Supplementary Derivations for
Nonparametric Stochastic Discount Factor Decomposition

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This note works through parametric derivations required for the simulations and empirical application in the paper "Nonparametric Stochastic Discount Factor Decomposition".

1 Recursive preferences with linear-Gaussian state

1.1 Univariate case

First assume that:

$$g_{t+1} - \mu = \kappa(g_t - \mu) + \sigma e_{t+1}\,, \quad e_{t+1} \sim \text{ i.i.d. N}(0,1). \tag{1}$$

Step 1: fixed-point problem.

Guess a solution of the form $h(g) = e^{b_0 + b_1 g}$ to the fixed-point equation:

$$h(g_t) = \mathbb{E}[G_{t+1}^{1-\gamma}\big(h(g_{t+1})\big)^{\beta}|g_t]\,. \tag{2}$$

Then we have:

$$\begin{aligned} e^{b_0+b_1 g_t} &= \mathbb{E}[e^{(1-\gamma)g_{t+1}+\beta b_0+\beta b_1 g_{t+1}}|g_t] & (3)\\ &= e^{\beta b_0}\mathbb{E}[e^{(1-\gamma+\beta b_1)g_{t+1}}|g_t] & (4)\\ &= e^{\beta b_0} e^{(1-\gamma+\beta b_1)(1-\kappa)\mu+(1-\gamma+\beta b_1)\kappa g_t+\frac{1}{2}(1-\gamma+\beta b_1)^2\sigma^2}\,. & (5)\end{aligned}$$

Collecting coefficients of g_t:

$$b_1 = \kappa(1-\gamma+\beta b_1) \quad \Rightarrow \quad b_1 = \frac{\kappa(1-\gamma)}{1-\kappa\beta}\,. \tag{6}$$

Collecting constants:

$$b_0 = \beta b_0 + (1-\gamma+\beta b_1)(1-\kappa)\mu + \frac{1}{2}(1-\gamma+\beta b_1)^2\sigma^2 \tag{7}$$

$$\Rightarrow b_0 = \frac{1}{1-\beta}\left(b_2(1-\kappa)\mu + \frac{1}{2}b_2^2\sigma^2\right) \tag{8}$$

with $b_2 = 1-\gamma+\beta b_1$.

Also notice that unconditionally $g_t \sim N(\mu, \frac{\sigma^2}{1-\kappa^2})$. Therefore:

$$\|h\|^2 = e^{2b_0}\mathbb{E}[e^{2b_1 g_t}] = e^{2b_0+2b_1\mu+2b_1^2\frac{\sigma^2}{1-\kappa^2}} \quad \Rightarrow \quad \|h\| = e^{b_0+b_1\mu+b_1^2\frac{\sigma^2}{1-\kappa^2}} \tag{9}$$

and so:

$$\chi(g_t) = \frac{e^{b_0+b_1 g_t}}{e^{b_0+b_1\mu+b_1^2\frac{\sigma^2}{1-\kappa^2}}} = e^{-(b_1\mu+b_1^2\frac{\sigma^2}{1-\kappa^2})+b_1 g_t} = e^{c_0+b_1 g_t} \tag{10}$$

where $c_0 = -(b_1\mu + b_1^2\frac{\sigma^2}{1-\kappa^2})$. Finally,

$$\lambda = e^{(1-\beta)(b_0+b_1\mu+b_1^2\frac{\sigma^2}{1-\kappa^2})} = e^{(1-\beta)(b_0-c_0)}\,. \tag{11}$$

Step 2: Perron-Frobenius problem.

First, write the SDF as:

$$m(g_t, g_{t+1}) = \frac{\beta}{\lambda}G_{t+1}^{-\gamma}\frac{\chi(g_{t+1})^\beta}{\chi(g_t)} \tag{12}$$

$$= \frac{\beta}{\lambda}e^{-\gamma g_{t+1}+\beta(c_0+b_1 g_{t+1})-(c_0+b_1 g_t)} \tag{13}$$

$$= \frac{\beta}{\lambda}e^{(\beta-1)c_0}e^{(\beta b_1-\gamma)g_{t+1}-b_1 g_t}\,. \tag{14}$$

This gives:

$$\mathbb{E}[\log m(g_t, g_{t+1})] = \log\left(\frac{\beta}{\lambda}\right) + (\beta-1)c_0 + (\beta b_1 - \gamma - b_1)\mu \tag{15}$$

which is needed for the entropy calculation.

