Solution Algorithm

Monetary Policy, Redistribution, and Risk Premia

This document describes the numerical solution method used in the paper. We first outline the model's equilibrium conditions and then the computational algorithm.

1 Equilibrium conditions

We first list the equilibrium conditions of the infinite horizon environment outlined in appendix C of the paper. We denote real variables in lower-case (except for the nominal rate i_t) and further define the re-scaled variables

$$\tilde{c}_t^i \equiv \frac{c_t^i}{z_t}, \tilde{v}_t^i \equiv \frac{v_t^i}{z_t}, \tilde{ce}_t^i \equiv \frac{ce_t^i}{z_t}, \tilde{b}_t^i \equiv \frac{b_t^i + \nu^i b^g}{z_t}, \tilde{k}_t^i \equiv \frac{k_t^i}{z_t}, \tilde{q}_t^{\bar{\ell},i} \equiv \frac{\tilde{q}_t^{\bar{\ell},i}}{z_t}, \tilde{k}_t \equiv \frac{k_t}{z_t}, \tilde{w}_t \equiv \frac{w_t}{z_t}. \quad (1)$$

We note that $\tilde{b}_t^i$ is the productivity-adjusted bond position of representative household i accounting for the implicit position that household also has through the government.

The optimality conditions for the representative household $i \in \{a, b, c\}$ are:

$$1 = \mathbb{E}_t m_{t,t+1}^i (1 + r_{t+1}), \quad (2)$$

$$1 - \kappa_t = \mathbb{E}_t m_{t,t+1}^i (1 + r_{t+1}^k), \quad (3)$$

$$\kappa_t^i \left(\tilde{k}_t^i - \underline{k} \right) = 0, \;\; \tilde{k}_t^i \geq \underline{k}, \;\; \kappa_t^i \geq 0, \quad (4)$$

given the stochastic discount factor and certainty equivalent

$$m_{t,t+1}^i = \beta \exp\left(-\gamma^i \left[\epsilon_{t+1}^z + \varphi_{t+1}\right]\right) \times$$
$$\left(\mu_{t,t+1}^i\right)^{-\gamma^i} \left(\tilde{ce}_t^i\right)^{\gamma^i - 1/\psi} \left(\tilde{v}_{t+1}^i\right)^{1/\psi - \gamma^i} \frac{\left(\tilde{c}_{t+1}^i\right)^{-1/\psi} \Phi(\ell_{t+1}^i)^{1-1/\psi}}{\left(\tilde{c}_t^i\right)^{-1/\psi} \Phi(\ell_t^i)^{1-1/\psi}}, \quad (5)$$

$$\tilde{ce}_t^i = \mathbb{E}_t \left[\exp\left(\left(1 - \gamma^i\right) \left[\epsilon_{t+1}^z + \varphi_{t+1}\right]\right) \left(\mu_{t,t+1}^i\right)^{1-\gamma^i} \left(\tilde{v}_{t+1}^i\right)^{1-\gamma^i} \right]^{\frac{1}{1-\gamma^i}}, \quad (6)$$

where $\mu_{t,t+1}^i$ is the growth rate of incumbents' wealth, equal to the relative wealth of incumbents relative to the average group member. The value function of the representative household is

$$\tilde{v}_t^i = \left((1-\beta) \left(\tilde{c}_t^i \Phi \left(\ell_t^i \right) \right)^{1-1/\psi} + \beta \left(\tilde{ce}_t^i \right)^{1-1/\psi} \right)^{\frac{1}{1-1/\psi}}. \tag{7}$$

Its resource constraint and financial wealth inclusive of taxes and transfers, accounting for government budget balance, are

$$\tilde{c}_t^i + \tilde{b}_t^i + q_t \tilde{k}_t^i = \tilde{w}_t \ell_t^i + \frac{1}{\lambda^i} s_t^i \left(\pi_t + (1-\delta) q_t \right) \frac{\tilde{k}_{t-1}}{\exp(\epsilon_t^z)}. \tag{8}$$

