

Supplement to “Identification of time and risk preferences in buy price auctions”: Appendix

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This supplementary appendix contains proofs for some results in the paper and additional discussion of identification of utility functions from certainty equivalents.

APPENDIX SA: PROOFS OF SELECTED RESULTS FROM THE MAIN TEXT

Proof of Proposition 2

Consider the term

$$e^{-\gamma}U(v-r) + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} F_V^n(v) E_n[U(v - \max\{r, Y\}) | Y \leq v]. \quad (\text{SA.1})$$

Note that

$$\begin{aligned} & F_V^n(v) E_n[U(v - \max(r, Y)) | Y \leq v] \\ &= \int_0^v U(v - \max\{r, y\}) n F_V^{n-1}(y) f_V(y) dy \\ &= \int_0^r U(v-r) n F_V^{n-1}(y) f_V(y) dy + \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\ &= U(v-r) F_V^n(r) + \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy. \end{aligned}$$

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So we can write (SA.1) as

$$\begin{aligned}
& e^{-\gamma}U(v-r) + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} U(v-r) F_V^n(r) + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\
&= e^{-\gamma}U(v-r) \left[1 + \sum_{n=1}^{\infty} \frac{\gamma^n F_V^n(r)}{n!} \right] + \sum_{n=1}^{\infty} \frac{\gamma^n e^{-\gamma}}{n!} \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\
&= e^{-\gamma}U(v-r) \exp(\gamma F_V(r)) + e^{-\gamma} \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int_r^v U(v-y) n F_V^{n-1}(y) f_V(y) dy \\
&= U(v-r) \exp[\gamma F_V(r) - \gamma] + e^{-\gamma} \int_r^v U(v-y) f_V(y) \left[\sum_{n=1}^{\infty} \frac{\gamma^n n F_V^{n-1}(y)}{n!} \right] dy,
\end{aligned}$$

where the last equality follows from the dominated convergence theorem. We also have

$$\sum_{n=1}^{\infty} \frac{\gamma^n n F_V^{n-1}(y)}{n!} = \sum_{n=1}^{\infty} \frac{\gamma \gamma^{n-1} F_V^{n-1}(y)}{(n-1)!} = \gamma \exp(\gamma F_V(y)),$$

so

$$U^R(v, t) = \delta(T-t) \left\{ U(v-r) \exp[\gamma F(r) - \gamma] + \int_r^v U(v-y) \exp(\gamma F_V(y) - \gamma) \gamma f_V(y) dy \right\}.$$

The other parts of the proposition are straightforward to verify.

Proof of Proposition 4

We show properties of the inverse cutoff function defined by

$$\begin{aligned}
p(c, r, \tau, t) &= c - U^{-1} \left(\delta(\tau) \left(\alpha(r, \tau, t) U(c-r) + \int_r^c U(c-y) h(y, \tau, t) dy \right) \right) \\
&= c - M(c, r, \tau, t)
\end{aligned}$$

over the support $r \in [0, \infty)$, $c \in [r, \infty)$, $\tau \in (0, \infty)$, and $t \in (0, \infty)$.

We start by deriving some useful properties of $U^{-1\prime}(x)$ and $U^{-1\prime\prime}(x)$ given Assumption 1. Starting with the identity

$$z = U^{-1}(U(z)),$$

differentiate w.r.t. z to get

$$1 = U^{-1\prime}(U(z)) U'(z).$$

Evaluating this expression at $z = U^{-1}(x)$, we obtain

$$U^{-1\prime}(x) = \frac{1}{U'(U^{-1}(x))} = (U'(U^{-1}(x)))^{-1}.$$

Differentiating this results in

$$\begin{aligned}
 U^{-1''}(x) &= -(U'(U^{-1}(x)))^{-2}U''(U^{-1}(x))U^{-1'}(x) \\
 &= -(U'(U^{-1}(x)))^{-2}U''(U^{-1}(x))(U'(U^{-1}(x)))^{-1} \\
 &= -(U'(U^{-1}(x)))^{-3}U''(U^{-1}(x)).
 \end{aligned}$$

Given our assumptions on $U(x)$, these results imply the following statements:

- (i) We have $U^{-1'}(0) = 1$.
- (ii) We have $U^{-1''}(0) = -U''(0)$.
- (iii) We have that $U^{-1'}(\cdot) > 0$ and is bounded away from 0 and ∞ .
- (iv) We have that $U^{-1''}(\cdot) \geq 0$ and is bounded away from ∞ .

With these results in hand, consider the statements in the proposition one by one. First, $p_c(c, r, \tau, t) > 0$ because, by Assumption 3, the derivative of M w.r.t. its first argument is strictly less than 1.

Second, $p_c(c, r, \tau, t) < 1$, since

$$\begin{aligned}
 p_c(c, r, \tau, t) &= 1 - U^{-1'}\left(\delta(\tau)\left(\alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t) dy\right)\right) \\
 &\quad \cdot \delta(\tau)\left(\alpha(r, \tau, t)U'(c-r) + \int_r^c U'(c-y)h(y, \tau, t) dy\right),
 \end{aligned}$$

and because under our assumptions, $U^{-1'}(\cdot) > 0$, $\delta(\cdot) > 0$, $U'(\cdot) > 0$, $\alpha(y, \tau, t) > 0$, and $h(y, \tau, t) > 0$ for $y > r$.

