Econometrica Supplementary Material

SUPPLEMENT TO "POWER IN HIGH-DIMENSIONAL TESTING PROBLEMS"

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This supplement contains all appendices to the main article. In particular, the supplement contains the proofs of all results in the paper.

CONTENTS

THROUGHOUT, GIVEN A RANDOM variable (or vector) x defined on a probability space $(F, \mathcal{F}, \mathbb{Q})$, the image measure induced by x is denoted by $\mathbb{Q} \circ x$. Furthermore, $\Rightarrow$ denotes weak convergence.

APPENDIX A: ADDITIONAL MATERIAL FOR SECTION 1.1

The following lemma shows that the test ϕ_n considered in Section 1.1 is consistent against θ_n if and only if $d(n)^{-1/2}n\|\theta_n\|_2^2 \to \infty$. The result is probably well known, but difficult to pinpoint in the literature in this form; therefore, we provide a direct argument for completeness and for the convenience of the reader.

LEMMA A.1: *Let $\alpha \in (0,1)$, let $d(n)$ diverge to ∞, and let $X_1, \dots, X_n$ be i.i.d. $N_{d(n)}(\theta, I_{d(n)})$. Then the test ϕ_n, which rejects the null hypothesis $H_0 : \theta = 0$ if the squared $\|\cdot\|_2$ norm of $Z_n = n^{-1/2}\sum_{i=1}^n X_i$ exceeds the $1-\alpha$ quantile of a χ^2 distribution with $d(n)$ degrees of freedom, has* (i) *size α for every $n \in \mathbb{N}$ and* (ii) *is consistent against a sequence θ_n, where $\theta_n \in \mathbb{R}^{d(n)}$ for every n, if and only if*

$$\rho_n := d(n)^{-1/2}n\|\theta_n\|_2^2 \to \infty. \tag{A.1}$$

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PROOF: Part (i) is trivial, because under the null $\|Z_n\|_2^2$ is χ^2 distributed with $d(n)$ degrees of freedom.

Consider now part (ii). Denote the $1-\alpha$ quantile of a χ^2 distribution with $d(n)$ degrees of freedom by κ_n. Observe that ϕ_n rejects if and only if

$$\big(\|Z_n\|_2^2 - d(n)\big)/\sqrt{2d(n)} > \big(\kappa_n - d(n)\big)/\sqrt{2d(n)}. \tag{A.2}$$

It follows immediately from the central limit theorem that under the null, $(\|Z_n\|_2^2 - d(n))/\sqrt{2d(n)} \Rightarrow N(0,1)$. Consequently, we obtain from part (i) that $(\kappa_n - d(n))/\sqrt{2d(n)}$ must converge to the $1-\alpha$ quantile η, say, of a standard normal distribution. Let $\theta_n \neq 0$ be a sequence of alternatives. Writing $\|Z_n\|_2^2 = \|G_n\|_2^2 + \sqrt{n}2G_n'\theta_n + n\|\theta_n\|_2^2$ with $G_n := Z_n - n^{1/2}\theta_n \sim N_{d(n)}(0, I_{d(n)})$, we have

$$\begin{aligned}\big(\|Z_n\|_2^2 - d(n)\big)/\sqrt{2d(n)} &= \big(\|G_n\|_2^2 - d(n)\big)/\sqrt{2d(n)} \\ &\quad + \big(\sqrt{n}2G_n'\theta_n + n\|\theta_n\|_2^2\big)/\sqrt{2d(n)}.\end{aligned} \tag{A.3}$$

The distribution of the first summand to the right in the previous display does not depend on θ_n and converges weakly to $N(0,1)$; the second summand to the right is $N(\mu_n, \sigma_n^2)$ distributed with

$$\mu_n := 2^{-1/2}\rho_n \quad \text{and} \quad \sigma_n^2 := \frac{2}{d^{1/2}(n)}\rho_n.$$

To prove sufficiency, suppose that θ_n satisfies (A.1). Obviously, ϕ_n rejects if and only if

$$\rho_n^{-1}\big(\|Z_n\|_2^2 - d(n)\big)/\sqrt{2d(n)} > \rho_n^{-1}\big(\kappa_n - d(n)\big)/\sqrt{2d(n)}. \tag{A.4}$$

Since $\rho_n \to \infty$ and because the sequence $(\kappa_n - d(n))/\sqrt{2d(n)} \to \eta$, as pointed out above, the right hand side converges to 0. From (A.3) and the observations succeeding it, we conclude that the sequence of random variables to the left in (A.4) converges in probability to $2^{-1/2}$. This, together with the Portmanteau theorem, implies that the test under consideration is consistent against θ_n. Next, we establish necessity. Suppose ρ_n converges to ρ, say, along a subsequence n'. Then $N(\mu_n, \sigma_n^2) \Rightarrow \delta_{2^{-1/2}\rho}$ along n', and by Slutzky's lemma and (A.3), the sequence of random variables to the left in (A.2) converges weakly to $N(2^{-1/2}\rho, 1)$ along n'. From $(\kappa_n - d(n))/\sqrt{2d(n)} \to \eta$ and the Portmanteau theorem it then immediately follows that the sequence of tests under consideration is not consistent against such a sequence of alternatives θ_n. *Q.E.D.*