Guess a solution of the form $\phi(g) = e^{a_0+a_1 g}$ and plugging into $\mathbb{M}\phi = \rho\phi$, we obtain:

$$\rho e^{a_1 g_t} = \frac{\beta}{\lambda}e^{(\beta-1)c_0}e^{-b_1 g_t}\mathbb{E}[e^{(a_1+\beta b_1-\gamma)g_{t+1}}|g_t] \tag{16}$$

$$= \frac{\beta}{\lambda}e^{(\beta-1)c_0}e^{-b_1 g_t}e^{(a_1+\beta b_1-\gamma)(1-\kappa)\mu+(a_1+\beta b_1-\gamma)\kappa g_t+\frac{1}{2}(a_1+\beta b_1-\gamma)^2\sigma^2}\,. \tag{17}$$

Collecting coefficients of g_t:

$$a_1 = -b_1 + (a_1 + \beta b_1 - \gamma)\kappa \quad \Rightarrow \quad a_1 = \frac{-b_1 + \kappa(\beta b_1 - \gamma)}{1-\kappa} \tag{18}$$

and so $a_0 = -a_1 \mu - a_1^2 \frac{\sigma^2}{1-\kappa^2}$.

Moreover,

$$\rho = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{(a_1+\beta b_1-\gamma)(1-\kappa)\mu + \frac{1}{2}(a_1+\beta b_1-\gamma)^2\sigma^2} = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{\nu(1-\kappa)\mu + \frac{1}{2}\nu^2\sigma^2} \tag{19}$$

where $\nu = \beta b_1 - \gamma + a_1$.

Step 2b: time-reversed Perron-Frobenius problem.

Now we guess a solution of the form $\phi(g) = e^{a_0^* + a_1^* g}$. Plugging into $\mathbb{M}^*\phi^* = \rho\phi^*$ and using reversibility of $\{g_t\}$ (Weiss, 1975), we obtain:

$$\rho e^{a_1^* g_{t+1}} = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{(\beta b_1-\gamma)g_{t+1}} \mathbb{E}[e^{(a_1^*-b_1)g_t} | g_{t+1}] \tag{20}$$

$$= \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{(\beta b_1-\gamma)g_{t+1} + (a_1^*-b_1)(1-\kappa)\mu + (a_1^*-b_1)\kappa g_{t+1} + \frac{1}{2}(a_1^*-b_1)^2\sigma^2} \,. \tag{21}$$

Collecting coefficients of g_{t+1}:

$$a_1^* = \beta b_1 - \gamma + (a_1^* - b_1)\kappa \quad \Rightarrow \quad a_1^* = \frac{(\beta-\kappa)b_1 - \gamma}{1-\kappa} \,. \tag{22}$$

We choose a_0^* so that:

$$1 = \mathbb{E}[e^{a_0 + a_0^* + (a_1+a_1^*)g_t}] = e^{a_0 + a_0^* + (a_1+a_1^*)\mu + \frac{1}{2}(a_1+a_1^*)^2 \frac{\sigma^2}{1-\kappa^2}} \tag{23}$$

which implies $a_0^* = -(a_0 + (a_1 + a_1^*)\mu + \frac{1}{2}(a_1 + a_1^*)^2 \frac{\sigma^2}{1-\kappa^2})$.

Step 3: Covariance of the permanent and transitory components.