The representative union's optimality condition is:

$$\begin{aligned}
&\sum_i \lambda^i \left(\tilde{v}_t^i \right)^{\frac{1}{\psi}} \left(\tilde{c}_t^i \right)^{-\frac{1}{\psi}} \Phi(\ell_t^i)^{1-\frac{1}{\psi}} \phi^i \Bigg[\tilde{w}_t + \tilde{c}_t^i \frac{\Phi'(\ell_t^i)}{\Phi(\ell_t^i)} \\
&+ \tilde{w}_t \frac{1}{\phi^i} \frac{\chi^W}{\epsilon} \Bigg[\frac{\tilde{w}_t}{\tilde{w}_{t-1}/\exp(\epsilon_t^z)} \frac{P_t}{P_{t-1}} \left(\frac{\tilde{w}_t}{\tilde{w}_{t-1}/\exp(\epsilon_t^z)} \frac{P_t}{P_{t-1}} - 1 \right) \\
&- \mathbb{E}_t m_{t,t+1}^i \exp\left(\epsilon_{t+1}^z + \varphi_{t+1} \right) \frac{\tilde{w}_{t+1}\ell_{t+1}}{\tilde{w}_t \ell_t} \frac{\tilde{w}_{t+1}}{\tilde{w}_t/\exp(\epsilon_{t+1}^z)} \frac{P_{t+1}}{P_t} \left(\frac{\tilde{w}_{t+1}}{\tilde{w}_t/\exp(\epsilon_{t+1}^z)} \frac{P_{t+1}}{P_t} - 1 \right) \Bigg] \Bigg] \\
&= 0,
\end{aligned} \tag{9}$$

and the allocation of labor is

$$\ell_t^i = \phi^i \ell_t. \tag{10}$$

The representative producer's optimality condition and flow of funds are:

$$\tilde{w}_t = (1-\alpha) \ell_t^{-\alpha} \left(\frac{\tilde{k}_{t-1}}{\exp(\epsilon_t^z)} \right)^{\alpha}, \tag{11}$$

$$q_t = \left(\frac{\tilde{k}_t}{\tilde{k}_{t-1}/\exp(\epsilon_t^z)} \right)^{\chi^x}, \tag{12}$$

$$\pi_t = \alpha \ell_t^{1-\alpha} \left(\frac{\tilde{k}_{t-1}}{\exp(\epsilon_t^z)} \right)^{\alpha-1}. \tag{13}$$

The definitions of real returns are:

$$1 + r_{t+1} \equiv (1 + i_t)\frac{P_t}{P_{t+1}}, \tag{14}$$

$$1 + r^k_{t+1} \equiv \frac{(\pi_{t+1} + (1-\delta)q_{t+1})\exp(\varphi_{t+1})}{q_t}. \tag{15}$$

The specifications of fiscal and monetary policy imply:

$$s^i_{t+1} = \lambda^i(1-\xi)\frac{(1+r_{t+1})\tilde{b}^i_t + (1+r^k_{t+1})q_t\tilde{k}^i_t}{(1+r^k_{t+1})q_t\tilde{k}_t} + \bar{s}^i\xi, \tag{16}$$

$$1 + i_t = (1+\bar{i})\left(\frac{P_t}{P_{t-1}}\right)^{\phi} m_t. \tag{17}$$

The relative wealth of incumbent members to the average member of each group (inclusive of time endowment) is:

$$\mu^i_{t,t+1} = \frac{(1+r_{t+1})\tilde{b}^i_t + (1+r^k_{t+1})q_t\tilde{k}^i_t + \tilde{q}^{\bar{\ell},i}_{t+1}\exp(\epsilon^z_{t+1} + \varphi_{t+1})}{\frac{1}{\lambda^i}s^i_{t+1}(1+r^k_{t+1})q_t\tilde{k}_t + \tilde{q}^{\bar{\ell},i}_{t+1}\exp(\epsilon^z_{t+1} + \varphi_{t+1})}. \tag{18}$$

The equilibrium prices of time endowments satisfy

$$\tilde{q}^{\bar{\ell},i}_t = \tilde{w}_t\ell^i_t + \mathbb{E}_t m^i_{t,t+1}\exp\left(\epsilon^z_{t+1} + \varphi_{t+1}\right)\tilde{q}^{\bar{\ell},i}_{t+1}. \tag{19}$$

