)\big(U_Y\big(\tau_B, g_B, G\big(b'_B\big); b, \theta, R\big)$$
$$- U_Y\big(\tau_A, g_A, G\big(b'_A\big); b, \theta, R\big)\big)$$
$$+ \omega\big(U_O(g_B; b, \theta, R) - U_O(g_A; b, \theta, R)\big).$$

Hence, the two candidates' platforms converge in equilibrium to the same fiscal policy maximizing the weighted-average utility of the young and

old,

$$
\begin{aligned}
\left(b_A^{\prime *}, \tau_A^*, g_A^*\right) &= \left(b_B^{\prime *}, \tau_B^*, g_B^*\right) \\
&= \max_{b^{\prime} \tau, g}\big((1-\omega) U_Y\big(\tau, g, G(b^{\prime}); b, \theta, R\big) \\
&\quad + \omega U_O(g; b, \theta, R)\big),
\end{aligned}
$$

subject to the government budget constraint. This is the political objective function given in the body of the paper.

Note that the parameter ω has a structural interpretation: it is a measure of the relative variability within the old group of the candidates' appeal. As shown above, ϕ^Y/ϕ^O (and, hence, ω) affects the number of swing voters in each group. For instance, suppose that $\phi^O > \phi^Y$. Intuitively, this means that the old are more "responsive," in electoral terms, to fiscal policy announcements in favor of or against them. An alternative interpretation is that $1/\phi^J$ measures the extent of group J heterogeneity with respect to other policy dimensions that are orthogonal to fiscal policy. For example, the young might work in different sectors and cast their votes also based on the sectoral policy proposed by each candidate. As a result, the vote of the young is less responsive to fiscal policy announcements, and the young have effectively less political power than the old. This interpretation is consistent with Mulligan and Sala-i-Martin (1999) and Hassler et al. (2005). In the extreme case of $\omega = 1$, the old only care about fiscal policy ($\phi^O \to \infty$) and the distribution has a mass point at $\sigma^O = 0$. In this case, the young have no influence and the old dictate the fiscal policy choice.

Suppose, next, that the groups have different relative size, and that there are N_O old voters and N_Y young voters. Proceeding as above, the political objective function is then modified to

$$
\begin{aligned}
\left(b_A^{\prime *}, \tau_A^*, g_A^*\right) &= \left(b_B^{\prime *}, \tau_B^*, g_B^*\right) \\
&= \max_{b^{\prime} \tau, g}\big\{(1-\omega) N_Y U_Y\big(\tau, g, G(b^{\prime}); b, \theta, R\big) \\
&\quad + \omega N_O U_O(g; b, \theta, R)\big\}.
\end{aligned}
$$

We conclude by noting that the probabilistic voting outlined in this appendix applies equally to both static and dynamic models (under the assumption of MPE). The political model entails some important restrictions. First, agents only condition their voting strategy on the payoff-relevant state variable (here, debt). Second, the shock δ is i.i.d. over time; otherwise, the previous realization of δ becomes a state variable, complicating the analysis substantially. Third, although the assumption of uniform distributions can be relaxed, it is necessary to impose regularity conditions on the density function to ensure that the maximization problem is well behaved.

B.2. *Derivation of Equation (9)*

Following the logic of the proof of Proposition 1, write the problem as

$$\max_{\{b'\in[\underline{b},\overline{b}],\tau\in[0,\overline{\tau}]\}} U\big(\tau, b' - Rb + \tau w(R)H(\tau), G\big(b'; \theta, R\big); \theta, R\big)$$

$$= (1 - \omega)\big((1 + \beta)\log\big(A(\tau; R)\big)$$

$$+ \theta\log\big(b' - Rb + \tau w(R)H(\tau)\big) + \beta\lambda\theta\log\big(G\big(b'; \theta, R\big)\big)\big)$$

$$+ \omega\big(\log\big(A(\tau_{-1}; R)\big) + \lambda\theta\log\big(b' - Rb + \tau w(R)H(\tau)\big)\big).$$

