WEB APPENDIX FOR:

"The Analytic Theory of a Monetary Shock"*

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F Sticky Price Multiproduct firms

Multiproduct models consider a firm that produces n different products and that faces increasing returns in the price adjustment: if a pays a fixed cost it can adjust simultaneously the n prices. Variations on this model have been studied by Midrigan (2011) and Bhattarai and Schoenle (2014). These models are appealing because they match several empirical regularities: synchronization among price changes within a store and the coexistence of both small and large price changes. Their economic analysis is of interest because in an economy populated by multiproduct firms the monetary shocks have more persistent real effects. In Alvarez and Lippi (2014) we derived results for impulse responses to this multidimensional setup and explore the sense in which such a model is realistic. Here we show that the characterization of the selection effect, as the difference between the survival function and the output IRF holds in this model, with the number of products n serving as the parameter that control selection. We also show that in this case a single eigenvalue gives a poor characterization of the output IRF.

In the multiproduct model the price gap is given by a vector of n price gaps, each of them given by an independently standard BM's $(p_1, p_2, \ldots p_n)$, driftless and with innovation variance σ^2. We are interested only on two functions of this vector, the sum of its squares and it sum:

$$y = \sum_{i=1}^{n} p_i^2 \text{ and } z = \sum_{i=1}^{n} p_i$$

It is interesting to notice that while the original state is n dimensional, (y, z) can be described as a two dimensional diffusion –see Alvarez and Lippi (2014) and Appendix F.1 for details.

*Appendix to be posted online.

We are interested in the sum of its squares y because in Alvarez and Lippi (2014) under the assumption of symmetric demand the optimal decision rule is to adjust the firm time that y hits a critical value $\bar{y}$. We are interested in z, the sum of the price gaps, because this give the contribution of firm to the deviation of the price level relative to the steady state value, and hence $-z$ is proportional to its contribution to output. Note that the domain of (y, z) is $0 \leq y \leq \bar{y}$ and $-\sqrt{ny} \leq z \leq \sqrt{n\,y}$. In Alvarez and Lippi (2014) we show that the expected number of adjustments per unit of time is given by $N = \frac{n\sigma^2}{\bar{y}}$ and also give a characterization of $\bar{y}$ in terms of the parameters for the firm's problem. For the purpose in this paper we find it convenient to rewrite the state as (x, w) defined as

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}.$$

In Lemma 1 in Appendix F.1 we analyze the behavior of the $(x, w) \in [0, \bar{x}] \times [-1, 1]$ process with $\bar{x} \equiv \sqrt{\bar{y}}$. Clearly we can recover (y, z) from (x, w). For instance, $z = w\sqrt{nx}$. Yet with this change on variables, even though the original problem is n dimensional, we define a two dimensional process for which we can analytically find its associated eigenfunctions and eigenvalues for the operator:

$$\mathcal{G}(f)\left((x, w), t\right) = \mathbb{E}\left[f\left(x(t), w(t)\right)\, 1_{y \geq \bar{y}} \,\middle|\, (x(0), w(0)) = (x, w)\right]$$

where $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$. The relevant p.d.e. is defined and its solution via eigenfunctions and eigenvalues, is characterized in Proposition 3 in Appendix F.1. Moreover the eigenfunctions and eigenvalues are indexed by a countably double infinity indices $\{m, k\}$.

Eigenfunctions. The eigenfunctions φ have a multiplicative nature, so $\varphi_{m,k}(x, w) = h_m(w) g_{m,k}(x)$ where for each number of products n then h_m and $g_{m,k}$ are known analytic functions indexed by k and by (k, m) respectively. Indeed h_m are scaled Gegenbauer polynomials, and $g_{m,k}$ are scaled Bessel functions –see Proposition F.1 for the exact expressions and definition.[1]

Eigenvalues. For each n the eigenvalues can be also indexed by a countably double-infinity $\{\lambda_{m,k}\}$. As in the baseline case, the eigenvalues are proportional to N, the expected number of price changes per unit of time:

$$\lambda_{m,k} = -N\, \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \;\text{ for } m = 0, 1, \ldots, \text{ and } k = 1, 2, \ldots$$

$j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

The second sub-index k in the root of the Bessel function denote their ordering, so $k = 1$ is the smallest positive root. Also fixing k the roots $j_{m+\frac{n}{2}-1,k}$ are increasing in m. Thus, the *dominant* eigenvalue is given by $\lambda_{0,1}$. We will argue below that the smallest (in absolute value) eigenvalue that is featured in the (marginal) output IRF is $\lambda_{1,1}$. A very accurate approximation of the eigenvalues consists on using the first three leading terms in its expansion, as is given by: $j_{\nu,k} \approx \nu + \nu^{1/3} 2^{-1/3} a_k + (3/20)(a_k)^2 2^{1/3} \nu^{-1/3}$ where a_k are the zeros of the

[1]The Gegenbauer polynomials are orthogonal to each other, and so are the Bessel functions when using an appropriately weighted inner product, as defined in Appendix F.1.

Airy function. [2] Using this approximation into the expression for the eigenvalues, one can see that keeping fixed N, the absolute value both $\lambda_{0,1}$ and $\lambda_{1,1}$ go to infinity, and that the difference between the two decreases and converges to $N/2$. Figure 1 displays the difference between these two eigenvalues.

Impulse response. As before, we want to compute $G(t)$, the conditional expectation of $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$ for (x, w) following equation (A.1)-equation (A.2), integrated with respect to $p(w, x; 0)$. We are interested in functions $f : [0, \bar{x}] \times [-1, 1]$ that can be written as:

$$f(x, w) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{m,k}[f] \varphi_{m,k}(x, w)$$

Using the same logic as in the one dimensional case:[3]

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f]\, b_{m,k}\, [p(\cdot, 0)/\omega]\, \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

where the term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ appears because the have, as it is customary in this case, use an orthogonal, but not orthonormal base, and where $\omega(w, x)$ is a weighing function appropriately defined – see Appendix F.1. So that $b_{m,k}\, [p(\cdot, 0)/\omega]$ are the projections of the ratio of the functions $p(\cdot, 0)$ and ω.