The permanent component is:

$$\frac{M_{t+1}^P}{M_t^P} = \frac{\frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{(\beta b_1-\gamma)g_{t+1} - b_1 g_t}}{\frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{\nu(1-\kappa)\mu + \frac{1}{2}\nu^2\sigma^2}} e^{a_1 g_{t+1} - a_1 g_t} = e^{-\nu(1-\kappa)\mu - \frac{1}{2}\nu^2\sigma^2} e^{\nu g_{t+1} - (b_1+a_1)g_t} \,. \tag{24}$$

Therefore:

$$m_{t+1}^P = \log\left(\frac{M_{t+1}^P}{M_t^P}\right) = \text{const} - (b_1 + a_1)g_t + \nu g_{t+1} \,. \tag{25}$$

The transitory component is:

$$\frac{M_{t+1}^T}{M_t^T} = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{\nu(1-\kappa)\mu+\frac{1}{2}\nu^2\sigma^2} e^{a_1 g_t - a_1 g_{t+1}} \tag{26}$$

and so:

$$m_{t+1}^T = \log\left(\frac{M_{t+1}^T}{M_t^T}\right) = \text{const} + a_1 g_t - a_1 g_{t+1}\,. \tag{27}$$

Notice that:

$$\begin{pmatrix} g_t \\ g_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \underbrace{\begin{pmatrix} \frac{\sigma^2}{1-\kappa^2} & \kappa\frac{\sigma^2}{1-\kappa^2} \\ \kappa\frac{\sigma^2}{1-\kappa^2} & \frac{\sigma^2}{1-\kappa^2} \end{pmatrix}}_{\mathbb{V}}\right). \tag{28}$$

Therefore:

$$\operatorname{Cov}(m_{t+1}^P, m_{t+1}^T) = v_P' \mathbb{V} v_T \qquad \operatorname{Var}(m_{t+1}^P) = v_P' \mathbb{V} v_P \qquad \operatorname{Var}(m_{t+1}^T) = v_T' \mathbb{V} v_T \tag{29}$$

where:

$$v_P = \begin{pmatrix} -(a_1+b_1) \\ \nu \end{pmatrix} \qquad\qquad v_T = \begin{pmatrix} a_1 \\ -a_1 \end{pmatrix}. \tag{30}$$

1.2 Multivariate case

Now assume that $g_{t+1} = \Gamma' X_{t+1}$ where

$$X_{t+1} - \mu = A(X_t - \mu) + \sigma e_{t+1}\,, \quad e_{t+1} \sim \text{ i.i.d. N}(0, I) \tag{31}$$

and let $\Sigma = \sigma\sigma'$.

Step 1: fixed-point problem.

Guess a solution of the form $h(x) = e^{b_0 + b_1' x}$ to the fixed-point equation:

$$\begin{aligned}
h(X_t) &= \mathbb{E}[G_{t+1}^{1-\gamma}\big(h(X_{t+1})\big)^{\beta}|X_t] && (32)\\
\Rightarrow e^{b_0+b_1'X_t} &= \mathbb{E}[e^{(1-\gamma)\Gamma' X_{t+1}+\beta b_0+\beta b_1' X_{t+1}}|X_t] && (33)\\
&= e^{\beta b_0}\mathbb{E}[e^{((1-\gamma)\Gamma+\beta b_1)' X_{t+1}}|X_t] && (34)\\
&= e^{\beta b_0} e^{((1-\gamma)\Gamma+\beta b_1)'(I-A)\mu+((1-\gamma)\Gamma+\beta b_1)' A X_t+\frac{1}{2}((1-\gamma)\Gamma+\beta b_1)'\Sigma((1-\gamma)\Gamma+\beta b_1)}\,. && (35)
\end{aligned}$$

Collecting coefficients of X_t:

$$b_1 = A'((1-\gamma)\Gamma + \beta b_1) \quad \Rightarrow \quad b_1 = (1-\gamma)(I-\beta A')^{-1}A'\Gamma\,. \tag{36}$$

Collecting constants:

$$b_0 = \beta b_0 + ((1-\gamma)\Gamma + \beta b_1)'(I-A)\mu + \frac{1}{2}((1-\gamma)\Gamma + \beta b_1)'\Sigma((1-\gamma)\Gamma + \beta b_1) \tag{37}$$
$$\Rightarrow b_0 = \frac{1}{1-\beta}\left(b_2'(I-A)\mu + \frac{1}{2}b_2'\Sigma b_2\right) \tag{38}$$

where $b_2 = \beta b_1 + (1-\gamma)\Gamma$.