This yields

$$-\frac{(1 - \omega)(1 + \beta)}{A(\tau; R)}A_\tau(\tau; R) = \frac{\theta}{g}\big(1 + \omega(\lambda - 1)\big)\frac{\mathrm{d}}{\mathrm{d}\tau}\big(w(R)\tau H(\tau)\big).$$

Hence, using that fact that $A_\tau(\tau; R) = -w(R)H(\tau)$ and the definition of $e(\tau)$, we obtain

$$\frac{(1 - \omega)(1 + \beta)}{A(\tau; R)} = \big(1 - e(\tau)\big)\frac{\theta}{g}\big(1 + \omega(\lambda - 1)\big),$$

which is expression (9) in the text.

B.3. *Proof of Proposition 2*

Ignoring irrelevant terms, the planning problem can be expressed as

$$\mathcal{W} = \max_{\{g_t,\tau_t,b_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \big(\beta\big(\psi^t\lambda + \psi^{t+1}\big)\theta g_t + (1 + \beta)\psi^{t+1}A(\tau_t; R)\big)$$

subject to a period budget constraint,

$$b_{t+1} = g_t + Rb_t - \tau_t w(R)H(\tau_t),$$

a debt limit ($b_t \leq \overline{b}$ for all t), and a given b_0.

Write the problem as a standard Lagrange problem with multipliers ζ_t associated with the budget constraints. The first-order conditions for g_t, τ_t, and b_{t+1} yield

$$(12) \qquad 0 = \psi^t\frac{\beta\lambda\theta}{g_t} + \psi^{t+1}\frac{\theta}{g_t} - \zeta_t,$$

$$(13) \qquad 0 = -\psi^{t+1}(1 + \beta)\frac{\dfrac{dA(\tau_t; R)}{d\tau_t}}{A(\tau_t; R)} + w\zeta_t\Big(H(\tau_t) + \tau_t\frac{dH(\tau_t)}{d\tau_t}\Big),$$

$$(14) \qquad 0 = \zeta_t - \psi R\zeta_{t+1}.$$

Combining (12) and (13) and exploiting that $A_\tau = -w(R)H(\tau)$ and $e(\tau) = -(dH/d\tau)(\tau/H(\tau))$ yields (11). Combining (12) (for period t and $t+1$) and (14) yields $g'/g = \psi R$ as in the proposition. It is clear from this expression that $\lim_{t \to \infty} g_t = 0$ since the growth rate of g is constant and negative (i.e., $\psi R < 1$). In the case of elastic labor supply, the maximum tax rate is smaller than 100% ($\bar{\tau} < 1$), so the left-hand side of (11) must be bounded away from zero. Since $\lim_{t \to \infty} g_t = 0$, it follows that $\lim_{t \to \infty} e(\tau_t) = 1 = e(\bar{\tau})$ and, hence, $\lim_{t \to \infty} \tau_t = \bar{\tau}$. The budget constraint (1) then implies $\lim_{t \to \infty} b_t = \bar{b}$. In the case of inelastic labor supply, $e(\tau) = 0$ for all τ, so $\lim_{t \to \infty} g_t = 0$ and (11) imply $\lim_{t \to \infty} \tau_t = 1$ and, hence, $\lim_{t \to \infty} b_t = \bar{b}$.   *Q.E.D.*

B.4. *Calibrated Economy: Robustness Using the Krusell et al. (2002) Method*

The main numerical approach to compute the equilibrium in the calibrated economies is based on a projection method with Chebyshev collocation to approximate the policy function based on Equations (4), (9), and (7). To assess the robustness of our quantitative results, we also solve for the equilibrium using an alternative algorithm proposed by Krusell, Kuruscu, and Smith (2002; KKS henceforth) to compute the equilibrium policy functions. As opposed to the global nature of the projection method, KKS is based on the calculation of higher-order derivatives of (4), (9), and (7) around steady state. We find that when using derivatives up to the fourth order, the KKS algorithm identifies—up to the fourth decimal point for debt—the same (internal) steady state as the one we found using projection methods. As illustrated in Figure 1.a (the analogue of Figure 1), even outside of the steady state, the two alternative solutions for the policy rule are quantitatively similar.