APPENDIX B: PROOF OF THEOREM 4.1

The statement trivially holds for $\alpha = 1$. Let $\alpha \in (0,1)$. Suppose we could construct a sequence of tests $\varphi_n^* : \Omega_{n,d} \to [0,1]$ with the property that for some $\varepsilon > 0$ such that $B(\varepsilon) = \{z \in \mathbb{R}^d : \|z\|_2 < \varepsilon\} \subsetneq \Theta_d$ (recall that Θ_d is assumed throughout to contain an open neighborhood of the origin), $\mathbb{E}_{n,d,0}(\varphi_n^*) \to \alpha$ holds, and for any sequence $\theta_n \in B(\varepsilon)$ such that $n^{1/2}\|\theta_n\|_2 \to \infty$, it holds that $\mathbb{E}_{n,d,\theta_n}(\varphi_n^*) \to 1$. Given such a sequence of tests, we could define tests $\varphi_n = \min(\varphi_n^* + \psi_{n,d}(\varepsilon), 1)$ (cf. Assumption 1), and note that φ_n has asymptotic size α, and has the property that $\mathbb{E}_{n,d,\theta_n}(\varphi_n) \to 1$ for any sequence $\theta_n \in \Theta_d$ such that $n^{1/2}\|\theta_n\|_2 \to \infty$. But tests with the latter property are certainly not asymptotically enhanceable, because tests $\nu_n : \Omega_{n,d} \to [0,1]$ can satisfy $\mathbb{E}_{n,d,0}(\nu_n) \to 0$ and $\mathbb{E}_{n,d,\theta_n}(\nu_n) \to 1$

only if $\theta_n \in \Theta_d$ satisfies $n^{1/2}\|\theta_n\|_2 \to \infty$. To see this, use Remark 3.1 and recall that convergence of $n^{1/2}\|\theta_n\|_2$ along a subsequence n' together with the maintained i.i.d. and $\mathbb{L}_2$-differentiability assumption implies contiguity of $\mathbb{P}_{n',d,\theta_{n'}}$ w.r.t. $\mathbb{P}_{n',d,0}$ (this can be verified easily using, for example, results in Section 1.5 of Liese and Miescke (2008) and Theorem 6.26 in the same reference). It hence remains to construct such a sequence φ_n^*. To this end, denote by $L : \Omega \to \mathbb{R}^d$ (measurable) an $\mathbb{L}_2$-derivative of $\{\mathbb{P}_{d,\theta} : \theta \in \Theta_d\}$ at 0. In the following argument, we denote expectation w.r.t. $\mathbb{P}_{d,\theta}$ by $\mathbb{E}_{d,\theta}$. By assumption, the information matrix $\mathbb{E}_{d,0}(LL') = \mathsf{I}_d$ is positive definite. Let $C > 0$ and define $L_C = L\mathbf{1}\{\|L\|_2 \le C\}$. Since $\mathbb{E}_{d,0}(L_C L')$ and $M(C) = \mathbb{E}_{d,0}((L_C - \mathbb{E}_{d,0}(L_C))(L_C - \mathbb{E}_{d,0}(L_C))')$ converge to I_d as $C \to \infty$ (by the dominated convergence theorem and $\mathbb{E}_{d,0}(L) = 0$; for the latter, see Proposition 1.110 in Liese and Miescke (2008)), there exists a C^* such that $\mathbb{E}_{d,0}(L_{C^*}L')$ and $M := M(C^*)$ are nonsingular. Now, by the $\mathbb{L}_2$-differentiability assumption (again using Proposition 1.110 in Liese and Miescke (2008)), there exists an $\varepsilon > 0$ and a $c > 0$ such that $B(\varepsilon) \subsetneq \Theta_d$ and such that

$$\left\|\mathbb{E}_{d,\theta}(L_{C^*}) - \mathbb{E}_{d,0}(L_{C^*})\right\|_2 \ge c\|\theta\|_2 \quad \text{holds for every } \theta \in B(\varepsilon). \tag{B.1}$$

Define on $\times_{i=1}^n \Omega$ the functions $Z_n(\theta) := n^{-1/2}\sum_{i=1}^n (L_{C^*}(\omega_{i,n}) - \mathbb{E}_{d,\theta}(L_{C^*}))$ for $\theta \in \Theta_d$, where $\omega_{i,n}$ denotes the ith coordinate projection on $\times_{i=1}^n \Omega$, and set $Z_n(0) = Z_n$. It is easy to verify that $\mathbb{P}_{n,d,\theta_n} \circ Z_n(\theta_n)$ is tight for any sequence $\theta_n \in \Theta_d$ and that, by the central limit theorem, $\mathbb{P}_{n,d,0} \circ Z_n \Rightarrow N_d(0, M)$. Finally, let $\varphi_n^* : \Omega_{n,d} \to [0, 1]$ be the indicator function of the set $\{\|Z_n\|_2 \ge Q_\alpha\}$, where Q_α denotes the $1-\alpha$ quantile of the distribution of the Euclidean norm of an $N_d(0, M)$ distributed random vector. By construction, $\mathbb{E}_{n,d,0}(\varphi_n^*) \to \alpha$. It remains to verify $\mathbb{E}_{n,d,\theta_n}(\varphi_n^*) \to 1$ for any sequence $\theta_n \in B(\varepsilon)$ such that $n^{1/2}\|\theta_n\|_2 \to \infty$. Let θ_n be such a sequence. By the triangle inequality,

$$\|Z_n\|_2 \ge n^{1/2}\left\|\mathbb{E}_{d,\theta_n}(L_{C^*}) - \mathbb{E}_{d,0}(L_{C^*})\right\|_2 - \left\|Z_n(\theta_n)\right\|_2.$$