The marginal of X_t is $N(\mu, V)$ where V solves $V = AVA' + \Sigma$. Therefore:

$$\|h\|^2 = e^{2b_0}\mathbb{E}[e^{2b_1'X_t}] = e^{2b_0+2b_1'\mu+2b_1'Vb_1} \quad \Rightarrow \quad \|h\| = e^{b_0+b_1'\mu+b_1'Vb_1} \tag{39}$$

and so

$$\chi(X_t) = \frac{e^{b_0+b_1'x}}{e^{b_0+b_1'\mu+b_1'Vb_1}} = e^{-(b_1'\mu+b_1'Vb_1)+b_1'X_t} = e^{c_0+b_1'X_t} \tag{40}$$

where $c_0 = -(b_1'\mu + b_1'Vb_1)$. Finally,

$$\lambda = e^{(1-\beta)(b_0+b_1'\mu+b_1'Vb_1)} = e^{(1-\beta)(b_0-c_0)}\,. \tag{41}$$

Step 2: Perron-Frobenius problem.

First, write the SDF as:

$$m(X_t, X_{t+1}) = \frac{\beta}{\lambda}G_{t+1}^{-\gamma}\frac{\chi(X_{t+1})^\beta}{\chi(X_t)} \tag{42}$$
$$= \frac{\beta}{\lambda}e^{-\gamma\Gamma'X_{t+1}+\beta(c_0+b_1'X_{t+1})-(c_0+b_1'X_t)} \tag{43}$$
$$= \frac{\beta}{\lambda}e^{(\beta-1)c_0}e^{(\beta b_1-\gamma\Gamma)'X_{t+1}-b_1'X_t}\,. \tag{44}$$

This gives:

$$\mathbb{E}[\log m(X_t, X_{t+1})] = \log\left(\frac{\beta}{\lambda}\right) + (\beta-1)c_0 + (\beta b_1 - \gamma\Gamma - b_1)'\mu \tag{45}$$

which is needed for the entropy calculation.

Guess a solution of the form $\phi(x) = e^{a_0 + a_1' x}$. Plugging into $\mathbb{M}\phi = \rho\phi$, we obtain:

$$\rho e^{a_1' X_t} = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{-b_1' X_t} \mathbb{E}[e^{(a_1+\beta b_1-\gamma\Gamma)' X_{t+1}} | X_t] \tag{46}$$

$$= \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{-b_1' X_t} e^{(a_1+\beta b_1-\gamma\Gamma)'(I-A)\mu + (a_1+\beta b_1-\gamma\Gamma)' A X_t + \frac{1}{2}(a_1+\beta b_1-\gamma\Gamma)'\Sigma(a_1+\beta b_1-\gamma\Gamma)}\,. \tag{47}$$

Collecting coefficients of X_t:

$$a_1 = -b_1 + A'(a_1 + \beta b_1 - \gamma\Gamma) \quad \Rightarrow \quad a_1 = (I - A')^{-1}(-b_1 + A'(\beta b_1 - \gamma\Gamma))\,. \tag{48}$$

Moreover,

$$\rho = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{(a_1+\beta b_1-\gamma\Gamma)'(I-A)\mu + \frac{1}{2}(a_1+\beta b_1-\gamma\Gamma)'\Sigma(a_1+\beta b_1-\gamma\Gamma)} = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{\nu'(I-A)\mu + \frac{1}{2}\nu'\Sigma\nu} \tag{49}$$

where $\nu = \beta b_1 - \gamma\Gamma + a_1$.

Step 3: Covariance of the permanent and transitory components.