Functions of interest. We analyze two important functions of interest f. The first one a constant, $f(w, x) = 1$ which is used to compute the measure of firms that have not adjusted, or the survival function $S(t)$. The second one is the one that gives the average price gap among the n product of the firm, i.e. $f(w, z) = -z/n = -wx/\sqrt{n}$. This is, as before, the negative of the average across the n products of the price gaps. This is the function f used for the impulse response of output to a monetary shock. An important property of the Gegenbauer polynomials is that the $m = 0$ equals a constant, for $m = 1$ is proportional to w, and in general for m odd are antisymmetric on w and symmetric for even m. Thus for $f = 1$ we can use just the Gegenbauer polynomial with $m = 0$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$. Instead for $f(w, x) = wx/sqrtn = z$ we can use just the Gegenbauer polynomial with $m = 1$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$.

Initial shifted distribution for a small shock. We have derived the invariant distribution of (z, y) in Alvarez and Lippi (2014). Using the change in variables (y, z) to $y = x^2$ and $z = \sqrt{yn}\, w =$ we can define the steady state density as $\bar{p}(w, x) = \bar{h}(w)\bar{g}(x)$ – see Appendix F.1 for the expressions. We perturb this density with a shock of size δ in each of the n price gaps. We want to subtract δ to each component of $(p_1, ..., p_n)$. This means that the density for each $x = ||p||$ just after the shock becomes the density of $x(\delta) = ||(p_1 + \delta, \ldots, p_n + \delta)||$ just before. Likewise the density corresponding to each w becomes th one for $w(\delta) = (z + n\delta)/(\sqrt{n}\, x(\delta))$. We consider the initial condition given by density $p_0(w, x; \delta) = \bar{h}(w(\delta))\bar{g}(x(\delta))$. We will use the first order terms, which are appropriate for the case of a small shock δ. The expressions can be found in Appendix F.1.

[2]In our case, we are interested in $k = 1$ which is about $a_1 = -2.33811$. See Figure 2 where we plot both eigenvalues, as well as its approximation for several n.

[3]See Appendix F.1 for a derivation

Interpretation of dominant eigenvalue, and irrelevance for the marginal IRF. We are now ready to generalize our interpretation of the dominant eigenvalue (as well as those corresponding to symmetric functions of z), as well as its irrelevance for the marginal output IRF.

PROPOSITION 1. The coefficient of the marginal impulse response of output for a monetary shock are a function of the $\{\lambda_{1,k}, \varphi_{1,k}\}_{k=1}^{\infty}$ eigenvalue-eigenfunctions pairs, so that:

$$Y(t) = \sum_{k=1}^{\infty} \beta_{1,k}\, e^{\lambda_{1,k}\, t} \quad \text{and} \quad -\lambda_{1,1} = \lim_{t\to\infty} \frac{\log|Y(t)|}{t}$$

where $\beta_{1,k} = b_{1,k}\,[wx/\sqrt{n}]\; b_{1,k}\,[\bar{p}'(w,x)]$. In particular, the dominant eigenvalue $\lambda_{0,1}$ does not characterize the limiting behavior of the impulse response. Instead the survival function for price changes $S(t)$, can be written in terms of $\{\lambda_{0,k}, \varphi_{0,k}\}_{k=1}^{\infty}$, and hence the asymptotic hazard rate is equal to the dominant eigenvalue $\lambda_{0,1}$, i.e.

$$S(t) = \sum_{k=1}^{\infty} \beta_{0,k}\, e^{\lambda_{0,k}\, t} \quad \text{and} \quad -\lambda_{0,1} = \lim_{t\to\infty} \frac{\log S(t)}{t}$$

where $\beta_{0,k} = b_{0,k}\,[1]\; b_{0,k}\,[\delta_0]$ where δ_0 is the Dirac delta function for $(p_1, \ldots, p_n)$ transformed to the (x, w) coordinates. Recall that $0 > \lambda_{0,1} > \lambda_{1,1}$.

Given the importance of the difference between the eigenvalues $\lambda_{1,k}$ and $\lambda_{0,k}$ we show that for a fixed k they both increase with n, but it difference decreases to asymptote to $1/2$.

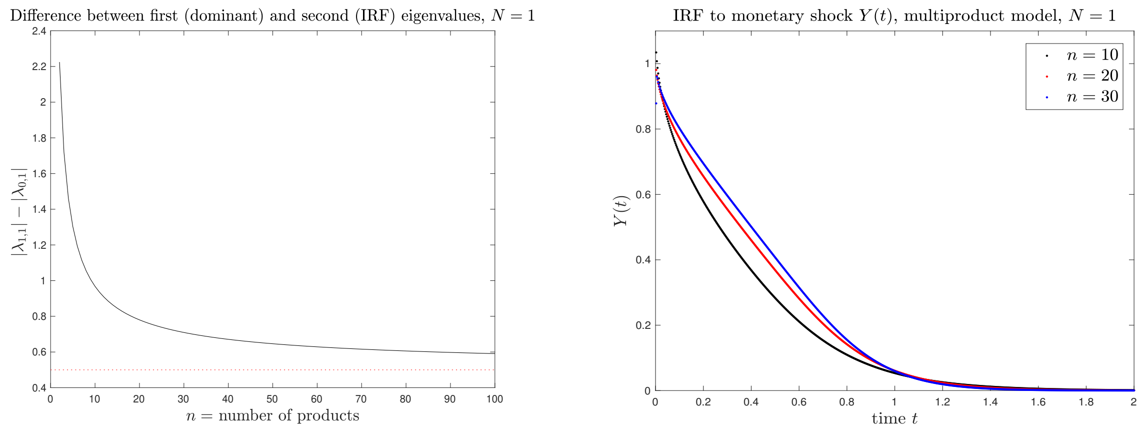
PROPOSITION 2. Fixing $k \geq 1$, the k^{th} eigenvalue for the IRF $Y(\cdot)$ given by $\lambda_{1,k}$ and the k^{th} eigenvalue for the survival function $S(\cdot)$ given by $\lambda_{1,k}$ both increase with the number of products n, diverging towards $-\infty$ as $n \to \infty$. The difference $\lambda_{0,k} - \lambda_{1,k} > 0$ decreases with n, converging to $1/2$ as $n \to \infty$.