Hence, $1 - \mathbb{E}_{n,d,\theta_n}(\varphi_n^*)$ is not greater (cf. (B.1)) than $\mathbb{P}_{n,d,\theta_n}(cn^{1/2}\|\theta_n\|_2 - Q_\alpha \le \|Z_n(\theta_n)\|_2) \to 0$, the convergence following from $\mathbb{P}_{n,d,\theta_n} \circ Z_n(\theta_n)$ being tight, and $cn^{1/2}\|\theta_n\|_2 \to \infty$. *Q.E.D.*

APPENDIX C: THEOREM C.1

In this section we present our second result concerning asymptotic enhanceability in the fixed-dimensional case, which was already referred to in Section 4.

THEOREM C.1: *Let $d(n) \equiv d$ for some $d \in \mathbb{N}$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Assume that a sequence of estimators $\hat{\theta}_n : \Omega_{n,d} \to \Theta_d$ (measurable) satisfies the following conditions*:

(i) *Uniform consistency*: *For every $\varepsilon > 0$, $\sup_{\theta\in\Theta_d} \mathbb{P}_{n,d,\theta}(\|\hat{\theta}_n - \theta\| > \varepsilon) \to 0$.*

(ii) *Contiguity rate*: *There exists a nondecreasing sequence $s_n > 0$ diverging to ∞ such that for every sequence $\theta_n \in \Theta_d$ such that $s_n\|\theta_n\|$ is bounded, the sequence $\mathbb{P}_{n,d,\theta_n}$ is contiguous w.r.t. $\mathbb{P}_{n,d,0}$.*

(iii) *Local uniform tightness*: *There exists a $\delta > 0$ such that for every sequence θ_n in Θ_d satisfying $\|\theta_n\| \le \delta$, the sequence of (image) measures $\mathbb{P}_{n,d,\theta_n} \circ [s_n(\hat{\theta}_n - \theta_n)]$ is tight.*

Then, for every $\alpha \in (0, 1]$, there exists a $C = C(\alpha) \ge 0$ such that the sequence of tests $\varphi_n = \mathbf{1}\{s_n\|\hat{\theta}\| \ge C\}$ is not asymptotically enhanceable and has asymptotic size not greater than α.

PROOF: If $\alpha = 1$, set $C = 0$ and note that $\varphi_n := \mathbf{1}\{s_n\|\hat{\theta}_n\| \geq 0\} \equiv 1$, which is obviously not asymptotically enhanceable and has size 1. Next, consider the case where $\alpha \in (0, 1)$. The existence of a C ensuring the size requirement follows immediately from the local tightness assumption applied to the sequence $\theta_n = 0$. It remains to show that $\varphi_n := \mathbf{1}\{s_n\|\hat{\theta}_n\| \geq C\}$ is not asymptotically enhanceable. We claim that it suffices to verify that if $s_n\|\theta_n\|$ diverges to ∞ for $\theta_n \in \Theta_d$, then $\mathbb{E}_{n,d,\theta_n}(\varphi_n) \to 1$. This claim easily follows from the contiguity rate assumption, together with Remark 3.1. Now let $s_n\|\theta_n\|$ diverge to ∞. To show that $\mathbb{E}_{n,d,\theta_n}(\varphi_n) \to 1$, it suffices to verify that for every subsequence n' of n there exists a subsequence n'' of n' along which $\mathbb{E}_{n,d,\theta_n}(\varphi_n) \to 1$. Let n' be a subsequence of n. Then (i) there exists a subsequence n'' of n' such that $\|\theta_{n''}\| < \delta$ holds for every n'' or (ii) there exists a subsequence n'' of n' such that $\|\theta_{n''}\| \geq \delta$ holds for every n''. Consider first case (i). By the local uniform tightness assumption, the sequence of image measures $\mathbb{P}_{n'',d,\theta_{n''}} \circ [s_{n''}(\hat{\theta}_{n''} - \theta_{n''})]$ is then tight. Let $\varepsilon \in (0, 1)$ and choose $K > 0$ such that $\mathbb{P}_{n'',d,\theta_{n''}} \circ [s_{n''}(\hat{\theta}_{n''} - \theta_{n''})](\bar{B}_{\|.\|}(K)) \geq 1 - \varepsilon$ holds for every n'', where $\bar{B}_{\|.\|}(K) := \{z \in \mathbb{R}^d : \|z\| \leq K\}$. We write