Third, $p_r(c, r, \tau, t) > 0$, since

$$\begin{aligned}
 p_r(c, r, \tau, t) &= -U^{-1'}\left(\delta(\tau)\left(\alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t) dy\right)\right) \\
 &\quad \cdot \delta(\tau)\left(\frac{\partial\alpha(r, \tau, t)}{\partial r}U(c-r) - \alpha(r, \tau, t)U'(c-r) + U(c-r)h(r, \tau, t)\right) \\
 &= U^{-1'}\left(\delta(\tau)\left(\alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t) dy\right)\right) \\
 &\quad \cdot \delta(\tau)\alpha(r, \tau, t)U'(c-r).
 \end{aligned}$$

The second line follows since $\frac{\partial\alpha(r, \tau, t)}{\partial r} = h(r, \tau, t)$, and the term is strictly positive since under our assumptions, $U^{-1'}(\cdot) > 0$, $\delta(\cdot) > 0$, $U'(\cdot) > 0$, and $\alpha(y, \tau, t) > 0$.

Fourth, $p_\tau(c, r, \tau, t) \geq 0$, since

$$p_\tau(c, r, \tau, t) = -U^{-1'} \left(\delta(\tau) \left(\alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t) dy \right) \right) \\ \cdot \left[\begin{array}{l} \delta'(\tau) \left(\alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t) dy \right) \\ + \delta(\tau) \frac{\partial \left(\alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t) dy \right)}{\partial \tau} \end{array} \right].$$

The first term in the square brackets is weakly negative since Assumption 1 implies $\delta'(\cdot) < 0$, $\alpha(\cdot, \cdot, \cdot) > 0$, $h(\cdot, \cdot, \cdot) > 0$, and $U(\cdot) \geq 0$. The second term in the square brackets is weakly negative since $\delta(\tau) > 0$ and the derivative of the expected utility from rejecting the BP w.r.t. τ is weakly negative (since the distribution of the highest competitor valuation is stochastically increasing in the length of the bidding phase τ ; this derivative is zero when $c = r$). Since $U^{-1'}(x) > 0$, this implies $p_\tau(c, r, \tau, t) \geq 0$.

Fifth, $p(c, r, \tau, t) = r$ iff $c = r$, because

$$p(c, c, \tau, t) = c - U^{-1} \left(\delta(\tau) \left(\alpha(c, \tau, t)U(c-c) + \int_c^c U(c-y)h(y, \tau, t) dy \right) \right) \\ = c - U^{-1}(0) \\ = c = r.$$

The “only if” follows because $p_c(c, r, \tau, t) > 0$ and because $p(c, r, \tau, t)$ is only defined for $c \geq r$.

Sixth, $p(c, r, \tau, t) \geq r$ from a similar argument, since $p(c, r, \tau, t) = r$ when $c = r$ and $p_c(c, r, \tau, t) > 0$.

Seventh, $p(c, r, \tau, t) \leq c$, since

$$p(c, r, \tau, t) = c - U^{-1} \left(\delta(\tau) \left(\alpha(r, \tau, t)U(c-r) + \int_r^c U(c-y)h(y, \tau, t) dy \right) \right)$$

and $U^{-1}(\cdot) \geq 0$.

Next, $p_c(z, z, \tau, t) = 1 - \delta(\tau)\alpha(z, \tau, t)$, since

$$p_c(z, z, \tau, t) \\ = 1 - U^{-1'} \left(\delta(\tau) \left(\alpha(z, \tau, t)U(z-z) + \int_z^z U(z-y)h(y, \tau, t) dy \right) \right) \\ \cdot \delta(\tau) \left(\alpha(z, \tau, t)U'(z-z) + \int_z^z U'(z-y)h(y, \tau, t) dy \right) \\ = 1 - U^{-1'}(0)\delta(\tau)\alpha(z, \tau, t)U'(0) \\ = 1 - \delta(\tau)\alpha(z, \tau, t).$$

Finally, $p_r(z, z, \tau, t) = \delta(\tau)\alpha(z, \tau, t)$, since

$$\begin{aligned}
 p_r(z, z, \tau, t) &= -U^{-1'}\left(\delta(\tau)\left(\alpha(z, \tau, t)U(z-z) + \int_z^z U(z-y)h(y, \tau, t) dy\right)\right) \\
 &\quad \cdot \delta(\tau)\left(\frac{\partial\alpha(z, \tau, t)}{\partial z}U(z-z) - \alpha(z, \tau, t)U'(z-z) + U(z-z)h(z, \tau, t)\right) \\
 &= U^{-1'}\left(\delta(\tau)\left(\alpha(z, \tau, t)U(z-z) + \int_z^z U(z-y)h(y, \tau, t) dy\right)\right) \\
 &\quad \cdot \delta(\tau)\alpha(z, \tau, t)U'(z-z) \\
 &= U^{-1'}(0)\delta(\tau)\alpha(z, \tau, t)U'(0) \\
 &= \delta(\tau)\alpha(z, \tau, t),
 \end{aligned}$$

where the last line follows because $U^{-1'}(0) = U'(0) = 1$.