Figure 1 illustrates Proposition 2 for the case of $k = 1$, i.e. the eigenvalue that dominates the long run behaviour of the survival and IRF functions. Proposition 2 extends the result for all k. Increasing the number of products n in the multi product model decreases the selection effect at the time of a price change. As n goes to infinity, the eigenvalues that control the duration of the price changes (S) and those that control the marginal output IRF (Y) converge. This result shows that the characterization of selection effect in terms of dynamics controlled by two different types of eigenvalues is present not only in the Calvo$^+$ model, but also in this setup.

In Appendix F.1 we include Proposition 4 which gives a closed form solution for $\bar{p}'(w, x; 0)$ and for the coefficients for $b_{1,k}$ of the output impulse response function. All these expressions depends only of the number of products n. Instead we include a figure of the impulse responses for three values of n. It is clear both the output IRF and the survival function cannot be well described using one eigenfunction-eigenvalue for large n. For instance, as $n \to \infty$ the output's IRF Y becomes a linearly declining function until it hits zero at $t = 1/N$, and the survival function S is zero until it becomes infinite at $t = 1/N$.

Figure 1: Shock propagation in Multiproduct models

Selection effect | Impulse response



Keeping fixed $N = 1$ for all n

F.1 Details of the multiproduct model

Law of motion for y, z.

$$dy = \sigma^2 n\, dt + 2\sigma\sqrt{y}\, dW^a$$
$$dz = \sigma\sqrt{n}\left[\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1 - \left(\frac{z}{\sqrt{ny}}\right)^2}\, dW^b\right]$$

where W^a, W^b are independent standard BM's.

Lemma 1. Define

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}$$

so that the domain is $0 \le x \le \bar{x} \equiv \sqrt{\bar{y}}$ and $-1 \le w \le 1$. They satisfy:

$$dx = \sigma^2 \frac{n-1}{2\,x} dt + \sigma dW^a \tag{A.1}$$
$$dw = \frac{w}{x^2}\left(\frac{1-n}{2}\right) dt + \frac{\sqrt{1-w^2}}{x}\, dW^b \tag{A.2}$$

We look for a solution to the eigenvalue-eigenfunction problem (λ, φ) given by equation (A.1) and equation (A.2). They must satisfy

$$\lambda\varphi(w,x) = \varphi_x(w,x)\sigma^2\left(\frac{n-1}{2x}\right) + \varphi_w(w,x)\frac{w}{x^2}\left(\frac{1-n}{2}\right) + \frac{1}{2}\varphi_{ww}(w,x)\frac{(1-w^2)}{x^2} + \frac{1}{2}\sigma^2\varphi_{xx}(w,x)$$

for all $(x,w) \in [0,\bar{x}] \times [-1,1]$, with $\varphi(\bar{x},w) = 0$, all w and φ^2 integrable.

Proposition 3. The eigenfunctions-eigenvalues of (w,x) satisfying equation (A.1)-equation (A.2) denoted by $\{\varphi_{m,k}(\cdot), \lambda_{m,k}\}$ for $k = 1,2,\ldots$ and $m = 0,1,\ldots$ are given by:

$$\begin{aligned}
\varphi_{m,k}(x,w) &= h_m(w)\, g_{m,k}(x) \quad \text{where} \\
h_m(w) &= C_m^{\frac{n}{2}-1}(w) \quad \text{for } m = 0,1,2,\ldots \text{ and} \\
g_{m,k}(x) &= x^{1-n/2} J_{\frac{n}{2}-1+m}\left(j_{\frac{n}{2}-1+m,k}\,\frac{x}{\bar{x}}\right) \quad \text{for } k = 1,2,\ldots \text{ and} \\
\lambda_{m,k} &= -N\,\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \quad \text{for } m = 0,1,\ldots, \text{ and } k = 1,2,\ldots
\end{aligned}$$

where $C_m^{\frac{n}{2}-1}(\cdot)$ denote the Gegenbauer polynomials, and where $J_{\frac{n}{2}-1+m}(\cdot)$ denote the Bessel function of the first kind, $j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

Note that the expressions for the eigenfunctions are only valid only for $n > 2$. For $n = 2$ the expression take a different special form, which we skip to save space. The expressions for the eigenvalues are valid for $n \geq 2$.

We remind the reader how the Gegenbauer polynomial and Bessel function, which form an orthogonal base, are defined. The Gegenbauer polynomial $C_m^{\frac{n}{2}-1}(w)$ is given by:

$$C_m^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{\Gamma(m-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(m-2k)!}(2w)^{m-2k} \tag{A.3}$$

For a fixed n, the polynomials are orthogonal on with respect to the weighting function $(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}$ so that:[4]

$$\int_{-1}^{1} C_m^{(\frac{n}{2}-1)}(w)\, C_j^{(\frac{n}{2}-1)}(w)\,(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\,dw = 0 \text{ for } m \neq j \tag{A.4}$$

and for $m = j$ we get

$$\int_{-1}^{1} \left[C_m^{(\frac{n}{2}-1)}(w)\right]^2 (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\,dw = \frac{\pi\, 2^{1-2(\frac{n}{2}-1)}\Gamma(m+2(\frac{n}{2}-1))}{m!(m+\frac{n}{2}-1)[\Gamma(\frac{n}{2}-1)]^2} \tag{A.5}$$

[4]By this we mean that we define the inner product between functions a, b from $[-1,1]$ to $\mathbb{R}$ as : $\langle a,b\rangle = \int_{-1}^{1} a(w)b(w)\,(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\,dw$.