B.5. *CRRA Utility*

In this section, we provide a complete characterization of the equilibrium under general CRRA utility. We consider the case of inelastic labor supply. The analysis can be extended to the case of elastic labor supply, though, as in the case of logarithmic utility, a full analytical solution is not available in this case.

PROPOSITION 3: *Assume that agents have CRRA utility*:

$$U_Y = \frac{c_Y^{1-\sigma} - 1}{1 - \sigma} + \theta \frac{g^{1-\sigma} - 1}{1 - \sigma} + \beta \left(\frac{(c_O')^{1-\sigma} - 1}{1 - \sigma} + \lambda \theta \frac{(g')^{1-\sigma} - 1}{1 - \sigma} \right),$$

where $\sigma > 1/2$. Then there exists a SSMPPE equilibrium with policy functions given by

$$(15) \qquad \tau = T(b; \theta, R)$$

$$= 1 - \left(1 + \beta(\beta R)^{(1-\sigma)/\sigma}\right) \left(\frac{1 - \omega}{\theta(1 + \omega(\lambda - 1))} \right)^{1/\sigma} \frac{\gamma(\theta, R)}{w(R)} (\bar{b} - b),$$
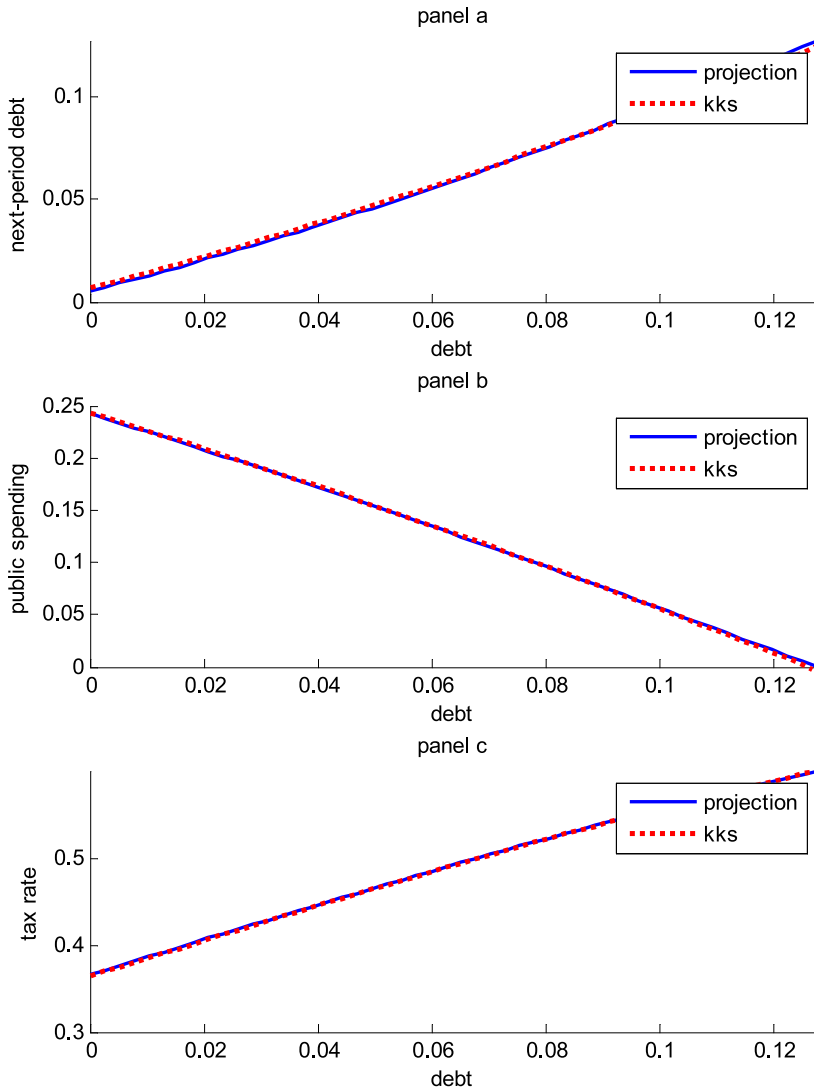
FIGURE 1.A.—Projection method versus Krusell, Kuruscu, and Smith (2002). The figure shows policy rules computed using two different numerical methods: the projection method (solid line) and the Krusell–Kuruscu–Smith (KKS) method (dotted line), respectively. Panels a, b, and c show the equilibrium policy rules for debt $B(b)$, public good provision $G(b)$, and taxes $T(b)$, respectively. Parameter values are the same as in the calibrated economy (see Table I).

