Econometrica Supplementary Material

SUPPLEMENT TO "MISINTERPRETING OTHERS AND THE FRAGILITY OF SOCIAL LEARNING"

(*Econometrica*, Vol. 88, No. 6, November 2020, 2281–2328)

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THIS SUPPLEMENTARY APPENDIX contains the proofs for Section 6, as well as other material omitted from the main manuscript.

APPENDIX D: PROOFS FOR SECTION 6

D.1. *Proof of Proposition 1*

We omit the proof of the first part, as it follows the same steps as in Appendix A (for details, see Appendix A of the previous working paper version, Frick, Iijima, and Ishii (2019)). To prove the second part, define for each $F, \hat{F} \in \mathcal{F}$ and $\omega \in \Omega$ the set of steady states

$$\mathrm{SS}(F, \hat{F}, \omega) := \{\hat{\omega}_\infty \in \Omega : \hat{\omega}_\infty \in \operatorname*{argmin}_{\hat{\omega} \in \Omega} \mathrm{KL}\big(\alpha F\big(\theta^*(\hat{\omega}_\infty)\big) + (1-\alpha) F\big(\theta^*(\omega)\big), \hat{F}\big(\theta^*(\hat{\omega})\big)\big). \quad (7)$$

The following lemma shows that whenever $\mathrm{SS}(F, \hat{F}, \omega)$ is finite, incorrect agents' long-run beliefs correspond to steady states.

LEMMA D.1: *Fix any F, $\hat{F}$ such that $\mathrm{SS}(F, \hat{F}, \omega)$ is finite for each ω. Then in all states ω, there exists some state $\hat{\omega}_\infty(\omega) \in \mathrm{SS}(F, \hat{F}, \omega)$ such that almost all incorrect agents' beliefs converge to a point mass on $\hat{\omega}_\infty(\omega)$.*

PROOF: Since Lemma B.2 continues to characterize incorrect agents' inferences from observed actions, the proof proceeds in an analogous manner to that of Proposition B.1. Let $q_t^C(\omega), q_t^I(\omega) \in [0, 1]$ denote the actual fraction of action 0 among correct and incorrect agents in period t and state ω, and let $\overline{q}_t^C(\omega) := \frac{1}{t}\sum_{\tau=1}^t q_\tau^C(\omega)$ and $\overline{q}_t^I(\omega) := \frac{1}{t}\sum_{\tau=1}^t q_\tau^I(\omega)$ denote the corresponding time averages.

Note that since by the first part of Proposition 1 almost all correct agents learn the true state as $t \to \infty$, it follows that $\lim_{t\to\infty} \overline{q}_t^C(\omega) = \lim_{t\to\infty} q_t^C(\omega) = F(\theta^*(\omega))$ for all ω.

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 https://doi.org/10.3982/ECTA16981

Moreover, since $\mathrm{SS}(F, \hat{F}, \omega, \alpha)$ is finite, we can follow the same argument as in the proof of Lemma B.3 to show (using Lemma B.2) that the limit $R^I(\omega) := \lim_{t\to\infty} \overline{q}_t^I(\omega)$ exists for all ω. For each ω, let

$$\hat{\omega}_\infty(\omega) := \operatorname*{argmin}_{\hat{\omega}\in\Omega} \mathrm{KL}\big(\alpha R^I(\omega) + (1-\alpha)F\big(\theta^*(\omega)\big), \hat{F}\big(\theta^*(\hat{\omega})\big)\big).$$

Then by the same argument as in the proof of Proposition B.1, we obtain that, conditional on each state ω, almost all incorrect agents' beliefs converge to a point mass on $\hat{\omega}_\infty(\omega)$. But then $R^I(\omega) = F(\theta^*(\hat{\omega}_\infty(\omega)))$, whence $\hat{\omega}_\infty(\omega) \in \mathrm{SS}(F, \hat{F}, \omega)$. *Q.E.D.*

Combined with Lemma D.1, the following lemma completes the proof of the proposition.

LEMMA D.2: *Fix any analytic $F \in \mathcal{F}$ and $\delta > 0$. There exists $\varepsilon > 0$ such that for any analytic $\hat{F} \neq F$ with $\|F - \hat{F}\| < \varepsilon$ and every $\omega \in \Omega$:*

1. $\mathrm{SS}(F, \hat{F}, \omega)$ *is finite.*
2. $|\omega - \hat{\omega}| < \delta$ *for every* $\hat{\omega} \in \mathrm{SS}(F, \hat{F}, \omega)$.