The Bessel function of the first kind is given by :

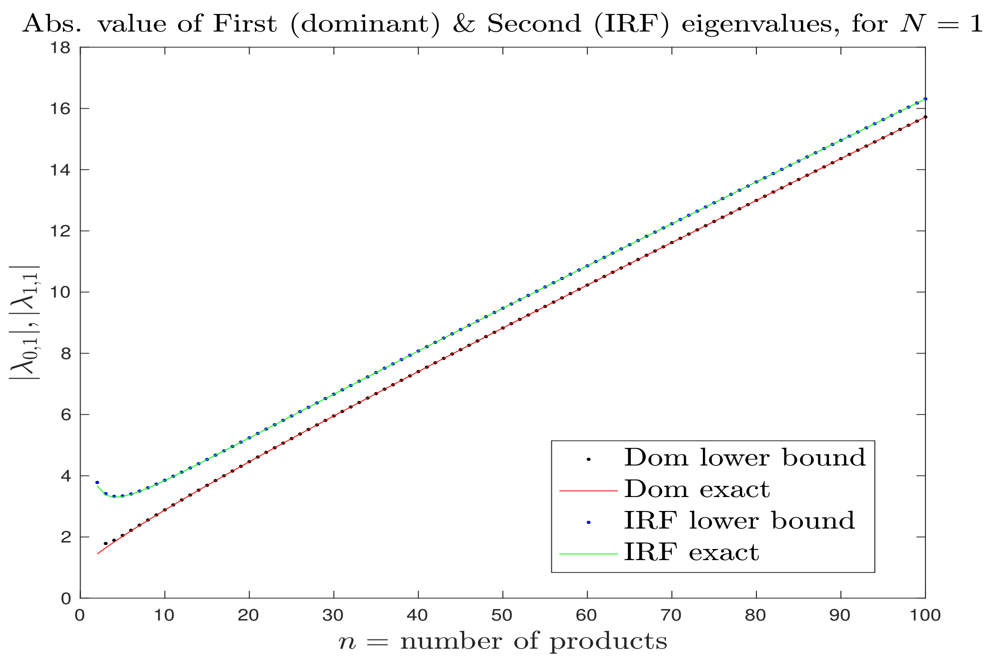
$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{A.6}$$

For a given ν, the following functions are orthogonal, using the weighting function x^{n-1} so that:[5]

$$\begin{aligned} &\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right] \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right)\right] x^{n-1}\, dx \\ =&\int_0^{\bar{x}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right) J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right) x\, dx = 0 \text{ if } k \neq s \in \{1,2,3,\dots\} \text{ and} \\ &\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 x^{n-1}\, dx = \bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}} \qquad \text{(A.7)} \\ =&\ \frac{1}{2}\left(\bar{x}\, J_{\nu+1}\left(j_{\nu,k}\right)\right)^2 \text{ for all } k \in \{1,2,3\dots\} \qquad \text{(A.8)} \end{aligned}$$

where $j_{\nu,k}$ and $j_{\nu,s}$ are two zeros of $J_\nu(\cdot)$.

Figure 2: Eigenvalues for multiproduct model



Kepping fixed $N = 1$ for all n

[5] By this we mean that we define the inner product between functions a, b from $[0, \bar{x}]$ to $\mathbb{R}$ as: $\langle a, b\rangle = \int_0^{\bar{x}} a(x)b(x)x^{n-1}dx$.

Derivation of IRF. Thus we have

$$G(t) \equiv \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G}(f)(x,w,t)\, p(x,w;0) dw\, dx$$

As in Section 3, we can write this expected value as:

$$\begin{aligned} Y(t) &= \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G}\left(\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, \varphi_{k,m} \right)(x,w,t)\ p(x,w;0) dw\, dx \\ &= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, \mathcal{G}\left(\varphi_{k,m}\right)(x,w,t)\ p(x,w;0) dw\, dx \\ &= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, e^{\lambda_{m,k}\, t}\, \varphi_{m,k}(x,w)\, p(x,w;0) dw\, dx \\ &= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f] \int_0^{\bar{x}} \int_{-1}^{1} \varphi_{m,k}(x,w)\, p(x,w;0) dw\, dx \end{aligned}$$

Then we get:

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f]\, b_{m,k}\left[p(\cdot,0)/\omega\right]\, \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

Inner product. We let $\omega(w,x) = x^{1-n}\,(1-w^2)^{\frac{n-3}{2}}$. The inner product of functions a, b from $[0,\bar{x}] \times [-1,1]$ to $\mathbb{R}$ is defied as

$$\langle a, b \rangle = \int_0^{\bar{x}} \int_{-1}^{1} a(x,w)\, b(x,w)\, x^{1-n}\,(1-w^2)^{\frac{n-3}{2}}\, dw\, dx$$

The term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ is given by the product of equation (A.5) and equation (A.8) found above. Indeed since the polynomials are orthogonal we have:

$$\begin{aligned} b_{m,k}[f] = \frac{\langle f, \varphi_{m,k} \rangle}{\langle \varphi_{m,k}, \varphi_{m,k} \rangle} &= \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) h_m(w) (1-w^2)^{\frac{n-3}{2}} dw \right] g_{m,k}(x)\, x^{n-1}\, dx}{\left[\int_{-1}^{1} \left(h_m(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw \right] \left[\int_0^{\bar{x}} \left(g_{m,k}(x)\right)^2 x^{n-1}\, dx \right]} \\ &= \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) C_m^{\frac{n}{2}-1}(w) (1-w^2)^{\frac{n-3}{2}} dw \right] J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\, \frac{x}{\bar{x}} \right) x^{\frac{n}{2}} dx}{\left[\int_{-1}^{1} \left(C_m^{\frac{n}{2}-1}(w) \right)^2 (1-w^2)^{\frac{n-3}{2}} dw \right] \left[\int_0^{\bar{x}} \left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\, \frac{x}{\bar{x}} \right) \right)^2 x\, dx \right]} \end{aligned}$$

Invariant Distribution. After the change in variables we have:

$$\bar{h}(w) = \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \; \left(1 - w^2\right)^{(n-3)/2} \quad \text{for} \;\; w \in (-1\,,\,1) \tag{A.9}$$

$$\bar{g}(x) = x \; (\bar{x})^{-n} \left(\frac{2n}{n-2}\right) \left[\,\bar{x}^{n-2} - x^{n-2}\right] \qquad \text{for } x \in [0, \bar{x}] \tag{A.10}$$