$$\mathbb{E}_{n'',d,\theta_{n''}}(\varphi_{n''}) = \mathbb{P}_{n'',d,\theta_{n''}} \circ \left[s_{n''}(\hat{\theta}_{n''} - \theta_{n''})\right]\left(\left\{z \in \mathbb{R}^d : \|z + s_{n''}\theta_{n''}\| \geq C\right\}\right)$$

and note that $\{z \in \mathbb{R}^d : \|z + s_{n''}\theta_{n''}\| \geq C\}$ contains $\bar{B}_{\|.\|}(K)$ for all n'' large enough, recalling that $s_n\|\theta_n\| \to \infty$. Hence, the expectation in the previous display is not smaller than $1 - \varepsilon$ for n'' large enough. Since ε was arbitrary, it follows that $\mathbb{E}_{n,d,\theta_n}(\varphi_n) \to 1$ along n''. Next we consider the case (ii). In this case, we write

$$\mathbb{E}_{n'',d,\theta_{n''}}(\varphi_{n''}) = \mathbb{P}_{n'',d,\theta_{n''}}\left(\|\hat{\theta}_{n''}\| \geq s_{n''}^{-1}C\right) \geq \mathbb{P}_{n'',d,\theta_{n''}}\left(\|\hat{\theta}_{n''}\| \geq s_{n''}^{-1}C, \|\hat{\theta}_{n''} - \theta_{n''}\| < \delta/2\right).$$

For n'' large (since s_n increases to ∞ and $\|\theta_{n''}\| \geq \delta$ for every n''), the right hand side equals $\mathbb{P}_{n'',d,\theta_{n''}}(\|\hat{\theta}_{n''} - \theta_{n''}\| < \delta/2)$, which converges to 1 by the uniform consistency assumption. *Q.E.D.*

The contiguity rate in Theorem C.1 is often given by $s_n = \sqrt{n}$. For an extensive discussion of primitive conditions sufficient for the consistency and tightness assumptions imposed in the previous result, we refer the reader to Sections 4 and 5 in Chapter 1 in Ibragimov and Has'minskii (1981), respectively; cf. also pp. 144–146 in van der Vaart (2000) and Section 5.4 in Pfanzagl (2017). We also emphasize that in the i.i.d. case, the local tightness assumption required in Theorem C.1 is satisfied by the maximum likelihood estimator (MLE) under standard regularity conditions including smoothness and integrability properties of the log-likelihood function over a *neighborhood* of 0; cf., for example, the discussion at the end of Section 7 in Chapter 1 in Ibragimov and Has'minskii (1981) or the results in Section 7.5 in Pfanzagl (1994) (these regularity conditions, however, are stronger than the $\mathbb{L}_2$-differentiability condition at the point 0 required by Theorem 4.1; thus Theorem C.1 is not more general than Theorem 4.1 in this respect). In the context of our running example, $s_n = \sqrt{n}$ and the ordinary least squares (OLS) estimator satisfies conditions (i) and (iii) in Theorem C.1 under standard assumptions on the distribution F of the errors and on the regressors.

APPENDIX D: PROOF OF PROPOSITION 5.1

The proof is divided into three steps. First we construct a sequence $p(n)$. Then we verify that the first and second parts of Proposition 5.1 are satisfied for this sequence.

D.1. *Step 1: Construction of the Sequence* $p(n)$

Assumption 2 asserts (cf. Definition 6.63 of Liese and Miescke (2008)) that for every fixed $d \in \mathbb{N}$, there exists a sequence of measurable functions (a "central sequence") $Z_{n,d} : \Omega_{n,d} \to \mathbb{R}^d$ and a (positive definite and symmetric) information matrix I_d, such that $\mathbb{P}_{n,d,0} \circ Z_n \Rightarrow N_d(0, \mathsf{I}_d)$ (as $n \to \infty$) and such that for every $h \in \mathbb{R}^d$, the (eventually well defined) log-likelihood ratio of $\mathbb{P}_{n,d,s_n^{-1}h}$ w.r.t. $\mathbb{P}_{n,d,0}$ equals $h'Z_{n,d} - h'\mathsf{I}_d h/2 + r_{n,d}(h)$ for a measurable sequence $r_{n,d}(h) : \Omega_{n,d} \to \bar{\mathbb{R}}$ that converges to 0 in $\mathbb{P}_{n,d,0}$ probability (as $n \to \infty$). By Theorem 6.76 in Liese and Miescke (2008), the following statement holds for every *fixed* $d \in \mathbb{N}$: there exists a sequence $c(n, d) > 0$ satisfying $c(n, d) \to \infty$ as $n \to \infty$, such that the family of probability measures $\{\mathbb{Q}_{n,d,h} : h \in H_{n,d}\}$ on $(\Omega_{n,d}, \mathcal{A}_{n,d})$ defined via

$$\frac{d\mathbb{Q}_{n,d,h}}{d\mathbb{P}_{n,d,0}} = \exp\bigl(h'Z^*_{n,d} - K_{n,d}(h)\bigr), \tag{D.1}$$

where $K_{n,d}(h) = \log(\int_{\Omega_{n,d}} \exp(h'Z^*_{n,d})d\mathbb{P}_{n,d,0})$ and $Z^*_{n,d} = Z_{n,d}\mathbf{1}\{\|Z_{n,d}\|_2 \leq c(n, d)\}$, satisfies

$$\lim_{n\to\infty} \bigl|K_{n,d}(h) - 0.5h'\mathsf{I}_d h\bigr| = 0 \quad \text{for every } h \in \mathbb{R}^d \tag{D.2}$$

and

$$\lim_{n\to\infty} d_1(\mathbb{P}_{n,d,s_n^{-1}h}, \mathbb{Q}_{n,d,h}) = 0 \quad \text{for every } h \in \mathbb{R}^d. \tag{D.3}$$