$$b' = B(b; \theta, R) = \overline{b} - \left(\frac{(1-\omega)\lambda\beta}{1 + \omega(\lambda-1)} \gamma(\theta, R) \right)^{1/\sigma} (\overline{b} - b),$$

$$g = G(b; \theta, R) = \gamma(\theta, R)(\overline{b} - b),$$

where $\gamma(\theta, R)$ is the unique nonnegative solution to the polynomial

$$(16) \qquad \left(\frac{(1-\omega)\lambda\beta}{1 + \omega(\lambda-1)} \gamma(\theta, R) \right)^{1/\sigma}$$

$$= R - \gamma(\theta, R) \left(1 + \left(1 + \beta^{1/\sigma}(R)^{(1-\sigma)/\sigma} \right) \left(\frac{1-\omega}{\theta(1 + \omega(\lambda-1))} \right)^{1/\sigma} \right).$$

The world interest rate is pinned down by the unique solution of

$$(17) \qquad R = 1 + \Gamma(R; \omega, \lambda, \beta, \sigma, \theta),$$

where

$$\Gamma(R; \omega, \lambda, \beta, \sigma, \theta)$$

$$\equiv \frac{1 + \omega(\lambda-1)}{(1-\omega)\lambda\beta} \left(1 + \left(1 + \beta(\beta R)^{(1-\sigma)/\sigma} \right) \left(\frac{1-\omega}{\theta(1 + \omega(\lambda-1))} \right)^{1/\sigma} \right).$$

In the SSMPPE, $R = R^*(\overline{\theta})$, namely, the world interest rate is set by the countries with the strongest preference for the public good. All other countries have $R < R^*(\theta)$ and converge to immiseration. Finally, the average debt level for countries with $\theta = \overline{\theta}$ is unique and is given by

$$b(\overline{\theta}) = \overline{b} - \frac{(Q\alpha)^{1/(1-\alpha)}}{\upsilon}$$

$$\cdot \frac{\frac{1-\alpha}{\alpha} \frac{R}{(R-1)} (R)^{-\alpha/(1-\alpha)} + (R)^{-\alpha/(1-\alpha)}}{R + \left(\frac{(1-\omega)}{(1 + \omega(\lambda-1))} \right)^{(1-\sigma)/\sigma} \frac{1}{\lambda\beta} \left(\frac{\beta}{\theta} \right)^{1/\sigma} R^{1/\sigma}},$$

where $R = R^*(\overline{\theta})$.