The permanent component is:

$$\frac{M^P_{t+1}}{M^P_t} = \frac{\frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{(\beta b_1-\gamma\Gamma)' X_{t+1} - b_1' X_t}}{\frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{\nu'(I-A)\mu + \frac{1}{2}\nu'\Sigma\nu}} e^{a_1' X_{t+1} - a_1' X_t} = e^{-\nu'(I-A)\mu - \frac{1}{2}\nu'\Sigma\nu} e^{\nu' X_{t+1} - (a_1+b_1)' X_t} \tag{50}$$

and so:

$$m^P_{t+1} = \text{const} - (a_1 + b_1)' X_t + \nu' X_{t+1}\,. \tag{51}$$

The transitory component is:

$$\frac{M^T_{t+1}}{M^T_t} = \frac{\beta}{\lambda} e^{(\beta-1)c_0} e^{\nu'(I-A)\mu + \frac{1}{2}\nu'\Sigma\nu} e^{a_1' X_t - a_1' X_{t+1}} \tag{52}$$

and so:

$$m^T_{t+1} = \text{const} + a_1' X_t - a_1' X_{t+1}\,. \tag{53}$$

Notice that:

$$\begin{pmatrix} X_t \\ X_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \underbrace{\begin{pmatrix} V & VA' \\ AV & V \end{pmatrix}}_{\mathbb{V}} \right). \tag{54}$$

Therefore, the covariance of m^P_{t+1} and m^T_{t+1} are as in display (29) with $\mathbb{V}$ from (54) and:

$$v_P = \begin{pmatrix} -(a_1+b_1) \\ \nu \end{pmatrix} \qquad\qquad v_T = \begin{pmatrix} a_1 \\ -a_1 \end{pmatrix}. \tag{55}$$

2 Recursive preferences with stochastic volatility in consumption growth

2.1 Derivation

We use a specification of the consumption growth process similar to one studied in Backus, Chernov, and Zin (2014) but without habits. Assume that:

$$\begin{aligned} g_{t+1}-\mu &= \kappa(g_t-\mu)+v_t^{1/2}e_{t+1}\,, \quad e_{t+1}\sim \text{ i.i.d. N}(0,1) \\ v_{t+1} &= \text{ARG}(c_v,\varphi_v,\delta_v) \end{aligned}$$

where $\text{ARG}(c_v,\varphi_v,\delta_v)$ is an autoregressive gamma (ARG) process of order 1 parameterized by (c_v,φ_v,δ_v) that is independent of the sequence of $N(0,1)$ innovations. We refer the reader to Gourieroux and Jasiak (2006) for background material on ARG processes.

The state vector is $X_t=(g_t,v_t)'$. The following derivations are essentially a special case of Appendix H in Backus et al. (2014) but are included for compelteness. Throughout, we will use the fact that:

$$\mathbb{E}[e^{sv_{t+1}}|v_t] = e^{\frac{\varphi_v s}{1-sc_v}v_t-\delta_v\log(1-sc_v)} \tag{56}$$

(see Appendix H in Backus et al. (2014)).

Step 1: fixed-point problem.

Guess a solution of the form $h(g,v)=e^{a_0+b_1g+a_2v}$ to the fixed-point equation:

$$\begin{aligned} h(g_t,v_t) &= \mathbb{E}[G_{t+1}^{1-\gamma}\big(h(g_{t+1},v_{t+1})\big)^\beta|g_t,v_t] &(57)\\ \Rightarrow e^{a_0+a_1g_t+a_2v_t} &= \mathbb{E}[e^{(1-\gamma)g_{t+1}+\beta a_0+\beta a_1 g_{t+1}+\beta a_2 v_{t+1}}|g_t,v_t] &(58)\\ &= e^{\beta a_0}\mathbb{E}[e^{(1-\gamma+\beta a_1)g_{t+1}+\beta a_2 v_{t+1}}|g_t,v_t] &(59)\\ &= e^{\beta a_0}e^{(1-\gamma+\beta a_1)(1-\kappa)\mu+(1-\gamma+\beta a_1)\kappa g_t+\frac{1}{2}(1-\gamma+\beta a_1)^2 v_t}\mathbb{E}[e^{\beta a_2 v_{t+1}}|v_t] &(60)\\ &= e^{\beta a_0}e^{(1-\gamma+\beta a_1)(1-\kappa)\mu+(1-\gamma+\beta a_1)\kappa g_t+\frac{1}{2}(1-\gamma+\beta a_1)^2 v_t}e^{\frac{\varphi_v\beta a_2}{1-\beta a_2 c_v}v_t-\delta_v\log(1-\beta a_2 c_v)}. &(61) \end{aligned}$$