Here d_1 denotes the total variation distance; cf. Definition 2.1 of Strasser (1985). Furthermore (see, e.g., Theorem 6.72 in Liese and Miescke (2008)), for every fixed $d \in \mathbb{N}$ and as $n \to \infty$,

$$\mathbb{P}_{n,d,s_n^{-1}h} \circ Z_{n,d} \Rightarrow N_d(\mathsf{I}_d h, \mathsf{I}_d) \quad \text{for every } h \in \mathbb{R}^d. \tag{D.4}$$

Next define the sequence

$$a_i = \max\bigl(\bigl[0.5\log(i)\bigr]^{1/2}, 1\bigr) \quad \text{for } i \in \mathbb{N},$$

which (i) is positive, (ii) diverges to ∞, and satisfies (iii) $i^{-1}\exp(a_i^2) \to 0$. Now let $\tilde{H}_d = \{0, a_d v_{1,d}, \dots, a_d v_{d,d}\}$ and $H_d = a_d^{-2}\tilde{H}_d \setminus \{0\}$. By $H_{n,d} \uparrow \mathbb{R}^d$ (as $n \to \infty$) and by (D.2), (D.3), and (D.4) (and the continuous mapping theorem together with $e'\mathsf{I}_d e = a_d^{-2}$ for every $e \in H_d$), for every $d \in \mathbb{N}$, there exists an $N(d) \in \mathbb{N}$ such that $n \geq N(d)$ implies (first)

$$\tilde{H}_d + \tilde{H}_d \subseteq H_{n,d},$$

where, for $A \subseteq \mathbb{R}^d$, the set $A + A$ denotes $\{a + b : a \in A, b \in A\}$, and (second)

$$\max_{h\in(\tilde{H}_d+\tilde{H}_d)} \bigl|K_{n,d}(h) - 0.5h'\mathsf{I}_d h\bigr| + \max_{h\in\tilde{H}_d} d_1(\mathbb{P}_{n,d,s_n^{-1}h}, \mathbb{Q}_{n,d,h})$$
$$+ \max_{(h,e)\in\tilde{H}_d\times H_d} d_w\bigl(\mathbb{P}_{n,d,s_n^{-1}h} \circ \bigl(e'Z_{n,d}\bigr), N_1\bigl(e'\mathsf{I}_d h, a_d^{-2}\bigr)\bigr) \leq d^{-1}.$$

Here $d_w(\cdot, \cdot)$ denotes a metric on the set of probability measures on the Borel sets of $\mathbb{R}$ that generates the topology of weak convergence; cf. Dudley (2002, p. 393) for specific examples. Note also that we can (and do) choose $N(1) < N(2) < \cdots$. Obviously, there exists a nondecreasing unbounded sequence $p(n)$ in $\mathbb{N}$ that satisfies $N(p(n)) \leq n$ for every $n \geq N(1) =: M$. Hence, the two previous displays still hold for $n \geq M$ when d is replaced by $p(n)$. Moreover, the two previous displays also hold for $n \geq M$ when d is replaced by any sequence of nondecreasing natural numbers $d(n) \leq p(n)$. This implies that for any such sequence $d(n)$ that is also unbounded, we have

$$\tilde{H}_{d(n)} + \tilde{H}_{d(n)} \subseteq H_{n,d(n)} \quad \text{for } n \geq M \tag{D.5}$$

and that (as $n \to \infty$)

$$\max_{h \in (\tilde{H}_{d(n)} + \tilde{H}_{d(n)})} \left| K_{n,d(n)}(h) - 0.5 h' \mathsf{I}_{d(n)} h \right| \to 0, \tag{D.6}$$

$$\max_{h \in \tilde{H}_{d(n)}} d_1(\mathbb{P}_{n,d(n),s_n^{-1}h}, \mathbb{Q}_{n,d(n),h}) \to 0, \tag{D.7}$$

and

$$\max_{(h,e) \in \tilde{H}_{d(n)} \times H_{d(n)}} d_w\big(\mathbb{P}_{n,d(n),s_n^{-1}h} \circ \big(e' Z_{n,d(n)}\big), N_1\big(e' \mathsf{I}_{d(n)} h, a_{d(n)}^{-2}\big)\big) \to 0. \tag{D.8}$$

We shall now verify that the sequence $p(n)$ and the natural number M defined above have the required properties. Let $d(n) \leq p(n)$ be an unbounded nondecreasing sequence of natural numbers.