Initial distribution after a small monetary shock.

$$\begin{aligned} &p(w, x; 0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w, x; 0)\delta + o(\delta) \text{ with} \\ &\bar{p}'(w, x; 0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \end{aligned}$$

where:

$$\begin{aligned} &\frac{\partial}{\partial\delta}x(\delta)|_{\delta=0} = x'(0) = \sqrt{n}\,w \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w)w'(0) \\ &\frac{\partial}{\partial\delta}w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n}\,(1-w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x)x'(0) \end{aligned}$$

Proposition 4. The expressions for $\bar{p}'(x, w; 0)$ and the coefficients $b_{1,k}(n)$ for the impulse response of output are given by:

$$\begin{aligned} \bar{p}'(w, x; 0) &= \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \\ &= \frac{w\,(1-w^2)^{(n-3)/2}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\frac{[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2}]}{\bar{x}^n} \end{aligned}$$

and the coefficients for the impulse response $b_{1,k}(n) = b_{1,k}[f]\, b_{1,k}[\bar{p}'(\cdot, 0)/\omega]\, \langle \varphi_{1,k}, \varphi_{1,k} \rangle$ are given by

$$\begin{aligned} b_{1,k}(n) = -\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2n}{(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} &\left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} - \frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right. \\ &\left. -(4+n)2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right] \end{aligned}$$

where ${}_1\tilde{F}_2(a_1; b_1, b_2; z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1; b_1, b_2; z) = {}_1F_2(a_1; b_1, b_2; z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function and $j_{\frac{n}{2},k}$ is the k^{th} ordered zero of the Bessel function $J_{\frac{n}{2}}(\cdot)$.

Note that, as our notation emphasizes, the coefficients $b_j(n)$ depends only on the number of products.

F.2 Proofs for the Multiproduct model of Appendix F.1

Proof. (of Proposition 1) First take $f(w,x) = w\,x/\sqrt{n} = \frac{1}{n}\sum_{i=1}^{n} p_i$. But note that the Gegenbauer polynomial of degree 1 is

$$C_1^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor 1/2 \rfloor}(-1)^k \frac{\Gamma(1-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(1-2k)!}(2w)^{1-2k} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-1)}(2w) = (n-2)w$$

Thus for $f(w,x) = wx/\sqrt{n}$ we can simply write:

$$f(x,w) = \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w)$$

since

$$b_{m,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\left[\int_{-1}^{1} C_1^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}} x\, J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx\right]}{\left[\int_{-1}^{1}\left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}}\left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]}$$

and thus $b_{m,k}[f] = 0$ for all $m \neq 1$, since the polynomials are orthogonal, and

$$b_{1,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\int_0^{\bar{x}} x J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\int_0^{\bar{x}}\left(J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx} \text{ for all } k \geq 1$$

Now we argue that fixing x the function $p(w,x;0)$ is odd (antisymmetric) viewed as a function of w. This is because $\bar{h}$ is even and $x'(0)$ is odd, so $\bar{h}'(w)x'(0)$ is odd. Also $\bar{h}'$ is odd and $w'(0)$ is even, hence $\bar{h}'(w)w'(0)$ is odd. Hence $p(w,x;0)$ is not orthogonal to the $C_1^{\frac{n}{2}-1}(\cdot)$. Thus $b_{1,k}[\bar{p}'] \neq 0$.

Finally, to represent the survival function, take $f(w,x) = 1$. Note that this also coincides with a Gegenbauer polynomial for $m = 0$, i.e. $C_0^{\frac{n}{2}-1}(w) = 1$. Thus:

$$f(x,w) = \sum_{k=1}^{\infty} b_{0,k}[f]\varphi_{1,k}(x,w)$$

since

$$b_{m,k}[f] = \frac{\left[\int_{-1}^{1} C_0^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}} J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx\right]}{\left[\int_{-1}^{1}\left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}}\left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]}$$

and since the Gegenbauer polynomials are orthogonal, and thus $b_{m,k}[f] = 0$ for all $m > 0$, and

$$b_{0,k}[f] = \frac{\int_0^{\bar{x}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\left[\int_0^{\bar{x}}\left(J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]} \text{ for all } k \geq 1$$

□

Proof. (of Proposition 2) Recall that for each $k \geq 1$:

$$\lambda_{1,k} = -N\frac{\left(j_{\frac{n}{2},k}\right)^2}{2n} \text{ and } \lambda_{0,k} = -N\frac{\left(j_{\frac{n}{2}-1,k}\right)^2}{2n}$$

and use $\nu = n/2$ in the first case and $\nu = n/2 - 1$ in the second. It is well know that $j_{\nu,k}$ is strictly increasing in both variables –see Elbert (2001). From here we know that $|\lambda_{1,k}| - |\lambda_{0,k}| > 0$ for all n and k. Also in Elbert (2001) we see that $\frac{\partial}{\partial v} j_{v,k} < 0$ for $\nu > -k$ and $k \geq 1/2$. Thus, the difference between $|\lambda_{1,k}| - |\lambda_{0,k}|$ is decreasing in n.

From Qu and Wong (1999) we have the lower and upper bound for the zeros of the Bessel function $J_\nu(\cdot)$:

$$\nu + \nu^{1/3}2^{-1/3}|a_k| \leq j_{\nu,k} \leq \nu + \nu^{1/3}2^{-1/3}|a_k| + \frac{3}{20}|a_k|^2 2^{1/3}\nu^{-1/3}$$

where a_k is the k^{th} zero of the Airy function. Thus, as $n \to \infty$ then $v \to \infty$ and thus both $\lambda_{1,k}$ and $\lambda_{0,k}$ diverge towards $-\infty$. From the same bounds we see that as $n \to \infty$, the difference $\lambda_{0,k} - \lambda_{1,k} \to 1/2$. □