Collecting coefficients of g_t:

$$a_1 = \kappa(1 - \gamma + \beta a_1) \quad \Rightarrow \quad a_1 = \frac{\kappa(1-\gamma)}{1 - \beta\kappa}\,. \tag{62}$$

Collecting coefficients of v_t:

$$a_2 = \frac{1}{2}(1 - \gamma + \beta a_1)^2 + \frac{\varphi_v \beta a_2}{1 - \beta a_2 c_v} \quad \Rightarrow a_2 - \nu = \frac{\varphi_v \beta a_2}{1 - \beta a_2 c_v} \tag{63}$$

where $\nu = \frac{1}{2}(1 - \gamma + \beta a_1)^2$. Rearranging:

$$0 = (a_2 - \nu)(1 - \beta c_v a_2) - \beta \varphi_v a_2 \tag{64}$$

$$= \beta c_v a_2^2 + (\beta(\varphi_v - c_v \nu) - 1) a_2 + \nu \tag{65}$$

so we solve the quadratic and take the smallest root to obtain:

$$a_2 = \frac{-(\beta(\varphi_v - c_v\nu) - 1) - \sqrt{(\beta(\varphi_v - c_v\nu) - 1)^2 - 4\beta c_v \nu}}{2\beta c_v} \tag{66}$$

provided $(\beta(\varphi_v - c_v\nu) - 1)^2 - 4\beta c_v \nu \geq 0$. Taking the smallest root ensures that $a_2 = 0$ if the variance of $\log g_t$ is zero (see Appendix H in Backus et al. (2014)).

Collecting the constant terms:

$$a_0 = \beta a_0 + (1 - \gamma + \beta a_1)(1 - \kappa)\mu - \delta_v \log(1 - \beta a_2 c_v) \tag{67}$$

$$\Rightarrow a_0 = \frac{1}{1 - \beta}\left((1 - \gamma + \beta a_1)(1 - \kappa)\mu - \delta_v \log(1 - \beta a_2 c_v)\right) \tag{68}$$

provided $\beta a_2 c_v < 1$.

Step 2: Perron-Frobenius problem.

First, write the SDF as:

$$m(X_t, X_{t+1}) = \beta G_{t+1}^{-\gamma} \frac{h(X_{t+1})^\beta}{h(X_t)} \tag{69}$$

$$= \beta e^{-\gamma g_{t+1} + \beta(a_0 + a_1 g_{t+1} + a_2 v_{t+1}) - (a_0 + a_1 g_t + a_2 v_t)} \tag{70}$$

$$= \beta e^{(\beta - 1)a_0 + (\beta a_1 - \gamma) g_{t+1} + \beta a_2 v_{t+1} - a_1 g_t - a_2 v_t} \tag{71}$$

$$= d_0 e^{d_1 g_{t+1} + d_2 v_{t+1} + d_3 g_t + d_4 v_t} \tag{72}$$

where

$$d_0 = \beta e^{(\beta-1)a_0} \qquad d_1 = \beta a_1 - \gamma \qquad d_2 = \beta a_2 \qquad d_3 = -a_1 \qquad d_4 = -a_2 \,. \tag{73}$$

In this notation we have:

$$\mathbb{E}[\log m(X_t, X_{t+1})] = \log(d_0) + (d_1 + d_3)\mu + (d_2 + d_4)\frac{\delta_v c_v}{1 - \varphi_v} \tag{74}$$

which is needed for the entropy of the permanent component.