PROOF: The optimal savings decision yields

$$c_Y = \frac{1}{1 + \beta^{1/\sigma} R^{(1-\sigma)/\sigma}} w(R)(1 - \tau),$$

$$c'_O = \frac{\beta^{1/\sigma} R^{(1-\sigma)/\sigma}}{1 + \beta^{1/\sigma} R^{(1-\sigma)/\sigma}} Rw(R)(1 - \tau).$$

Thus, ignoring irrelevant terms, we can write the political objective function as

$$U(\tau, b' - Rb + w\tau, G(b'); \theta, R)$$
$$= \frac{1}{1-\sigma}\Big[(1-\omega)\big(1+\beta(\beta R)^{(1-\sigma)/\sigma}\big)^{\sigma}\big(w(R)(1-\tau)\big)^{1-\sigma}$$
$$+ \theta\big(1+\omega(\lambda-1)\big)\big(b' - Rb + w(R)\tau\big)^{1-\sigma}$$
$$+ (1-\omega)\beta\lambda\theta\big(G(b'; \theta, R)\big)^{1-\sigma}\Big].$$

The first-order conditions yield

$$(1-\omega)\big(1+\beta(\beta R)^{(1-\sigma)/\sigma}\big)^{\sigma}\big(w(R)(1-\tau)\big)^{-\sigma}w(R)$$
$$= \theta\big(1+\omega(\lambda-1)\big)g^{-\sigma}w(R),$$
$$\theta\big(1+\omega(\lambda-1)\big)g^{-\sigma} + (1-\omega)\beta\lambda\theta(g')^{-\sigma}\frac{\partial G(b'; \theta, R)}{\partial b'} = 0.$$

Rearranging terms yields

$$(18) \qquad \frac{w(R)(1-\tau)}{g} = \big(1+\beta(\beta R)^{(1-\sigma)/\sigma}\big)\bigg(\frac{1-\omega}{\theta(1+\omega(\lambda-1))}\bigg)^{1/\sigma},$$

$$(19) \qquad \bigg(\frac{g'}{g}\bigg)^{\sigma} = -\frac{(1-\omega)\lambda\beta}{1+\omega(\lambda-1)}\frac{\partial G(b'; \theta, R)}{\partial b'}.$$

Next, the government budget constraint implies

$$\overline{b} - b' = \overline{b} - g - Rb - w(1-\tau) + w.$$

Plugging in (18), recalling that $w = (R-1)\overline{b}$, and rearranging terms yields

$$\overline{b} - b' = -g\bigg(1 + \big(1+\beta(\beta R)^{(1-\sigma)/\sigma}\big)\bigg(\frac{1-\omega}{\theta(1+\omega(\lambda-1))}\bigg)^{1/\sigma}\bigg)$$
$$+ R(\overline{b} - b).$$

Guess that $g = \gamma(\theta, R)(\overline{b} - b)$.
  The GEE implies that

$$(\overline{b} - b') = \bigg(\frac{(1-\omega)\lambda\beta}{1+\omega(\lambda-1)}\frac{\partial G(b'; \theta, R)}{\partial b'}\bigg)^{1/\sigma}(\overline{b} - b)$$
$$= \bigg(\frac{(1-\omega)\lambda\beta}{1+\omega(\lambda-1)}\gamma(\theta, R)\bigg)^{1/\sigma}(\overline{b} - b).$$

Hence, we can rewrite the budget constraint as

$$\left(\frac{(1-\omega)\lambda\beta}{1+\omega(\lambda-1)}\gamma(\theta,R)\right)^{1/\sigma}(\overline{b}-b)$$

$$= -\gamma(\theta,R)\left(1+\left(1+\beta(\beta R)^{(1-\sigma)/\sigma}\right)\right.$$

$$\left.\times\left(\frac{1-\omega}{\theta(1+\omega(\lambda-1))}\right)^{1/\sigma}\right)(\overline{b}-b)+R(\overline{b}-b).$$

Hence, $\gamma(\theta,R)$ is the solution to the polynomial equation (16). To see why (16) has a unique solution such that $\gamma>0$, note that the left-hand side of (16) is monotone increasing in γ (for $\gamma\geq0$), while the right-hand side is monotone decreasing in γ. Second, for $\gamma=0$, the left-hand side of (16) is zero, while the right-hand side is positive.