D.2. *Step 2: Verification of Part* (i)

The statement in the first display in Proposition 5.1 follows from (D.5), which implies $\tilde{H}_{d(n)} \subseteq H_{n,d(n)}$ for $n \geq M$ (cf. also Assumption 2). Now let $\varphi_n : \Omega_{n,d(n)} \to [0,1]$ be a sequence of tests. For $h \in H_{n,d(n)}$, abbreviate $\mathbb{P}_{n,d(n),s_n^{-1}h} = \mathbb{P}_{n,h}$ and $\mathbb{Q}_{n,d(n),h} = \mathbb{Q}_{n,h}$, and denote expectation w.r.t. $\mathbb{P}_{n,h}$ and $\mathbb{Q}_{n,h}$ by $\mathbb{E}^P_{n,h}$ and $\mathbb{E}^Q_{n,h}$, respectively. Furthermore, for $n \geq M$, define the probability measures $\mathbb{P}_n = \frac{1}{d(n)} \sum_{h \in \tilde{H}_{d(n)} \setminus \{0\}} \mathbb{P}_{n,h}$ and, similarly, $\mathbb{Q}_n = \frac{1}{d(n)} \sum_{h \in \tilde{H}_{d(n)} \setminus \{0\}} \mathbb{Q}_{n,h}$. Since for $n \geq M$,

$$\left| \mathbb{E}_{n,d(n),0}(\varphi_n) - d(n)^{-1} \sum_{h \in \tilde{H}_n \setminus \{0\}} \mathbb{E}^P_{n,h}(\varphi_n) \right| \leq d_1(\mathbb{P}_{n,0}, \mathbb{P}_n)$$

(cf. Strasser (1985, Lemma 2.3)), it suffices to verify that $d_1(\mathbb{P}_{n,0}, \mathbb{P}_n) \to 0$. From (D.7), we see that it suffices to show that $d_1(\mathbb{Q}_{n,0}, \mathbb{Q}_n) \to 0$. Since $\mathbb{Q}_n \ll \mathbb{Q}_{n,0} = \mathbb{P}_{n,0}$ by (D.1), $d_1^2(\mathbb{Q}_{n,0}, \mathbb{Q}_n)$ equals (e.g., Strasser (1985, Lemma 2.4))

$$\left(\frac{1}{2} \mathbb{E}^Q_{n,0} \left| \frac{d\mathbb{Q}_n}{d\mathbb{Q}_{n,0}} - 1 \right| \right)^2 \leq \mathbb{E}^Q_{n,0} \left(\frac{d\mathbb{Q}_n}{d\mathbb{Q}_{n,0}} - 1 \right)^2 = \mathbb{E}^P_{n,0} \left(\frac{d\mathbb{Q}_n}{d\mathbb{P}_{n,0}} \right)^2 - 1,$$

the first inequality following from Jensen's inequality.

It remains to verify that $\limsup_{n \to \infty} \mathbb{E}^P_{n,0}(\frac{d\mathbb{Q}_n}{d\mathbb{P}_{n,0}})^2 \leq 1$. Let $a_{d(n)} = a(n)$, $k_{n,i} = K_{n,d(n)}(a(n) v_{i,d(n)})$, $k_{n,i,j} = K_{n,d(n)}(a(n) v_{i,d(n)} + a(n) v_{j,d(n)})$, and $z^*_{n,i} = v'_{i,d(n)} Z^*_{n,d(n)}$. Let $n \geq M$. From (D.1), we see that

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}_{n,0}} = d(n)^{-1} \sum_{i=1}^{d(n)} \exp\big(a(n) z^*_{n,i} - k_{n,i}\big)$$

and

$$\mathbb{E}^P_{n,0}\big(\exp\big(a(n)z^*_{n,i} - k_{n,i}\big)\exp\big(a(n)z^*_{n,j} - k_{n,j}\big)\big) = \exp(k_{n,i,j} - k_{n,i} - k_{n,j}).$$

Thus, $\mathbb{E}^P_{n,0}(\frac{d\mathbb{Q}_n}{d\mathbb{P}_{n,0}})^2$ is not greater than the sum of

$$d(n)^{-1}\exp\big(a^2(n)\big)\max_{1\le i\le d(n)}\exp\big(k_{n,i,i} - 2k_{n,i} - a^2(n)\big) \quad \text{and}$$

$$\max_{1\le i<j\le d(n)}\exp(k_{n,i,j} - k_{n,i} - k_{n,j}).$$

But the first sequence converges to 0 and the second converges to 1. This follows from $i^{-1}\exp(a_i^2) \to 0$, and since the sequences $\max_{1\le i\le d(n)}|k_{n,i} - 0.5a^2(n)|$, $\max_{1\le i\le d(n)}|k_{n,i,i} - 2a^2(n)|$, and $\max_{1\le i<j\le d(n)}|k_{n,i,j} - a^2(n)|$ all converge to 0 by (D.6).