PROOF: Fix any analytic $F \in \mathcal{F}$ and $\delta > 0$, where we can assume that $\delta < \frac{\overline{\omega}-\underline{\omega}}{2}$. Choose $\varepsilon > 0$ sufficiently small such that $\frac{\varepsilon}{1-\alpha} < |F(\theta^*(\omega)) - F(\theta^*(\omega'))|$ for any pair of states ω, ω' with $|\omega - \omega'| \geq \delta$.

Consider any analytic $\hat{F} \neq F$ with $\|F - \hat{F}\| < \varepsilon$ and any ω. By (7), each $\hat{\omega} \in \mathrm{SS}(F, \hat{F}, \omega)$ satisfies one of the following three cases:

1. $\hat{\omega} \in (\underline{\omega}, \overline{\omega})$ and $\alpha F(\theta^*(\hat{\omega})) + (1-\alpha)F(\theta^*(\omega)) = \hat{F}(\theta^*(\hat{\omega}))$,
2. $\hat{\omega} = \overline{\omega}$ and $\alpha F(\theta^*(\overline{\omega})) + (1-\alpha)F(\theta^*(\omega)) \leq \hat{F}(\theta^*(\overline{\omega}))$,
3. $\hat{\omega} = \underline{\omega}$ and $\alpha F(\theta^*(\underline{\omega})) + (1-\alpha)F(\theta^*(\omega)) \geq \hat{F}(\theta^*(\underline{\omega}))$.

We first show that $|\omega - \hat{\omega}| < \delta$ for all $\hat{\omega} \in \mathrm{SS}(F, \hat{F}, \omega)$. We consider only the first case, as the remaining cases are analogous. Note that

$$\begin{aligned}
&\alpha F\big(\theta^*(\hat{\omega})\big) + (1-\alpha)F\big(\theta^*(\omega)\big) = \hat{F}\big(\theta^*(\hat{\omega})\big) \\
&\quad\Leftrightarrow\quad F\big(\theta^*(\omega)\big) - \hat{F}\big(\theta^*(\hat{\omega})\big) = \frac{\alpha}{1-\alpha}\big(\hat{F}\big(\theta^*(\hat{\omega})\big) - F\big(\theta^*(\hat{\omega})\big)\big),
\end{aligned}$$

so that $|F(\theta^*(\omega)) - \hat{F}(\theta^*(\hat{\omega}))| \leq \frac{\alpha}{1-\alpha}\varepsilon$. Thus,

$$\begin{aligned}
\big|F\big(\theta^*(\omega)\big) - F\big(\theta^*(\hat{\omega})\big)\big| &\leq \big|F\big(\theta^*(\omega)\big) - \hat{F}\big(\theta^*(\hat{\omega})\big)\big| + \big|\hat{F}\big(\theta^*(\hat{\omega})\big) - F\big(\theta^*(\hat{\omega})\big)\big| \\
&\leq \frac{\alpha}{1-\alpha}\varepsilon + \varepsilon = \frac{\varepsilon}{1-\alpha}.
\end{aligned}$$

By choice of ε, this implies $|\omega - \hat{\omega}| < \delta$.

To show that $\mathrm{SS}(F, \hat{F}, \omega)$ is finite, it suffices to show that the equality $\alpha F(\theta^*(\hat{\omega})) + (1-\alpha)F(\theta^*(\omega)) = \hat{F}(\theta^*(\hat{\omega}))$ admits at most finitely many solutions $\hat{\omega} \in [\underline{\omega}, \overline{\omega}]$. Since F and $\hat{F}$ are analytic and $[\underline{\omega}, \overline{\omega}]$ is compact, if this equality admits infinitely many solutions, then $\alpha F(\theta^*(\hat{\omega})) + (1-\alpha)F(\theta^*(\omega)) = \hat{F}(\theta^*(\hat{\omega}))$ holds for all $\hat{\omega} \in [\underline{\omega}, \overline{\omega}]$. But the latter is impossible since we have shown that $|\omega - \hat{\omega}| < \delta < \frac{\overline{\omega}-\underline{\omega}}{2}$ holds for any solution $\hat{\omega}$. *Q.E.D.*

D.2. *Proof of Proposition 2*

Fix any $F \in \mathcal{F}$, $\hat{\omega} \in \Omega$, $\hat{\alpha}, \alpha > 0$ with $\hat{\alpha} \neq \alpha$ and $\varepsilon > 0$. If $\hat{\alpha} < \alpha$, take $\hat{F} \in \mathcal{F}$ such that $\hat{F} - F$ crosses zero only once at $\theta^*(\hat{\omega})$ from below. If $\hat{\alpha} > \alpha$, take $\hat{F} \in \mathcal{F}$ such that $\hat{F} - F$ crosses zero only once at $\theta^*(\hat{\omega})$ from above. In either case, we can additionally require that $\|F - \hat{F}\| < \varepsilon$, as in the proof of Theorem 1. In addition, we can take $\hat{F}$ sufficiently close to F such that the inverse function $F \circ \hat{F}^{-1}$ has a Lipschitz constant less than $\frac{1}{\hat{\alpha}}$.