Proof. (of Lemma 1) Using Ito's lemma we have: $dx = (1/2)y^{-1/2}dy - (1/2)(1/4)y^{-3/2}dy^2$ which gives

$$dx = \frac{n-1}{2\,x}dt + dW^a$$

and $w = f(y, z) = z/\sqrt{ny}$. We have:

$$dw = f_y dy + f_z dz + \frac{1}{2}f_{yy}(dy)^2 + \frac{1}{2}f_{zz}(dz)^2 + f_{yz}dydz$$

where $f = (z/\sqrt{n})\, y^{-1/2}$, and thus

$$\begin{aligned}
f_y &= -\frac{z}{2\sqrt{n}}y^{-3/2} \\
f_z &= \frac{1}{\sqrt{n}}y^{-1/2} \\
f_{yy} &= \frac{3z}{4\sqrt{n}}y^{-5/2} \\
f_{zz} &= 0 \\
f_{yz} &= -\frac{1}{2\sqrt{n}}y^{-3/2}
\end{aligned}$$

We thus have:

$$\begin{aligned}
dw &= -\frac{z}{2\sqrt{n}} y^{-3/2} \left(ndt + 2\sqrt{y}\, dW^a\right) \\
&+ \frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \left(\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1 - \left(\frac{z}{\sqrt{ny}}\right)^2}\, dW^b \right) \\
&+ \frac{1}{2} \frac{3z}{4\sqrt{n}} y^{-5/2} 4y dt - \frac{1}{2\sqrt{n}} y^{-3/2} 2z dt
\end{aligned}$$

which we can rearrange as:

$$\begin{aligned}
dw &= \frac{z}{\sqrt{n}} y^{-3/2} \left(\frac{1-n}{2}\right) dt \\
&+ \left(\frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \frac{z}{\sqrt{n}\sqrt{y}} - \frac{z}{2\sqrt{n}} y^{-3/2} 2\sqrt{y} \right) dW^a \\
&+ \frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \sqrt{1 - \left(\frac{z}{\sqrt{n}\sqrt{y}}\right)^2}\, dW^b
\end{aligned}$$

or

$$\begin{aligned}
dw &= \frac{w}{x^2} \left(\frac{1-n}{2}\right) dt + \left(\frac{z}{\sqrt{ny}} - \frac{z}{\sqrt{ny}} \right) dW^a + \frac{1}{x} \sqrt{1 - (w)^2}\, dW^b \\
&= \frac{w}{x^2} \left(\frac{1-n}{2}\right) dt + \frac{1}{x} \sqrt{1 - w^2}\, dW^b
\end{aligned}$$

□

Proof. (of Proposition 3) We try a multiplicative solution of the form:

$$\varphi(w, x) = h(w)\, g(x)$$

To simplify the proof we set $\sigma^2 = 1$. Thus

$$\begin{aligned}
\lambda h(w) g(x) = h(w) g'(x) \left(\frac{n-1}{2x}\right) + h'(w) g(x) \frac{w}{x^2} \left(\frac{1-n}{2}\right) \\
+ \frac{1}{2} h''(w) g(x) \frac{(1-w^2)}{x^2} + \frac{1}{2} h(w) g''(x)
\end{aligned}$$

Dividing by $h(w)$ in both sides we have:

$$\begin{aligned}
\lambda g(x) = g'(x) \left(\frac{n-1}{2x}\right) + \frac{h'(w) w}{h(w)} \frac{g(x)}{x^2} \left(\frac{1-n}{2}\right) \\
+ \frac{1}{2} \frac{h''(w)\,(1-w^2)}{h(w)} \frac{g(x)}{x^2} + \frac{1}{2} g''(x)
\end{aligned}$$

Collecting terms:

$$\lambda g(x) = \frac{g(x)}{x^2}\left[\frac{h'(w)w}{h(w)}\left(\frac{1-n}{2}\right) + \frac{1}{2}\frac{h''(w)\left(1-w^2\right)}{h(w)}\right] + g'(x)\left[\frac{n-1}{2x}\right] + \frac{1}{2}g''(x)$$

Which suggests to try the following separating variable:

$$\mu = \frac{h'(w)w}{h(w)}\left(\frac{1-n}{2}\right) + \frac{1}{2}\frac{h''(w)\left(1-w^2\right)}{h(w)}$$

or

$$0 = -2\mu h(w) + h'(w)w\left(1-n\right) + h''(w)\left(1-w^2\right)$$

The solution of this equation is given by the Gegenbauer polynomials $C_m^{\alpha}(w)$. The Gegenbauer polynomials are the solution to the following o.d.e.:

$$(1-w^2)h(w)'' - (2\alpha+1)wh'(w) + m(m+2\alpha)h(w) = 0 \text{ for } w \in [-1,1]$$

for integer $m \geq 0$. Matching coefficients we have:[6]

$$-2\mu = m(m+2\alpha) \text{ and } -(2\alpha+1) = (1-n)$$

which gives

$$\alpha = \frac{n}{2} - 1 \text{ and } \mu = -\frac{m}{2}(m+n-2)$$

Then given $\mu = -(m/2)(m+n-2)$ the o.d.e. for g is:

$$\lambda g(x) = \frac{g(x)}{x^2}\mu + g'(x)\left[\frac{n-1}{2x}\right] + \frac{1}{2}g''(x)$$

or

$$0 = g(x)\left(\mu - x^2\lambda\right) + g'(x)\left[\frac{n-1}{2}\right]x + \frac{1}{2}g''(x)x^2$$

or

$$0 = g(x)\left(2\mu - x^2 2\lambda\right) + g'(x)x\left(n-1\right) + g''(x)x^2$$

with boundary condition $g(\bar{x}) = 0$. The solution of this o.d.e., which does not explode at $x = 0$ is given by a Bessel function of the first kind. This is because the following o.d.e.:

$$g(x)(c+bx^2) + g'(x)x\,a + g''(x)x^2 = 0$$

[6] See https://en.wikipedia.org/wiki/Gegenbauer_polynomials, which is bases on Abramowitz, Milton; Stegun, Irene Ann, eds. (1983) [June 1964], Chapter 22, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematics Series. 55, Dover Publications.