D.3. *Step 3: Verification of Part* (ii)

Given a sequence $1 \le i(n) \le d(n)$, define $t_n = a(n)^{-1}v'_{i(n),d(n)}Z_{n,d(n)}$ and let $\nu_n = \mathbf{1}\{t_n \ge 1/2\}$. By definition (using the same notation as in Step 2),

$$\mathbb{E}^P_{n,0}(\nu_n) = \mathbb{P}_{n,0}\circ t_n\big([0.5,\infty)\big). \tag{D.9}$$

Since $0 \in \tilde{H}_{d(n)}$ and $a(n)^{-1}v_{i(n),d(n)} \in H_{d(n)}$, it follows from (D.8) that

$$d_w\big(\mathbb{P}_{n,0}\circ t_n, N_1\big(0, a(n)^{-2}\big)\big) \to 0.$$

But $a(n) \to \infty$ thus implies (via the triangle inequality, together with d_w continuity of $(\mu,\sigma^2) \mapsto N_1(\mu,\sigma^2)$ on $\mathbb{R}\times[0,\infty)$, $N_1(\mu,0)$ being interpreted as δ_μ, i.e., point mass at μ) that $\mathbb{P}_{n,0}\circ t_n \Rightarrow \delta_0$. From the Portmanteau theorem it hence follows that the sequence in (D.9) converges to $\delta_0([0.5,\infty)) = 0$. Concerning asymptotic power, let $v_n = a(n)v_{i(n),d(n)}$. Note that $v_n \in \tilde{H}_{d(n)}$, $a(n)^{-1}v_{i(n),d(n)} \in H_{d(n)}$, and (D.8) implies $d_w(\mathbb{P}_{n,v_n}\circ t_n, N_1(1, a(n)^{-2})) \to 0$; hence, $\mathbb{P}_{n,v_n}\circ t_n \Rightarrow \delta_1$ and, thus, $\mathbb{E}^P_{n,v_n}(\nu_n) = \mathbb{P}_{n,v_n}\circ t_n([0.5,\infty)) \to 1$. *Q.E.D.*

APPENDIX E: PROOF OF THEOREM 5.2

To prove Theorem 5.2, choose for each $d \in \mathbb{N}$ an arbitrary orthogonal basis as in Proposition 5.1 to obtain a corresponding sequence $p(n)$, and let $d(n) \le p(n)$ be nondecreasing and unbounded. Let the sequence of tests $\varphi_n : \Omega_{n,d(n)} \to [0,1]$ be of asymptotic size $\alpha < 1$, that is, $\limsup_{n\to\infty}\mathbb{E}_{n,d(n),0}(\varphi_n) = \alpha < 1$. According to Definition 3.1, we need to show that $\liminf_{n\to\infty}\mathbb{E}_{n,d(n),\theta_n}(\varphi_n) < 1$ for a sequence $\theta_n \in \Theta_{d(n)}$ for which a sequence of tests $\nu_n : \Omega_{n,d(n)} \to [0,1]$ exists such that

$$\lim_{n\to\infty}\mathbb{E}_{n,d(n),0}(\nu_n) = 0 \quad \text{and} \quad \lim_{n\to\infty}\mathbb{E}_{n,d(n),\theta_n}(\nu_n) = 1. \tag{E.1}$$

But part (i) of Proposition 5.1 implies the existence of a sequence $1 \le i(n) \le d(n)$ such that

$$\limsup_{n\to\infty}\mathbb{E}_{n,d(n),\theta_{i(n),n}}(\varphi_n) \le \alpha < 1,$$

and part (ii) of Proposition 5.1 verifies the existence of a sequence of tests ν_n as in (E.1) for $\theta_n = \theta_{i(n),n}$. *Q.E.D.*

Note that the above proof actually exploits a power enhancement component for a sequence θ_n against which φ_n has asymptotic power not only smaller than 1, but in fact at most α.

APPENDIX F: VERIFICATION OF ASSUMPTION 3 FOR THE RANDOM COVARIATES CASE IN OUR RUNNING EXAMPLE

We show that Assumption 3 is satisfied for $F(\theta) = (\theta', 0)' \in \mathbb{R}^{d_2}$. For convenience, denote a generic element of $\Omega_{n,d} = \times_{i=1}^{n}(\mathbb{R} \times \mathbb{R}^d)$ by $z_d = (y, x^{(1)}, \dots, x^{(d)})$ for $y, x^{(1)}, \dots, x^{(d)} \in \mathbb{R}^n$. Let $d_1 < d_2$ and n be natural numbers. Consider the experiment

$$\big(\Omega_{n,d_2}, \mathcal{A}_{n,d_2}, \{\mathbb{P}_{n,d_2,F(\theta)} : \theta \in \Theta_{d_1}\}\big), \tag{F.1}$$

define the map $T : \Omega_{n,d_2} \to \Omega_{n,d_1}$ as $T(z_{d_2}) = z_{d_1}$, and note that T is sufficient for (F.1) (e.g., Theorem 20.9 in Strasser (1985)). Note further that $\mathbb{P}_{n,d_2,F(\theta)} \circ T = \mathbb{P}_{n,d_1,\theta}$ holds for every $\theta \in \Theta_{d_1}$ under our additional assumption that $K_{d_1} = K_{d_1,d_2}$. That Assumption 3 is satisfied now follows from Corollaries 22.4 and 22.6 in Strasser (1985).