Let $\hat{q}_t^I(\omega)$ and $\hat{q}_t^C(\omega)$ denote incorrect and quasi-correct agents' perceived population fractions of action 0 in period t and state ω. The proof of Lemma 1 applied to incorrect agents' perceptions implies that $\hat{q}_t^I(\omega)$ is strictly decreasing in ω with $\hat{q}_\infty^I(\omega) := \lim_{t\to\infty} \hat{q}_t^I(\omega) = \hat{F}(\theta^*(\omega))$. Likewise, the proof of Proposition 1 applied to quasi-correct agents' perceptions implies that $\hat{q}_\infty^C(\omega) := \lim_{t\to\infty} \hat{q}_t^C(\omega)$ exists, is strictly decreasing, and satisfies

$$\hat{q}_\infty^C(\omega) = \hat{\alpha} F\big(\theta^*(\hat{\omega}_\omega)\big) + (1-\hat{\alpha}) F\big(\theta^*(\omega)\big)$$
$$\text{where } \hat{\omega}_\omega = \operatorname*{argmin}_{\hat{\omega}'} \mathrm{KL}\big(\hat{q}_\infty^C(\omega), \hat{F}\big(\theta^*\big(\hat{\omega}'\big)\big)\big). \tag{8}$$

LEMMA D.3: *If $\hat{\alpha} < \alpha$ (resp. $\hat{\alpha} > \alpha$), then $\hat{F}(\theta^*(\omega)) - q_\infty^C(\omega)$ crosses zero only once from below (resp. above) at $\omega = \hat{\omega}$.*

PROOF: Note that since by construction of $\hat{F}$ the Lipschitz constant of the RHS of (8) is less than 1, there is a unique solution $\hat{q}_\infty^C(\omega)$ to (8). Given this, we have $\hat{q}_\infty^C(\hat{\omega}) = \hat{F}(\theta^*(\hat{\omega}))$ as $F(\theta^*(\hat{\omega})) = \hat{F}(\theta^*(\hat{\omega}))$. For the remaining claim, we focus on the case $\hat{\alpha} < \alpha$, as the case $\hat{\alpha} > \alpha$ follows a symmetric argument.

Take any $\omega < \hat{\omega}$. Then $\hat{q}_\infty^C(\omega) > \hat{q}_\infty^C(\hat{\omega}) = \hat{F}(\theta^*(\hat{\omega}))$, so that $\hat{\omega}_\omega = \operatorname{argmin}_{\hat{\omega}'} \mathrm{KL}(\hat{q}_\infty^C(\omega), \hat{F}(\theta^*(\hat{\omega}')))$ must satisfy $\hat{\omega}_\omega < \omega$ and $\hat{F}(\theta^*(\hat{\omega}_\omega)) \leq \hat{q}_\infty^C(\omega)$. But since $F(\theta) < \hat{F}(\theta)$ for all $\theta > \theta^*(\hat{\omega})$, this implies $F(\theta^*(\hat{\omega}_\omega)) \in (F(\theta^*(\hat{\omega})), \hat{q}_\infty^C(\omega))$. Since by (8), $\hat{q}_\infty^C(\omega) = \hat{\alpha} F(\theta^*(\hat{\omega}_\omega)) + (1-\hat{\alpha}) F(\theta^*(\omega))$, this implies $F(\theta^*(\hat{\omega}_\omega)) < \hat{q}_\infty^C(\omega) < F(\theta^*(\omega)) < \hat{F}(\theta^*(\omega))$, as required. Likewise if $\omega > \hat{\omega}$, then an analogous argument shows $\hat{q}_\infty^C(\omega) > \hat{F}(\theta^*(\omega))$. *Q.E.D.*

Let $q_t(\omega)$ denote the actual population fraction of action 0 in period t at state ω, and let $\bar{q}_t(\omega) := \frac{1}{t}\sum_{\tau=1}^t q_\tau(\omega)$ be its time average. The following lemma uses a similar argument as in Lemma B.3 to show that $\bar{q}_t$ converges to $F(\theta^*(\hat{\omega}))$.