Moreover, as long as $\sigma\geq1$ (sufficient condition), the solution for γ is decreasing in R. To see why, note that the RHS of (16) is, in this case, increasing in R. Moreover, the LHS is increasing in γ while the RHS is decreasing in γ. Thus, an increase in R would increase the RHS of (16), which requires an offsetting increase in γ.

Next, we consider the stationary GE solution. In a steady state, $\overline{b}-b'=\overline{b}-b$, implying that

$$1=\left(\frac{(1-\omega)\lambda\beta}{1+\omega(\lambda-1)}\gamma(\theta,R)\right)^{1/\sigma}$$

$$\Rightarrow\quad\gamma(\theta,R)=\frac{1+\omega(\lambda-1)}{(1-\omega)\lambda\beta}.$$

Substituting this expression of $\gamma(\theta,R)$ into Equation (16) yields an implicit expression for the steady-state level of R (given θ),

$$R=1+\frac{1+\omega(\lambda-1)}{(1-\omega)\lambda\beta}$$

$$\times\left(1+\left(1+\beta^{1/\sigma}(R)^{(1-\sigma)/\sigma}\right)\left(\frac{1-\omega}{\theta(1+\omega(\lambda-1))}\right)^{1/\sigma}\right),$$

which is equivalent to (17). Consider the above definition of $\Gamma(R;\omega,\lambda,\beta,\sigma,\theta)$. Note that $\Gamma>0$. If $\sigma>1$, then Γ is decreasing in R. If $\sigma\in(1/2,1)$, then Γ is increasing and strictly concave in R. If $\sigma\in(0,1/2)$, then Γ is increasing and convex in R. Thus, $\sigma>1/2$ is a sufficient condition for the existence and uniqueness of a SSMPPE. Moreover, Γ is decreasing in θ. This ensures

that (assuming $\sigma > 1/2$) $R^*(\theta)$ is monotone decreasing in θ. The proof that $R = R^*(\bar{\theta})$ is the SSMPPE equilibrium interest rate proceeds as in the case of logarithmic utility discussed in Section 1. Finally, to compute the steady-state debt in high-θ economies (so as to obtain fiscal policy), note that, given the fiscal policy, savings can be computed as a function of b:

$$
\begin{aligned}
s &= w(1-\tau) - \frac{w(1-\tau)}{1+\beta^{1/\sigma}R^{(1-\sigma)/\sigma}} \\
&= \left(1 - \frac{1}{1+\beta^{1/\sigma}R^{(1-\sigma)/\sigma}}\right)\left(\frac{(1-\omega)}{(1+\omega(\lambda-1))\theta}\right)^{1/\sigma} \\
&\quad \times \left(1 + \beta^{1/\sigma}(R)^{(1-\sigma)/\sigma}\right)g \\
&= \left(\frac{(1-\omega)}{(1+\omega(\lambda-1))}\right)^{(1-\sigma)/\sigma}\frac{1}{\lambda\beta}\left(\frac{\beta}{\theta}\right)^{1/\sigma}R^{(1-\sigma)/\sigma}(\bar{b}-b).
\end{aligned}
$$

Recalling the equilibrium expression for τ yields an expression for savings in the high-θ countries:

$$
\begin{aligned}
s(T(b^*,R),R) &= \left(\frac{(1-\omega)}{(1+\omega(\lambda-1))}\right)^{(1-\sigma)/\sigma} \\
&\quad \times \frac{1}{\lambda\beta}\left(\frac{\beta}{\theta}\right)^{1/\sigma}R^{(1-\sigma)/\sigma}(\bar{b}-b^*).
\end{aligned}
$$