APPENDIX G: PROOF OF THEOREM 5.4

G.1. *A Weaker Version of Assumption* 3

Note that Assumption 3 imposes restrictions that hold for every $n \in \mathbb{N}$. Since asymptotic enhanceability concerns large-sample properties of tests, it is not surprising that a (weaker) asymptotic version of Assumption 3 suffices for establishing the same conclusion as in Theorem 5.4. The asymptotic (and weaker) version of Assumption 3 we subsequently work with is as follows.

ASSUMPTION G.1: *For all pairs of natural numbers $d_1 < d_2$, there exists a function $F = F_{d_1,d_2}$ from Θ_{d_1} to Θ_{d_2} satisfying $F(0) = 0$, and such that for any two nondecreasing unbounded sequences $r(n)$ and $d(n)$ in $\mathbb{N}$ such that $r(n) < d(n)$, the following statements hold, abbreviating $F_{r(n),d(n)}$ by F_n:*

(i) *For every sequence of tests $\varphi_n : \Omega_{n,d(n)} \to [0,1]$, there exists a sequence of tests $\varphi'_n : \Omega_{n,r(n)} \to [0,1]$ such that*

$$\sup_{\theta \in \Theta_{r(n)}} \big|\mathbb{E}_{n,d(n),F_n(\theta)}(\varphi_n) - \mathbb{E}_{n,r(n),\theta}(\varphi'_n)\big| \to 0 \quad \text{as } n \to \infty. \tag{G.1}$$

(ii) *For every sequence of tests $\varphi'_n : \Omega_{n,r(n)} \to [0,1]$, there exists a sequence of tests $\varphi_n : \Omega_{n,d(n)} \to [0,1]$ such that*

$$\sup_{\theta \in \Theta_{r(n)}} \big|\mathbb{E}_{n,r(n),\theta}(\varphi'_n) - \mathbb{E}_{n,d(n),F_n(\theta)}(\varphi_n)\big| \to 0 \quad \text{as } n \to \infty.$$

G.2. *Proof of Theorem* 5.4

We shall now prove the conclusion of Theorem 5.4 under slightly weaker conditions by replacing Assumption 3 by Assumption G.1. Theorem 5.4 then follows immediately as a corollary.

THEOREM G.1: *Suppose the double array of experiments* (2.1) *satisfies Assumptions* 2 *and* G.1. *Then, for every nondecreasing and unbounded sequence $d(n)$ in $\mathbb{N}$, every sequence of tests with asymptotic size smaller than* 1 *is asymptotically enhanceable.*

PROOF: Let $d(n)$ be a nondecreasing and unbounded sequence in $\mathbb{N}$, and let $\varphi_n : \Omega_{n,d(n)} \to [0,1]$ be of asymptotic size $\alpha < 1$. We apply Theorem 5.2 to obtain a sequence $p(n)$ as in that theorem. Let $r(n) \equiv \min(p(n), d(n)-1)$, a nondecreasing unbounded sequence that eventually satisfies $r(n) \in \mathbb{N}$ and $r(n) < d(n)$. By part (i) of Assumption G.1, there exists a sequence of tests $\varphi'_n : \Omega_{n,r(n)} \to [0,1]$ such that (G.1) holds. In particular, φ'_n also has asymptotic size α, recalling that $F_n(0) = 0$ holds by assumption. Therefore, by Theorem 5.2 (applied with $d(n) \equiv r(n)$), φ'_n is asymptotically enhanceable, that is, there exist tests $\nu'_n : \Omega_{n,r(n)} \to [0,1]$ and a sequence $\theta_n \in \Theta_{r(n)}$ such that $\mathbb{E}_{n,r(n),0}(\nu'_n) \to 0$ and

$$1 = \lim_{n\to\infty} \mathbb{E}_{n,r(n),\theta_n}(\nu'_n) > \liminf_{n\to\infty} \mathbb{E}_{n,r(n),\theta_n}(\varphi'_n) = \liminf_{n\to\infty} \mathbb{E}_{n,d(n),F_n(\theta_n)}(\varphi_n),$$

the second equality following from (G.1). By part (ii) of Assumption G.1, and again using $F_n(0) = 0$, tests $\nu_n : \Omega_{n,d(n)} \to [0,1]$ exist such that $\mathbb{E}_{n,d(n),0}(\nu_n) \to 0$ and $\mathbb{E}_{n,d(n),F_n(\theta_n)}(\nu_n) \to 1$. Hence, φ_n is asymptotically enhanceable. *Q.E.D.*

REFERENCES

DUDLEY, R. M. (2002): *Real Analysis and Probability*. Cambridge: Cambridge University Press. [5]
IBRAGIMOV, I. A., AND R. Z. HAS'MINSKII (1981): *Statistical Estimation*. New York: Springer. [4]
LIESE, F., AND K. J. MIESCKE (2008): *Statistical Decision Theory*. New York: Springer. [3,5]
PFANZAGL, J. (1994): *Parametric Statistical Theory*. Berlin: de Gruyter. [4]
——— (2017): *Mathematical Statistics: Essays on History and Methodology*. Berlin: Springer. [4]
STRASSER, H. (1985): *Mathematical Theory of Statistics*. Berlin: Walter de Gruyter. [5,6,8]
VAN DER VAART, A. W. (2000): *Asymptotic Statistics*. Cambridge: Cambridge University Press. [4]

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