LEMMA D.4: *For every ω, $\lim_{t\to\infty} \bar{q}_t(\omega) = F(\theta^*(\hat{\omega}))$.*

PROOF: Fix any ω. Let $\overline{R}(\omega) := \limsup_{t\to\infty} \bar{q}_t(\omega)$ and $\underline{R}(\omega) := \liminf_{t\to\infty} \bar{q}_t(\omega)$. Suppose for a contradiction that either $\overline{R}(\omega) > F(\theta^*(\hat{\omega}))$ or $\underline{R}(\omega) < F(\theta^*(\hat{\omega}))$. We consider only the first case, as the second case is analogous.

Consider any $R \in (F(\theta^*(\hat{\omega})), \overline{R}(\omega)]$. We first claim that, in state ω and any period t, if (i) almost all incorrect agents' beliefs assign probability 1 to $\hat{\omega}^I := \operatorname{argmin}_{\hat{\omega}'} \mathrm{KL}(R, \hat{F}(\theta^*(\hat{\omega}')))$ and (ii) almost all quasi-correct agents' beliefs assign probability 1 to $\hat{\omega}^C := \operatorname{argmin}_{\hat{\omega}'} \mathrm{KL}(R, \hat{q}_\infty^C(\hat{\omega}'))$, then $q_t(\omega) < R$.

To show this claim, we consider only the case $\hat{\alpha} < \alpha$, as the case $\hat{\alpha} > \alpha$ is analogous. By Lemma D.3, $\hat{q}_\infty^C(\omega) > \hat{F}(\theta^*(\hat{\omega}))$ iff $\omega < \hat{\omega}$. Hence, we have $\hat{\omega}^C < \hat{\omega}$ since $R > F(\theta^*(\hat{\omega})) =$

$\hat{F}(\theta^*(\hat{\omega}))$. Likewise, $\hat{\omega}^I < \hat{\omega}$. Thus, since $\hat{F}(\theta^*(\omega)) > \hat{q}^C_\infty(\omega)$ for all $\omega < \hat{\omega}$, it follows that $\hat{\omega} > \hat{\omega}^I > \hat{\omega}^C$.

By definition of $\hat{\omega}^C$, this leaves two cases to consider:

1. $R = \hat{q}^C_\infty(\hat{\omega}^C)$,
2. $R > \hat{q}^C_\infty(\hat{\omega}^C)$ and $\hat{\omega}^C = \underline{\omega}$.

In either case, $q_t(\omega) = \alpha F(\theta^*(\hat{\omega}^I)) + (1-\alpha)F(\theta^*(\hat{\omega}^C))$. Moreover, in case 1, (8) implies $R = \hat{\alpha}F(\theta^*(\hat{\omega}^I)) + (1-\hat{\alpha})F(\theta^*(\hat{\omega}^C))$, so that $R > q_t(\omega)$ because $\hat{\alpha} < \alpha$ and $\hat{\omega}^I > \hat{\omega}^C$. For case 2, we can extend the domain of function $\hat{q}^C_\infty$ from Ω to $\mathbb{R}$ by first extending the domain of function θ^* from Ω to $\mathbb{R}$ (in such a way that θ^* is still continuous, strictly decreasing, and has full range) and then defining $\hat{q}^C_\infty$ by (8) on the whole of $\mathbb{R}$. It is easy to show (using the same argument as above) that the extended $\hat{q}^C_\infty$ continues to satisfy Lemma D.3. Choosing $\tilde{\omega}^C < \overline{\omega}$ such that $R = \hat{q}^C_\infty(\tilde{\omega}^C)$ yields

$$R = \hat{\alpha}F(\theta^*(\hat{\omega}^I)) + (1-\hat{\alpha})F(\theta^*(\tilde{\omega}^C)) > \hat{\alpha}F(\theta^*(\hat{\omega}^I)) + (1-\hat{\alpha})F(\theta^*(\hat{\omega}^C)),$$

where the equality holds by (8) and the inequality holds since $\hat{\omega}^C = \underline{\omega}$. Thus, we again have $R > q_t(\omega)$ because $\hat{\alpha} < \alpha$ and $\hat{\omega}^I > \hat{\omega}^C$.

As a result, by continuity of u and F, there exist signals $\underline{s} < \overline{s}$, intervals of states $E^I \ni \hat{\omega}^I$, $E^C \ni \hat{\omega}^C$ with non-empty interior, and $\gamma > 0$ such that, in state ω and any period t, if (i') at least fraction $1-\gamma$ of incorrect agents with private signals $s \in [\underline{s}, \overline{s}]$ hold beliefs such that $H_t(E^I|a^{t-1}, s) \geq 1-\gamma$ and (ii') at least fraction $1-\gamma$ of quasi-correct agents with private signals $s \in [\underline{s}, \overline{s}]$ hold beliefs such that $H_t(E^C|a^{t-1}, s) \geq 1-\gamma$, then $q_t(\omega) < R-\gamma$.