The market clearing conditions are:

$$\sum_i \lambda^i\tilde{c}^i_t + \left(\frac{\tilde{k}_t}{\tilde{k}_{t-1}/\exp(\epsilon^z_t)}\right)^{\chi^x}\tilde{x}_t = \ell_t^{1-\alpha}\left(\frac{\tilde{k}_{t-1}}{\exp(\epsilon^z_t)}\right)^{\alpha}, \tag{20}$$

$$\sum_i \lambda^i\tilde{k}^i_t = \tilde{k}_t, \tag{21}$$

$$(1-\delta)\frac{\tilde{k}_{t-1}}{\exp(\epsilon^z_t)} + \tilde{x}_t = \sum_i \lambda^i\tilde{k}^i_t, \tag{22}$$

$$\sum_i \lambda^i\tilde{b}^i_t = 0. \tag{23}$$

Finally, the evolution of exogenous state variables is:

$$\log p_{t+1} - \log p = \rho^p \left(\log p_t - \log p\right) + \epsilon^p_{t+1}, \tag{24}$$

$$\log m_{t+1} = \rho^m \log m_t + \epsilon^m_{t+1}. \tag{25}$$

As is evident, this economy features six state variables:

$$S_t \equiv \left\{ \frac{\tilde{k}_{t-1}}{\exp(\epsilon^z_t)}, \frac{\tilde{w}_{t-1}}{\exp(\epsilon^z_t)}, s^a_t, s^c_t, p_t, m_t \right\}.$$

Any two of $\{s^a_t, s^b_t, s^c_t\}$ can be used as state variables (since they add to one); we use s^a_t and s^c_t in our code. After solving this transformed economy, we can simulate prices and quantities in the original economy by reversing the re-scaling in (1).

2 Computational algorithm

The model is solved over a sparse six-dimensional Smolyak grid given the above state variables.

Expectations over future values are formed as weighted sums over Gauss-Hermite quadrature nodes for the three normally distributed shocks ϵ^z_{t+1}, ϵ^m_{t+1} and ϵ^p_{t+1}, plus one additional node for the disaster shock φ_{t+1}. On a given grid point, each node is associated with a transition to a new state in the next period. Values of relevant variables at $t+1$ in that state are calculated using Chebyshev interpolation, as this state will generically lie off the grid.

It will be convenient for numerical purposes to define the following variables:

$$\hat{v}^i_{t+1} \equiv \tilde{v}^i_{t+1} \frac{\bar{\kappa}^i}{\frac{1}{\lambda^i} s^i_{t+1}(\pi_{t+1} + (1-\delta)q_{t+1})\tilde{k}_t / \exp(\epsilon^z_{t+1}) + \tilde{q}^{\bar{\ell},i}_{t+1}},$$

$$\hat{c}^i_{t+1} \equiv \tilde{c}^i_{t+1} \frac{\bar{\kappa}^i}{\frac{1}{\lambda^i} s^i_{t+1}(\pi_{t+1} + (1-\delta)q_{t+1})\tilde{k}_t / \exp(\epsilon^z_{t+1}) + \tilde{q}^{\bar{\ell},i}_{t+1}},$$

$$\hat{\mu}^i_{t,t+1} \equiv \frac{1}{\bar{\kappa}^i} \Big((1 + r_{t+1})\tilde{b}^i_t + (\pi_{t+1} + (1-\delta)q_{t+1})\tilde{k}^i_t \exp(\varphi_{t+1}) + \tilde{q}^{\bar{\ell},i}_{t+1} \exp(\epsilon^z_{t+1} + \varphi_{t+1}) \Big),$$

where $\bar{\kappa}^i$ is a normalizing constant roughly equal to households' wealth inclusive of the time endowment claim in deterministic steady state. Then note by (5), (6), (15), and (18) that the pricing kernel and certainty equivalent can be written

$$m^i_{t,t+1} = \beta \left(\hat{\mu}^i_{t,t+1}\right)^{-\gamma^i} \left(\tilde{ce}^i_t\right)^{\gamma^i - 1/\psi} \left(\hat{v}^i_{t+1}\right)^{1/\psi - \gamma^i} \frac{\left(\hat{c}^i_{t+1}\right)^{-1/\psi} \Phi(\ell^i_{t+1})^{1-1/\psi}}{\left(\tilde{c}^i_t\right)^{-1/\psi} \Phi(\ell^i_t)^{1-1/\psi}},$$

$$\tilde{ce}^i_t = \mathbb{E}_t \left[\left(\hat{\mu}^i_{t,t+1} \hat{v}^i_{t+1}\right)^{1-\gamma^i}\right]^{\frac{1}{1-\gamma^i}}.$$

The algorithm starts from an initial guess for value functions $\hat{v}^i$, policies $\{\tilde{c}^i, \ell, \tilde{k}, \left(\hat{c}^i\right)^{-1/\psi} \Phi(\ell^i)^{1-1/\psi}\}$ and prices $\{q, \tilde{q}^{\bar{\ell},i}, P/P_{-1}\}$ on each grid point, as well as state transition rules for each state and quadrature node. The model is solved recursively taking as given values at $t+1$, while solving for policies and market clearing prices at t. In each time step, the solution algorithm (`calc_equilibrium_and_update` in `mod_calc.f90`) performs the following steps for each grid point:

1. Given the current guess on aggregate labor supply ℓ_t and capital holdings $\tilde{k}_t$, calculate the real wage $\tilde{w}_t$, the price of capital q_t, and profits π_t based on (11)-(13). Given assumed prices and policies on the grid next period, and given assumed state transitions for each quadrature node, calculate next period's decision-relevant variables such as inflation P_{t+1}/P_t and the return to capital (15) using Chebyshev interpolation.