has solution:[7]

$$g(x) = x^{(1-a)/2} J_\nu\left(\sqrt{b}\,x\right) \text{ where } \nu = \frac{1}{2}\sqrt{(1-a)^2 - 4c}$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind. Matching coefficients we have:

$$\begin{aligned}
&a = n-1,\ b = -2\lambda,\ c = 2\mu \text{ and} \\
&\nu = \frac{1}{2}\sqrt{(n-2)^2 - 8\mu} = \frac{1}{2}\sqrt{(n-2)^2 + 8(m/2)(m+n-2)} = \frac{n}{2} - 1 + m
\end{aligned}$$

We argue that $\nu = n/2 - 1 + m$ to see that note we have

$$\begin{aligned}
4\nu^2 &= (n-2)^2 + 4m(m+n-2) \text{ and} \\
4\nu^2 &= 4\left(\frac{n-2+2m}{2}\right)^2 = (n-2)^2 + 4m(n-2) + 4m^2
\end{aligned}$$

which verifies the equality. So we have:

$$g(x) = x^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\,x\right)$$

We still have to determine the eigenvalue λ. For this we use the boundary condition $g(\bar{x}) = 0$ and that $J_\nu(\cdot)$ has infinitely strictly orderer positive zeros, denoted by $j_{\nu,k}$ for $k = 1, 2, \ldots$ so that $J_\nu(j_{\nu,k}) = 0$. Thus fixing μ, i.e. $m \geq 0$, we have:

$$0 = g(\bar{x}) = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\,\bar{x}\right)$$

so that:

$$0 = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\,\lambda_{m,k}}\,\bar{x}\right)$$

Hence

$$j_{\frac{n}{2}-1+m,k} = \sqrt{-2\,\lambda_{m,k}}\,\bar{x} \quad \text{or} \quad \lambda_{m,k} = -\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Collecting the terms for h, g and λ we obtain the desired result.

Since $\sigma^2 \neq 1$ changes the units of time, we need only to adjust the eigenvalue by its value, so that

$$\lambda_{m,k} = -\sigma^2 \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Using that $N = n\sigma^2/\bar{x}^2$ we get

$$\lambda_{m,k} = -\frac{n\sigma^2}{\bar{x}^2}\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n} = N\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n}$$

□

[7] See http://eqworld.ipmnet.ru/en/solutions/ode/ode0215.pdf which uses Polyanin, A. D. and Zaitsev, V. F., Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition, Chapman & Hall/CRC, Boca Raton, 2003.

Proof. (of Proposition 4) We start with the projections for $z/n = f(w,x) = wx/\sqrt{n}$. We are looking for:

$$
\begin{aligned}
f(x,w) = wx/\sqrt{n} \sim \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w) &= \sum_{k=1}^{\infty} b_{1,k}[f]h_1(w)g_{1,k}(x) \\
&= \sum_{k=1}^{\infty} b_{1,k}[f]\, C_1^{\frac{n}{2}-1(w)}\, J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\, x^{1-\frac{n}{2}} \\
&= w\, x^{1-\frac{n}{2}}\,(n-2)\sum_{k=1}^{\infty} b_{1,k}[f] J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)
\end{aligned}
$$

We can replace the expression we obtain below for $b_{1,k}[f]$ to get:

$$
\begin{aligned}
\sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w) &= w\, x^{1-\frac{n}{2}}\,(n-2)\sum_{k=1}^{\infty}\frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) \\
&= \frac{w\,x}{\sqrt{n}}\sum_{k=1}^{\infty}\frac{2\,(x/\bar{x})^{-\frac{n}{2}}\, J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)}{j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}
\end{aligned}
$$

To get the coefficients we start by computing

$$
\begin{aligned}
&\int_0^{\bar{x}} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}+1}dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2}\int_0^{\bar{x}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) j_{\frac{n}{2},k}\frac{dx}{\bar{x}} \\
&= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2}\int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1} J_{\frac{n}{2}}(z)\, dz
\end{aligned}
$$

Using that

$$
\int_a^b z^{\nu+1}J_\nu(z)dz = z^{\nu+1}J_{\nu+1}(z)\Big|_a^b
$$

then

$$
\begin{aligned}
&\int_0^{\bar{x}} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}+1}dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2}\int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1} J_{\frac{n}{2}}(z)\, dz \\
&= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2}\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)
\end{aligned}
$$

Using that

$$
\bar{x}^2\int_0^{\bar{x}}\frac{x}{\bar{x}}\left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2\frac{dx}{\bar{x}} = \frac{1}{2}\left(\bar{x}\,J_{\nu+1}\left(j_{\nu,k}\right)\right)^2 \text{ for all } k\in\{1,2,3\ldots\}
$$

we have

$$\int_0^{\bar{x}} \left(J_{\frac{n}{2}} \left(j_{\frac{n}{2},k} \, \frac{x}{\bar{x}} \right) \right)^2 x \, dx = \frac{1}{2} \left(\, \bar{x} \, J_{\frac{n}{2}+1} \left(j_{\frac{n}{2},k} \right) \, \right)^2$$

Thus:

$$\begin{aligned} b_{1,k}[f] &= \frac{2}{\sqrt{n}(n-2)} \frac{\left(\frac{\bar{x}}{j_{\frac{n}{2},k}} \right)^{\frac{n}{2}+2} \left(j_{\frac{n}{2},k} \right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1} \left(j_{\frac{n}{2},k} \right)}{\left(\, \bar{x} \, J_{\frac{n}{2}+1} \left(j_{\frac{n}{2},k} \right) \, \right)^2} \\ &= \frac{2 \, \bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2) \, j_{\frac{n}{2},k} \, J_{\frac{n}{2}+1} \left(j_{\frac{n}{2},k} \right)} \text{ for all } k \geq 1 \end{aligned}$$