To complete the proof, we consider separately the case where $\overline{R}(\omega) > \underline{R}(\omega)$ and the case where $\overline{R}(\omega) = \underline{R}(\omega)$. In the former case, we can choose $R \in (F(\theta^*(\hat{\omega}), \overline{R}(\omega))$ that additionally satisfies $R > \underline{R}(\omega)$. Then following a similar argument as in the proof of Lemma B.3 leads to a contradiction. Specifically, for any sufficiently small $\eta > 0$, by definition of $\overline{R}(\omega)$, $\underline{R}(\omega)$ and since $|\bar{q}_t(\omega) - \bar{q}_{t-1}(\omega)| < \eta$ for all large enough t, we can find an infinite sequence of times t_k such that $R - \frac{\eta}{2} \leq \bar{q}_{t_k-1}(\omega) \leq R + \frac{\eta}{2} < \bar{q}_{t_k}(\omega)$. Moreover, by choosing η small enough, the law of large numbers together with Lemma B.2 implies that, for all large enough t_k, hypotheses (i)' and (ii)' are satisfied. But then $q_{t_k}(\omega) < R-\gamma < R+\frac{\eta}{2}$, so that $\bar{q}_{t_k}(\omega) = \frac{t_k-1}{t_k}\bar{q}_{t_k-1}(\omega) + \frac{1}{t_k}q_{t_k}(\omega) < R+\frac{\eta}{2}$, a contradiction.

Finally, if $\overline{R}(\omega) = \underline{R}(\omega)$, then we choose $R = \overline{R}(\omega) = \underline{R}(\omega) > F(\theta^*(\hat{\omega}))$. In this case, by the law of large numbers and Lemma B.2, almost all incorrect agents' beliefs converge to a point mass on $\hat{\omega}^I := \operatorname{argmin}_{\hat{\omega}'} \mathrm{KL}(R, \hat{F}(\theta^*(\hat{\omega}')))$, and almost all quasi-correct agents' beliefs converge to a point mass on $\hat{\omega}^C := \operatorname{argmin}_{\hat{\omega}'} \mathrm{KL}(R, \hat{q}^C_\infty(\hat{\omega}'))$. Thus, hypotheses (i') and (ii') are satisfied for all large enough t, whence $\lim_{t\to\infty} q_t(\omega) \leq R-\gamma$. This contradicts $\lim_{t\to\infty} \bar{q}_t(\omega) = R$. *Q.E.D.*

To complete the proof of Proposition 2, let $\hat{\omega}^I := \operatorname{argmin}_{\hat{\omega}'} \mathrm{KL}(F(\theta^*(\hat{\omega})), \hat{q}^I_\infty(\hat{\omega}'))$ and $\hat{\omega}^C := \operatorname{argmin}_{\hat{\omega}'} \mathrm{KL}(F(\theta^*(\hat{\omega})), \hat{q}^C_\infty(\hat{\omega}'))$. Then Lemmas B.2 and D.4 imply that almost all incorrect agents' beliefs converge to a point mass on $\hat{\omega}^I$ and almost all quasi-correct agents' beliefs converge to a point mass on $\hat{\omega}^C$. Moreover, since $\hat{q}^I_\infty(\cdot) = \hat{F}(\theta^*(\cdot))$ and $\hat{F}(\theta^*(\hat{\omega})) = F(\theta^*(\hat{\omega}))$ by construction, we must have $\hat{\omega}^I = \hat{\omega}$. Likewise, by Lemma D.3, $\hat{q}^C_\infty(\theta^*(\hat{\omega})) = \hat{F}(\theta^*(\hat{\omega})) = F(\theta^*(\hat{\omega}))$, so that $\hat{\omega}^C = \hat{\omega}$.

APPENDIX E: OMITTED DETAILS

E.1. *Robustness of Single-Agent Active Learning*

Consider the active learning model discussed in Section 4.3, whose limit model belief process (see footnote 28) satisfies

$$\hat{\omega}_t = \underset{\hat{\omega}\in\Omega}{\operatorname{argmin}}\, \mathrm{KL}\big(q(x_t^*, \omega), \hat{q}(x_t^*, \hat{\omega})\big), \quad x_t^* = x^*(\hat{\omega}_{t-1}). \tag{9}$$

We measure the amount of misperception by a "bias" parameter $b \in \mathbb{R}$. Specifically, we write $\hat{q}(x, \omega) = r(x, \omega, b)$ for some C^1 function r that is strictly decreasing in (x, ω) and satisfies $q(x, \omega) = r(x, \omega, 0)$. We also assume that $x^*(\omega)$ is C^1.