Conjecture a solution of the form $\phi(g, v) = e^{c_0 + c_1 g + c_2 v}$. Plugging into $\mathbb{M}\phi = \rho\phi$ gives:

$$\begin{aligned}
\rho e^{c_1 g_t + c_2 v_t} &= d_0 e^{d_3 g_t + d_4 v_t} \mathbb{E}\big[e^{(d_1+c_1)g_{t+1} + (d_2+c_2)v_{t+1}} \big| g_t, v_t\big] && (75)\\
&= d_0 e^{d_3 g_t + d_4 v_t} e^{(d_1+c_1)(1-\kappa)\mu + (d_1+c_1)\kappa g_t + \frac{1}{2}(d_1+c_1)^2 v_t} \mathbb{E}\big[e^{(d_2+c_2)v_{t+1}} \big| v_t\big] && (76)\\
&= d_0 e^{(d_1+c_1)(1-\kappa)\mu} e^{[d_3 + (d_1+c_1)\kappa]g_t + [d_4 + \frac{1}{2}(d_1+c_1)^2]v_t} \mathbb{E}\big[e^{(d_2+c_2)v_{t+1}} \big| v_t\big] && (77)\\
&= d_0 e^{(d_1+c_1)(1-\kappa)\mu} e^{[d_3 + (d_1+c_1)\kappa]g_t + [d_4 + \frac{1}{2}(d_1+c_1)^2]v_t} e^{\frac{\varphi_v(d_2+c_2)}{1-c_v(d_2+c_2)}v_t - \delta_v \log(1 - c_v(d_2+c_2))} && (78)\\
&= d_0 e^{(d_1+c_1)(1-\kappa)\mu - \delta_v \log(1-c_v(d_2+c_2))} e^{[d_3 + (d_1+c_1)\kappa]g_t + [d_4 + \frac{1}{2}(d_1+c_1)^2 + \frac{\varphi_v(d_2+c_2)}{1-c_v(d_2+c_2)}]v_t} \,. && (79)
\end{aligned}$$

Collecting coefficients of g_t:

$$c_1 = d_3 + \kappa(d_1 + c_1) \quad \Rightarrow \quad c_1 = \frac{d_3 + \kappa d_1}{1 - \kappa} \,. \tag{80}$$

Collecting coefficients of v_t:

$$c_2 = d_4 + \frac{1}{2}(d_1 + c_1)^2 + \frac{\varphi_v(d_2 + c_2)}{1 - c_v(d_2 + c_2)} \,. \tag{81}$$

Let $c_3 = c_2 + d_2$. Then:

$$c_3 = d_2 + d_4 + \frac{1}{2}(d_1 + c_1)^2 + \frac{\varphi_v c_3}{1 - c_v c_3} = \xi + \frac{\varphi_v c_3}{1 - c_v c_3} \,. \tag{82}$$

where $\xi = d_2 + d_4 + \frac{1}{2}(d_1 + c_1)^2$. Rearranging:

$$\begin{aligned}
(c_3 - \xi)(1 - c_v c_3) &= \varphi_v c_3 && (83)\\
\Rightarrow 0 &= c_v c_3^2 + (\varphi_v - 1 - c_v \xi)c_3 + \xi \,. && (84)
\end{aligned}$$

Solving the quadratic and again taking the negative root, we obtain:

$$c_3 = \frac{-(\varphi_v - 1 - c_v \xi) - \sqrt{(\varphi_v - 1 - c_v \xi)^2 - 4 c_v \xi}}{2 c_v} \tag{85}$$

provided $(\varphi_v - 1 - c_v \xi)^2 - 4 c_v \xi \geq 0$. Finally, set $c_2 = c_3 - d_2$.

We also obtain

$$\rho = d_0 e^{(d_1 + c_1)(1-\kappa)\mu - \delta_v \log(1 - c_v(d_2 + c_2))} \tag{86}$$

provided $c_v(d_2 + c_2) < 1$.

Step 3: Covariance of the permanent and transitory components.