2. Taking as given households' consumption $\tilde{c}^i_t$ as well as next period's inflation and the return to capital, solve for the nominal rate i_t that clears the bond market (23) given the Fisher equation (14) and the solution to each household's portfolio choice problem (2)-(4). For an unconstrained household this requires finding a solution to

$$\mathbb{E}\left[\left(\hat{\mu}^i_{t,t+1}\right)^{-\gamma^i} \left(\hat{v}^i_{t+1}\right)^{1/\psi - \gamma^i} \left(\hat{c}^i_{t+1}\right)^{-1/\psi} \Phi(\ell^i_{t+1})^{1-1/\psi} \left(r^k_{t+1} - r_{t+1}\right)\right] = 0.$$

   If the solution violates household i's portfolio constraint (4), use the constraint directly to find the implied portfolio choice.

3. Calculate the marginal utilities $\left(\hat{c}^i_t\right)^{-1/\psi} \Phi(\ell^i_t)^{1-1/\psi}$ using (10), value functions $\hat{v}^i_t$ using (7), the transition rules for the wealth distribution using (16), and aggregate capital holdings $\tilde{k}_t$ using (21). The latter two also use the assumed $\tilde{c}^i_t$ and portfolio policies calculated in the prior step.

4. Holding fixed households' portfolio choice, solve for households' consumption-savings choice based on (2)-(4). Subject to the budget constraint (8), the consumption choice for an unconstrained household solves

$$\tilde{c}_t^i = \mathbb{E}\left[\beta\left(\hat{\mu}_{t+1}^i\right)^{-\gamma^i}\left(\tilde{c}e_t^i\right)^{\gamma^i-1/\psi}\left(\hat{v}_{t+1}^i\right)^{1/\psi-\gamma^i}\frac{\left(\hat{c}_{t+1}^i\right)^{-1/\psi}\Phi(\ell_{t+1}^i)^{1-1/\psi}}{\Phi(\ell_t^i)^{1-1/\psi}}r_{t+1}^i\right]^{-\psi},$$

   where r_{t+1}^i is the weighted portfolio return of household i. For a constrained household, r_{t+1} replaces r_{t+1}^i.

5. Calculate households' pricing kernel $m_{t,t+1}^i$ as defined above.

6. Calculate the labor endowment prices $\tilde{q}_t^{\bar{\ell},i}$ using (19).

7. Calculate the union's optimal labor supply ℓ_t using (9).

8. Given the nominal interest rate i_t obtained in step 2, calculate the implied inflation rate P_t/P_{t-1} using the monetary policy rule (17).

After these steps, we have obtained new values for the value functions $\hat{v}^i$, policies $\{\tilde{c}^i, \ell, \tilde{k}, (\hat{c}^i)^{-1/\psi}\,\Phi(\ell^i)^{1-1/\psi}\}$ and prices $\{q, \tilde{q}^{\bar{\ell},i}, P/P_{-1}\}$ on each grid point, as well as state transitions for each state and quadrature node. The assumed values, policies, prices, and transitions are updated using these solutions, dampened for stability. This procedure is repeated until the difference between assumed values and updates is sufficiently small. After convergence is obtained, investment at each grid point is implied by (22), and goods market clearing (20) is then implied by Walras' Law.

To calculate marginal propensities to consume and to take risk, we solve the households' optimization problems in each state over a range of financial wealth around the wealth level as given by the current state. The solution to this exercise are policy functions as a function of individual wealth, holding fixed the aggregate state of the economy including the aggregate wealth distribution. We then use numerical differentiation to derive the marginal changes in consumption and portfolio policies in response to changes in wealth. These policy functions are also used to decompose the effects of monetary policy shock on capital accumulation, since they allow us to determine the policy changes in response to the shock's wealth effects, holding fixed the aggregate state of the economy, as well as the policy changes in response to a change in the aggregate state, holding fixed households' wealth (see `mod_decomp.f90`).