PROPOSITION E.1: *Fix any $\varepsilon > 0$. There exists $\bar{b} > 0$ such that if $|b| < \bar{b}$, then at each $\omega \in \Omega$, process (9) admits a unique steady state $\hat{\omega}_\infty(\omega)$; moreover, $\hat{\omega}_\infty(\omega) \in [\omega - \varepsilon, \omega + \varepsilon]$ and is globally stable.*

PROOF: We first show that there exists $\bar{b} > 0$ such that at each $\omega \in \Omega$, process (9) satisfies $\hat{\omega}_t \in [\omega - \varepsilon, \omega + \varepsilon]$ for all $t \geq 2$ whenever $|b| \leq \bar{b}$. To see this, consider identity

$$r(x, \omega, 0) = r(x, \hat{\omega}, b) \tag{10}$$

as a function of $\hat{\omega}$. If $b = 0$, then for any x and ω, (10) admits $\hat{\omega} = \omega$ as the unique solution. Thus, by the implicit function theorem, $\dfrac{d\hat{\omega}}{db} = \dfrac{-\frac{\partial}{\partial b} r(x, \hat{\omega}, b)}{\frac{\partial}{\partial \hat{\omega}} r(x, \hat{\omega}, b)}$ holds at $b = 0$ and $\hat{\omega} = \omega$. But since r is C^1 and $X \times \Omega = [0, 1] \times [\underline{\omega}, \overline{\omega}]$ is compact, $\max_{(x,\omega)\in X\times\Omega} \left|\dfrac{-\frac{\partial}{\partial b} r(x, \omega, 0)}{\frac{\partial}{\partial \omega} r(x, \omega, 0)}\right| < \infty$. Hence, there exists $\bar{b} > 0$ such that for every $b \in [-\bar{b}, \bar{b}]$, x, and ω, (10) admits a unique solution $\hat{\omega} \in [\omega - \varepsilon, \omega + \varepsilon]$; that is, process (9) satisfies $\hat{\omega}_t \in [\omega - \varepsilon, \omega + \varepsilon]$ for all $t \geq 2$ from any initial point $\hat{\omega}_1$.

Finally, applying the implicit function theorem to $r(x^*(\hat{\omega}_t), \omega, 0) = r(x^*(\hat{\omega}_t), \hat{\omega}_{t+1}, b)$, we obtain

$$\frac{d\hat{\omega}_{t+1}}{d\hat{\omega}_t} = -\frac{x^{*\prime}(\hat{\omega}_t)\left(\dfrac{\partial r(x^*(\hat{\omega}_t), \omega, 0)}{\partial x^*} - \dfrac{\partial r(x^*(\hat{\omega}_t), \hat{\omega}_t, b)}{\partial a^*}\right)}{\dfrac{\partial r(x^*(\hat{\omega}_t), \hat{\omega}_{t+1}, b)}{\partial \hat{\omega}_{t+1}}}.$$

By uniform continuity of the derivatives (which holds by compactness of the domain $X \times \Omega$), we can choose $\bar{b}$ sufficiently small such that for all $|b| \leq \bar{b}$ and ω, the right-hand side is strictly less than 1 in absolute value at all $t \geq 2$. This guarantees that process (9) is a contraction on $[\omega - \varepsilon, \omega + \varepsilon]$. Hence, it admits a unique steady state $\hat{\omega}_\infty(\omega) \in [\omega - \varepsilon, \omega + \varepsilon]$, to which it converges from any initial point. *Q.E.D.*

E.2. *Misperceptions About Matching Technology*

Consider the assortative random matching model from Section 7.1. As in Section 4.2, we set up a limit model where each agent observes the actions of infinitely many matches at the end of each period. To simplify the exposition, we consider the unbounded state space $\Omega = \mathbb{R}$ and assume that $\theta^*(\cdot)$ is unbounded on Ω. Fix any true state ω. If $\hat{P} = P$, then agents learn the true state at the end of the first period; in period 2 and all subsequent periods, agents play a threshold strategy with cutoff type $\theta^*(\omega)$, and each type's observed fraction of action 0, $P(\theta^*(\omega)|\theta)$, matches his expectation.