Now we turn to compute: $b_{1,k}[\bar{p}'(\cdot, 0)]\langle \varphi 1, k, \varphi 1, k \rangle$. We start deriving an explicit expression for $\bar{p}'(\cdot, 0)$. We have

$$\begin{aligned} \bar{h}(w) &= \frac{1}{Beta \left(\frac{n-1}{2}, \frac{1}{2} \right)} \ \left(1 - w^2 \right)^{(n-3)/2} \quad \text{for} \ \ w \in (-1 \, , \, 1) \\ \bar{g}(x) &= x \ (\bar{x})^{-n} \left(\frac{2n}{n-2} \right) \left[\, \bar{x}^{n-2} - x^{n-2} \right] \qquad \text{for } x \in [0, \bar{x}] \end{aligned}$$

$$\begin{aligned} & p(w, x; 0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w, x; 0)\delta + o(\delta) \text{ with} \\ & \bar{p}'(w, x; 0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \end{aligned}$$

where:

$$\begin{aligned} & \frac{\partial}{\partial \delta} x(\delta)|_{\delta=0} = x'(0) = \sqrt{n} \, w \quad \text{and} \quad \frac{\partial}{\partial \delta} \bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w)w'(0) \\ & \frac{\partial}{\partial \delta} w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n} \, (1 - w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial \delta} \bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x)x'(0) \end{aligned}$$

so:

$$\begin{aligned} \bar{p}'(w, z; 0) &= \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \\ &= - \, (\bar{x})^{-n} \left(\frac{2n}{n-2} \right) \left[\, \bar{x}^{n-2} - x^{n-2} \right] \frac{(n-3)w \left(1 - w^2 \right)^{(n-3)/2}}{Beta \left(\frac{n-1}{2}, \frac{1}{2} \right)} \sqrt{n} \\ &+ \frac{w \left(1 - w^2 \right)^{(n-3)/2}}{Beta \left(\frac{n-1}{2}, \frac{1}{2} \right)} \ \bar{x}^{-n} \left[\left(\frac{2n}{n-2} \right) \left[\, \bar{x}^{n-2} - x^{n-2} \right] - 2n x^{n-2} \right] \sqrt{n} \\ &= \frac{w \left(1 - w^2 \right)^{(n-3)/2}}{Beta \left(\frac{n-1}{2}, \frac{1}{2} \right)} \frac{\sqrt{n}}{\bar{x}^n} \left(\frac{2n}{n-2} \right) \left[(4-n) \left(\bar{x}^{n-2} - x^{n-2} \right) - 2n x^{n-2} \right] \\ &= \frac{w \left(1 - w^2 \right)^{(n-3)/2}}{Beta \left(\frac{n-1}{2}, \frac{1}{2} \right)} \sqrt{n} \left(\frac{2n}{n-2} \right) \frac{\left[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2} \right]}{\bar{x}^n} \end{aligned}$$

We want to compute:

$$b_{1,k}[\bar{p}'(\cdot,0)/\omega]\langle\varphi_{1,k},\varphi_{1,k}\rangle = \int_0^{\bar{x}}\int_{-1}^{1}\bar{p}'(x,w;0)h_{1,k}(x)g_{m,k}(w)dwdx$$

So we split the integral in the product of two terms. The first term involves the integral over w given by:

$$\begin{aligned}
&\int_{-1}^{1}(n-2)w\,\frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\,dw\\
&=\frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\int_{-1}^{1}w^2\left(1-w^2\right)^{(n-3)/2}\,dw=\frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{2\,\Gamma\left(\frac{n}{2}+1\right)}\\
&=\frac{n\sqrt{n}\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{\Gamma\left(\frac{n}{2}+1\right)}=n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}=n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}
\end{aligned}$$

where we use that $C_1^{\frac{n}{2}-1}(w)=(n-2)w$, and properties of the $Beta$ and Γ functions.

The second term involves the integral over x and is given by:

$$\begin{aligned}
&\frac{1}{\bar{x}^n}\int_0^{\bar{x}}\left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right]J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\,x^{1-\frac{n}{2}}\,dx\\
&=\frac{\bar{x}^{1-\frac{n}{2}}\bar{x}^{n-2}}{\bar{x}^n}\int_0^{\bar{x}}\left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right]J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\,\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\,dx\\
&=\bar{x}^{-\frac{n}{2}}\int_0^{\bar{x}}\left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right]J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\,\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{dx}{\bar{x}}\\
&=\frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{\bar{x}}J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\,\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{j_{\frac{n}{2},k}dx}{\bar{x}}\\
&-\frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{\bar{x}}\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{n-2}\,J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\,\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{j_{\frac{n}{2},k}dx}{\bar{x}}\\
&=\frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{j_{\frac{n}{2},k}}z^{1-\frac{n}{2}}J_{\frac{n}{2}}\left(z\right)\,dz\\
&-\frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{j_{\frac{n}{2},k}}z^{\frac{n}{2}-1}\,J_{\frac{n}{2}}\left(z\right)\,dz
\end{aligned}$$

To find an expression for this integrals note that:

$$\int_0^a z^{1-\frac{n}{2}}J_{\frac{n}{2}}\left(z\right)\,dz=-\frac{2^{1-n/2}\left(-1+{}_0F_1\left(n/2,-a^2/4\right)\right)}{\Gamma(n/2)}=\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}-a^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(a)$$

and

$$\int_0^a z^{\frac{n}{2}-1}J_{\frac{n}{2}}\left(z\right)\,dz=2^{-1-\frac{n}{2}}a^n\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{a^2}{4}\right)$$

where ${}_1\tilde{F}_2(a_1; b_1, b_2; z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1; b_1, b_2; z) = {}_1F_2(a_1; b_1, b_2; z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function. Thus

$$\frac{1}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}}\, dx$$
$$= \bar{x}^{-\frac{n}{2}}\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right.$$
$$\left.-\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}} 2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^n \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]$$

Thus we have:

$$b_{1,k}[f]b_{1,k}[\bar{p}'(\cdot,0)]\langle\varphi_{1,k},\varphi_{1,k}\rangle$$
$$= n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}$$
$$\bar{x}^{-\frac{n}{2}}\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right.$$
$$\left.-\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}} 2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^n \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2n}{(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}$$
$$\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right.$$
$$\left.-\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}} 2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^n \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2n}{(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}$$
$$\left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} - \frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right.$$
$$\left.-(4+n)2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]$$

□

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