Imposing that $s_j = 0$ and $b_j = \bar{b}(R) = w(R)/(R-1)$ for all countries with $\theta_j < \bar{\theta}$, equilibrium condition 2 can be expressed as

$$
\upsilon \cdot s(T^*,R) = \upsilon b^* + (1-\upsilon)\bar{b} + K(R,1).
$$

Substituting in the savings yields an expression for the average debt in the high-θ economies:

$$
\begin{aligned}
&\left(1 + \left(\frac{(1-\omega)}{(1+\omega(\lambda-1))}\right)^{(1-\sigma)/\sigma}\frac{1}{\lambda\beta}\left(\frac{\beta}{\theta}\right)^{1/\sigma}R^{(1-\sigma)/\sigma}\right)\upsilon(\bar{b}-b^*) \\
&\quad = \bar{b} + K(R,1).
\end{aligned}
$$

Rearranging and substituting in the expressions for $K(R,1)$ and $w(R)$ yields

$$
\begin{aligned}
(20) \quad b^* &= \bar{b} - \frac{1}{\upsilon}\left(1 + \left(\frac{(1-\omega)}{(1+\omega(\lambda-1))}\right)^{(1-\sigma)/\sigma}\frac{1}{\lambda\beta}\left(\frac{\beta}{\theta}\right)^{1/\sigma}R^{(1-\sigma)/\sigma}\right)^{-1} \\
&\quad \cdot \left(\frac{(1-\alpha)Q^{1/(1-\alpha)}(\alpha)^{\alpha/(1-\alpha)}}{(R)^{\alpha/(1-\alpha)}(R-1)} + \left(\frac{Q\alpha}{R}\right)^{1/(1-\alpha)}\right)
\end{aligned}
$$

$$= \overline{b} - \frac{(Q\alpha)^{1/(1-\alpha)}}{\upsilon} \cdot \frac{\dfrac{1-\alpha}{\alpha} \dfrac{R}{(R-1)}(R)^{-\alpha/(1-\alpha)} + (R)^{-\alpha/(1-\alpha)}}{R + \left(\dfrac{(1-\omega)}{(1+\omega(\lambda-1))}\right)^{(1-\sigma)/\sigma} \dfrac{1}{\lambda\beta}\left(\dfrac{\beta}{\theta}\right)^{1/\sigma} R^{1/\sigma}},$$

where the numerator in the second term (on the right-hand side) of Equation (20) is monotone decreasing in R for $R > 1$, while the denominator is monotone increasing in R. *Q.E.D.*

B.6. *Calibrated Economy: Demographic Transition*

In this section, we consider a fully anticipated demographic transition such that, at $t = 0$, the economy is in the steady state described in the benchmark calibration of Table I with $N_0 = 1$, where N_t denotes the size of the cohort born at t. Then, at $t = 1$, there is an unexpected baby boom, with the size of the young population growing to $N_1 = 1.35$. This corresponds to an annualized population growth rate of 1%. Afterward, the population growth returns to zero ("baby bust"), so the cohort size stays constant at $N_t = 1.35$ for all $t \geq 2$. A falling population growth has two effects in the model. First, by increasing the relative size of the old cohort, it increases their political influence (recall that the political weight of the young and old are, respectively, $(1 - \omega)N_t$ and ωN_{t-1}). This causes a reduction in fiscal discipline and an increase in the current taste for the public good (since $\lambda \geq 1$). Second, the fall in population growth reduces the share of working population and the size of the tax base. Formally, the government budget constraint must be rewritten as $b_{t+1}(N_t + N_{t+1})/(N_{t-1} + N_t) = g_t + Rb_t - (N_t/(N_{t-1} + N_t))\tau_t wH(\tau_t)$, where b and g are now debt and public good per capita, respectively.