If $\hat{P} \neq P$, then for simplicity, we continue to assume that in period 2, agents play a threshold strategy according to some cutoff type θ_1^*.[1] Inductively, this induces the following sequence of cutoff types (θ_t^*) and type-dependent point-mass beliefs $(\hat{\omega}_t^\theta)$. At any $t \geq 2$, if agents play according to cutoff θ_{t-1}^*, then each type θ observes fraction $P(\theta_{t-1}^*|\theta)$ of action 0, and based on this, assigns a point mass to the state $\hat{\omega}_t^\theta = \operatorname{argmin}_{\hat{\omega}\in\mathbb{R}} \mathrm{KL}(P(\theta_{t-1}^*|\theta), \hat{P}(\theta^*(\hat{\omega})|\theta))$ that best explains this observation. Since $\theta^*(\cdot)$ is unbounded and $P(\cdot|\theta)$ is a continuous distribution with full support, $\hat{\omega}_t^\theta$ is uniquely given by

$$P(\theta_{t-1}^*|\theta) = \hat{P}(\theta^*(\hat{\omega}_t^\theta)|\theta). \tag{11}$$

Given this, we claim that in period $t+1$, agents follow a threshold strategy with cutoff type θ_t^* given by

$$P(\theta_{t-1}^*|\theta_t^*) = \hat{P}(\theta_t^*|\theta_t^*). \tag{12}$$

Note that (12) uniquely pins down θ_t^*, because by assumptions (i) and (ii) in Section 7.1, the left-hand side is weakly decreasing in θ_t^* but the right-hand side is strictly increasing in θ_t^*. To see that agents behave according to cutoff θ_t^* in period $t+1$, consider any $\theta > \theta_t^*$. Then $P(\theta_{t-1}^*|\theta) \leq P(\theta_{t-1}^*|\theta_t^*) = \hat{P}(\theta_t^*|\theta_t^*) < \hat{P}(\theta|\theta)$. Thus, (11) implies that $\theta^*(\hat{\omega}_t^\theta) < \theta$, whence type θ plays action 1 in period $t+1$. Analogously, we can verify that any type $\theta < \theta_t^*$ chooses action 0 in period $t+1$.

Note that by (12), θ_t^* is strictly increasing in θ_{t-1}^*. Indeed, for any $\eta > 0$, we have $P(\theta_{t-1}^* + \eta|\theta_t^*) > \hat{P}(\theta_t^*|\theta_t^*)$, and the left-hand side is decreasing in θ_t^* and the right-hand side is strictly increasing in θ_t^*. Given this, recursion (12) either converges to a steady state θ_∞^* with

$$P(\theta_\infty^*|\theta_\infty^*) = \hat{P}(\theta_\infty^*|\theta_\infty^*) \tag{13}$$

or diverges, and in the former case, each type θ's steady-state belief $\hat{\omega}_\infty^\theta$ satisfies

$$P(\theta_\infty^*|\theta) = \hat{P}(\theta^*(\hat{\omega}_\infty^\theta)|\theta). \tag{14}$$

The following example illustrates a natural misperception, assortativity neglect, under which the steady-state beliefs $\hat{\omega}_\infty^\theta$ are state-independent and increasing in types.

EXAMPLE 4—Assortativity Neglect in a Gaussian Setting: Suppose that P and $\hat{P}$ are symmetric bivariate Gaussian distributions whose mean, variance, and correlation coefficient are given by (μ, σ^2, ρ) and $(\hat{\mu}, \hat{\sigma}^2, \hat{\rho})$, respectively, with $\rho, \hat{\rho} \geq 0$ (reflecting assortativity). To model assortativity neglect, we suppose that $\hat{\rho} < \rho$, $\hat{\mu} = \mu$, and $\hat{\sigma} = \sigma$; that is, agents underestimate the correlation in the matching technology, but are correct about the marginal type distribution. Letting G denote the cdf of the standard Gaussian

[1]This simplifying assumption is satisfied whenever $\|\hat{P} - P\|$ is sufficiently small. Indeed, while different types θ might believe in different states $\hat{\omega}_1^\theta$ at the end of period 1, when $\|\hat{P} - P\|$ is sufficiently small, all $\hat{\omega}_1^\theta$ are sufficiently close to ω that $u(1, \theta, \hat{\omega}_1^\theta) - u(0, \theta, \hat{\omega}_1^\theta)$ is increasing in θ. Thus, agents follow a threshold strategy.

distribution, equation (13) yields $G[\sqrt{\frac{1-\rho}{1+\rho}}\frac{\theta^*_\infty-\mu}{\sigma}] = G[\sqrt{\frac{1-\hat\rho}{1+\hat\rho}}\frac{\theta^*_\infty-\mu}{\sigma}]$, which admits the unique solution $\theta^*_\infty = \mu$. Thus, by (14), each type θ's steady-state belief is a state-independent point mass $\hat\omega^\theta_\infty$ such that $\theta^*(\hat\omega^\theta_\infty) = \frac{\sqrt{1-\hat\rho}}{\sqrt{1-\rho}}(\mu - \rho\theta - (1-\rho)\mu) + \hat\rho\theta + (1-\hat\rho)\mu$. Since the right-hand side of the latter equation is decreasing in θ, beliefs $\hat\omega^\theta_\infty$ are increasing in types. *Q.E.D.*