The permanent component is:

$$\frac{M^P_{t+1}}{M^P_t} = \frac{d_0}{d_0 e^{(d_1 + c_1)(1-\kappa)\mu - \delta_v \log(1 - c_v(d_2 + c_2))}} e^{(d_1+c_1) g_{t+1} + (d_2 + c_2) v_{t+1} + (d_3 - c_1) g_t + (d_4 - c_2) v_t} \tag{87}$$

and so:

$$m^P_{t+1} = \text{const} + \left(\begin{array}{c} d_3 - c_1 \\ d_4 - c_2 \end{array} \right)' X_t + \left(\begin{array}{c} d_1 + c_1 \\ d_2 + c_2 \end{array} \right)' X_{t+1} \,. \tag{88}$$

The transitory component is:

$$\frac{M^T_{t+1}}{M^T_t} = d_0 e^{(d_1 + c_1)(1-\kappa)\mu - \delta \log(1 - c_v(d_2 + c_2))} e^{-c_1 g_{t+1} - c_2 v_{t+1} + c_1 g_t + c_2 v_t} \tag{89}$$

and so:

$$m^T_{t+1} = \text{const} + \left(\begin{array}{c} c_1 \\ c_2 \end{array} \right)' X_t - \left(\begin{array}{c} c_1 \\ c_2 \end{array} \right)' X_{t+1} \,. \tag{90}$$

It remains to construct the covariance matrix $\mathbb{V}$ of (X_t, X_{t+1}). From Gourieroux and Jasiak (2006) we have:

$$\mathbb{E}[v_t] = \frac{c_v \delta_v}{1 - \varphi_v} \tag{91}$$

$$\mathrm{Var}(v_t) = \frac{c_v^2 \delta_v}{(1 - \varphi_v)^2} \tag{92}$$

$$\mathrm{Cov}(v_t, v_{t+1}) = \varphi_v \mathrm{Var}(v_t) = \frac{\varphi_v c_v^2 \delta_v}{(1 - \varphi_v)^2} \,. \tag{93}$$

It follows that

$$\text{Var}(g_t) = \frac{c_v \delta_v}{(1-\varphi_v)(1-\kappa^2)} \tag{94}$$

$$\text{Cov}(g_t, g_{t+1}) = \kappa \text{Var}(g_t) = \frac{\kappa c_v \delta_v}{(1-\varphi_v)(1-\kappa^2)} . \tag{95}$$

Finally, note that $\text{Cov}(g_t, v_{t+1})$, $\text{Cov}(v_t, g_{t+1})$, and $\text{Cov}(v_{t+1}, g_{t+1})$ are all zero. The covariance matrix of $(X_t', X_{t+1}')'$ is therefore:

$$\mathbb{V} = \begin{pmatrix} \frac{c_v\delta_v}{(1-\varphi_v)(1-\kappa^2)} & 0 & \kappa\frac{c_v\delta_v}{(1-\varphi_v)(1-\kappa^2)} & 0 \\ 0 & \frac{c_v^2\delta_v}{(1-\varphi_v)^2} & 0 & \varphi_v\frac{c_v^2\delta_v}{(1-\varphi_v)^2} \\ \kappa\frac{c_v\delta_v}{(1-\varphi_v)(1-\kappa^2)} & 0 & \frac{c_v\delta_v}{(1-\varphi_v)(1-\kappa^2)} & 0 \\ 0 & \varphi_v\frac{c_v^2\delta_v}{(1-\varphi_v)^2} & 0 & \frac{c_v^2\delta_v}{(1-\varphi_v)^2} \end{pmatrix} . \tag{96}$$

The covariance of m_{t+1}^P and m_{t+1}^T are as in display (29) with this $\mathbb{V}$ and with:

$$v_P = \begin{pmatrix} d_3 - c_1 \\ d_4 - c_2 \\ d_1 + c_1 \\ d_2 + c_2 \end{pmatrix} \qquad v_T = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 \\ -c_2 \end{pmatrix} . \tag{97}$$

2.2 Estimation

We estimate the parameters $(\mu, \kappa, \varphi_v, c_v, \delta_v)$ by indirect inference. The auxiliary model for $\{g_t\}$ is an AR(1) process with a GARCH(1,1) process fit to the AR(1) residuals. We use 500 simulated series. In fitting the auxiliary model, we first estimate the AR(1) parameters μ and κ by regression then and fit the GARCH(1,1) process to the residuals by maximum likelihood.

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