Figure 4 shows the impulse response of debt per capita, public good per capita, and taxes along the demographic transition. For illustrative purposes, we assume initial debt per capita to be equal to the final steady-state debt per capita. Clearly, this choice is arbitrary, but the main qualitative results do not hinge on it. At $t = 1$, when the share of young agents is large, debt falls, as anticipated above. Taxes are low and public good provision is high due to the large tax base. From $t = 2$ onward, taxes grow and public good provision falls. Most interestingly, debt starts growing and eventually converges to the steady state. Thus, our theory predicts that ageing societies increase debt accumulation, in line with the empirical observation that, since the 1980s, an increasing share of old voters has been accompanied by a rising government debt, especially in quickly ageing societies like Japan.

The U-shaped behavior of debt in the example also resembles the post-war pattern for debt in the United States and most Western European countries. Debt was initially high due to the war shock (in 1946, the U.S. federal debt-GDP ratio was 122%) and fell gradually until the end of the 1970s, reaching a trough of 33% in 1981. During this period, the population share over 40 went
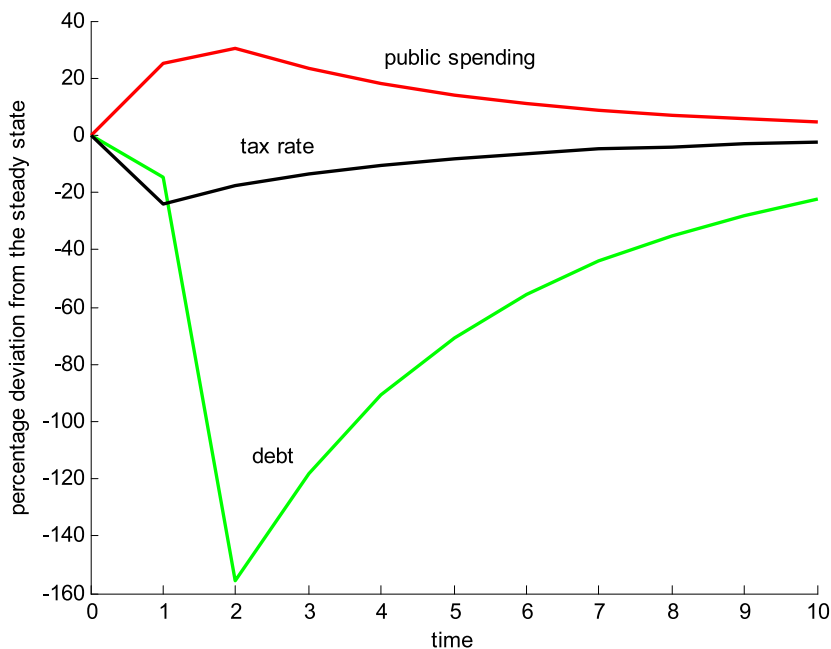
FIGURE 4.—Demographic transition. The figure shows impulse-response functions of a demographic boom-bust shock. The annualized population growth increases from zero to 1% in period $t = 1$ and reverts to zero thereafter. The initial debt at period 0 is that of steady state. The remaining parameter values are listed in Table I.

from 35% in 1948 to 36% in 1981. Thereafter, debt increased, reaching 68% in 2008, while the population share over 40 went up to 46%. In the same period, taxation grew and the share of government purchase of goods and services fell. Both facts are consistent with the impulse response of Figure 4.

ADDITIONAL REFERENCE

LINDBECK, A., AND J. W. WEIBULL (1987): "Balanced-Budget Redistribution as Political Equilibrium," *Public Choice*, 52, 273–297. [1]

University of Chicago Booth School of Business, Chicago, IL 60637, U.S.A.; zheng.song@chicagobooth.edu,
Dept. of Economics, University of Oslo, P.O. Box 1095 Blindern, 0317 Oslo, Norway; kjetil.storesletten@gmail.com,
                    and
Dept. of Economics, University of Zurich, Mühlebachstrasse 86, 8008 Zurich, Switzerland; fabrizio.zilibotti@econ.uzh.ch.