E.3. *Continuous Actions*

This section considers a continuous action space version of our model. We perform steady-state analysis (under the limit model) to illustrate why our main insights do not rely on a finite action space. Throughout, we assume that the action space is an interval $A = [\underline{a}, \overline{a}] \subseteq \mathbb{R}$, with $-\infty \leq \underline{a} < \overline{a} \leq \infty$. Let $u(a, \theta, \omega)$ denote type θ's utility to choosing action a in state ω. We assume that for every type $\theta \in \mathbb{R}$ and state $\omega \in \Omega := [\underline{\omega}, \overline{\omega}]$, there exists a unique optimal action $a^*(\theta, \omega) := \operatorname{argmax}_{a\in A} u(a, \theta, \omega)$ which is continuous and strictly increasing in (θ, ω) and such that $a^*(\cdot, \omega)$ has full range for all ω.

Given any true and perceived type distributions $F, \hat F \in \mathcal{F}$, we briefly analyze the set of steady states $\mathrm{SS}(F, \hat F)$ of this model. For each state ω, let $G(\cdot, \omega) \in \Delta(A)$ denote the true cdf over actions in the population when (almost all) agents assign probability 1 to state ω and let $g(\cdot, \omega)$ denote the corresponding density. Likewise, let $\hat G(\cdot, \omega)$ and $\hat g(\cdot, \omega)$ denote the corresponding perceived action distribution and density when agents assign probability 1 to ω. Note that $G(a, \omega) = F(\theta^*(a, \omega))$ and $\hat G(a, \omega) = \hat F(\theta^*(a, \omega))$, where $\theta^*(a, \omega)$ satisfies $a = a^*(\theta^*(a, \omega), \omega)$. Let $\mathrm{KL}(H, \hat H) := \int \log[\frac{h(a)}{\hat h(a)}]h(a)\,da$ denote the KL-divergence between continuous distributions H and $\hat H$ with densities h and $\hat h$. As in the binary action space setting, we define a steady state $\hat\omega^*$ to be a solution to

$$\hat\omega^* \in \operatorname*{argmin}_{\hat\omega} \mathrm{KL}\big(G\big(\cdot, \hat\omega^*\big), \hat G(\cdot, \hat\omega)\big).$$

Thus, as before, in a steady state, agents assign probability 1 to a state that minimizes the KL-divergence between the corresponding observed action distribution and agents' perceived action distribution. At interior steady states $\hat\omega^*$, the first-order condition yields

$$\int \frac{g\big(a, \hat\omega^*\big)}{\hat g\big(a, \hat\omega^*\big)} \frac{\partial \hat g\big(a, \hat\omega^*\big)}{\partial \hat\omega}\, da = 0. \tag{15}$$

Thus, the set of steady states $\mathrm{SS}(F, \hat F)$ is finite whenever there are at most finitely many $\hat\omega^*$ that satisfy (15). A sufficient condition for this is that the left-hand side of (15) is analytic in $\hat\omega^*$ and not constantly equal to 0; similar to the logic behind Theorem 2, this is ensured if $F \neq \hat F$ are analytic and $\theta^*(a, \cdot)$ is analytic. Moreover, similarly to the logic behind Theorem 1, it is easy to construct examples where $\hat F$ is arbitrarily close to F but there is only a single (state-independent) steady state, as the following illustrates:

EXAMPLE 5: Consider the quadratic-loss utility $u(a, \theta, \omega) = -(a - \theta - \omega)^2$, which implies that the optimal action takes the form $a^*(\theta, \omega) = \theta + \omega$. Suppose that F and $\hat F$ are cdfs of the Gaussian distributions $N(\mu, \sigma^2)$ and $N(\hat\mu, \hat\sigma^2)$. Then the left-hand side of (15) is given by $\int \frac{\hat\mu-\theta}{\hat\sigma^2}\frac{\exp[-\frac{(\theta-\mu)^2}{2\sigma^2}]}{\sqrt{2\pi\sigma^2}}d\theta = \frac{\hat\mu-\mu}{\hat\sigma^2}$. Thus, there is no interior steady state, and whenever $\mu > \hat{\mu}$ (respectively, $\mu < \hat{\mu}$), the unique steady state is given by $\overline{\omega}$ (respectively, $\underline{\omega}$), paralleling Example 1 in the binary action setting.

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Co-editor Alessandro Lizzeri handled this manuscript.

Manuscript received 15 January, 2019; final version accepted 27 May, 2020; available online 2 June, 2020.