Next, we consider the second derivatives of the inverse cutoff function w.r.t. c and r , that is, $p_{cc}(c, r)$, $p_{rr}(c, r)$, and $p_{cr}(c, r)$. We drop the τ and t arguments for compactness.

For $p_{cc}(c, r)$, we have

$$\begin{aligned}
 p_c(c, r) &= 1 - U^{-1'}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy\right)\right) \\
 &\quad \cdot \delta\left[\alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y) dy\right],
 \end{aligned}$$

so

$$\begin{aligned}
 p_{cc}(c, r) &= -U^{-1''}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy\right)\right) \\
 &\quad \cdot \delta^2\left[\alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y) dy\right]^2 \\
 &\quad - U^{-1'}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy\right)\right) \\
 &\quad \cdot \delta\left[\alpha(r)U''(c-r) + \int_r^c U''(c-y)h(y) dy + h(c)\right].
 \end{aligned}$$

Under our assumptions, all these terms are bounded away from ∞ and $-\infty$, so $p_{cc}(c, r)$ is bounded away from ∞ and $-\infty$. Moreover, if we evaluate this expression at $c = r = z$, we get

$$\begin{aligned}
 p_{cc}(z, z) &= -U^{-1''}\left(\delta\left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y) dy\right)\right) \\
 &\quad \cdot \delta^2\left[\alpha(z)U'(z-z) + \int_z^z U'(z-y)h(y) dy\right]^2 \\
 &\quad - U^{-1'}\left(\delta\left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y) dy\right)\right)
 \end{aligned}$$

$$\begin{aligned}
& \cdot \delta \left[\alpha(z)U''(z-z) + \int_z^z U''(z-y)h(y) dy + h(z) \right] \\
& = -U^{-1''}(0)\delta^2\alpha(z)^2 - U^{-1'}(0)\delta[\alpha(z)U''(0) + h(z)] \\
& = -U''(0)\delta\alpha(z)(1 - \delta\alpha(z)) - \delta h(z).
\end{aligned}$$

For $p_{rr}(c, r)$, we have

$$p_r(c, r) = U^{-1'} \left(\delta \left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy \right) \right) \delta\alpha(r)U'(c-r),$$

so

$$\begin{aligned}
p_{rr}(c, r) & = -U^{-1''} \left(\delta \left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy \right) \right) \delta^2\alpha(r)^2U'(c-r)^2 \\
& \quad + U^{-1'} \left(\delta \left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy \right) \right) \\
& \quad \cdot \delta[\alpha'(r)U'(c-r) - \alpha(r)U''(c-r)].
\end{aligned}$$

Again, under our assumptions, all the terms in this expression are bounded away from ∞ and $-\infty$, so $p_{rr}(c, r)$ is bounded away from ∞ and $-\infty$. If we evaluate this expression at $c = r = z$, we get

$$\begin{aligned}
p_{rr}(z, z) & = -U^{-1''} \left(\delta \left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y) dy \right) \right) \delta^2\alpha(z)^2U'(z-z)^2 \\
& \quad + U^{-1'} \left(\delta \left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y) dy \right) \right) \\
& \quad \cdot \delta[\alpha'(z)U'(z-z) - \alpha(z)U''(z-z)] \\
& = -U^{-1''}(0)\delta^2\alpha(z)^2U'(0)^2 + U^{-1'}(0)\delta[\alpha'(z)U'(0) - \alpha(z)U''(0)] \\
& = U''(0)\delta^2\alpha(z)^2 + \delta[\alpha'(z) - \alpha(z)U''(0)] \\
& = -U''(0)\delta\alpha(z)(1 - \delta\alpha(z)) + \delta\alpha'(z).
\end{aligned}$$

For $p_{rc}(c, r) = p_{cr}(c, r)$, we have

$$p_r(c, r) = U^{-1'} \left(\delta \left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy \right) \right) \delta\alpha(r)U'(c-r),$$

so

$$\begin{aligned}
p_{rc}(c, r) & = U^{-1'} \left(\delta \left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy \right) \right) \delta\alpha(r)U''(c-r) \\
& \quad + U^{-1''} \left(\delta \left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y) dy \right) \right) \delta\alpha(r)U'(c-r) \\
& \quad \cdot \left[\delta \left(\alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y) dy \right) \right].
\end{aligned}$$

Again, all the terms are bounded away from ∞ and $-\infty$, so $p_{rc}(c, r)$ is bounded away from ∞ and $-\infty$. Evaluated at $c = r = z$, we get

$$\begin{aligned}
p_{rc}(z, z) &= U^{-1'} \left(\delta \left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y) dy \right) \right) \delta\alpha(z)U''(z-z) \\
&\quad + U^{-1''} \left(\delta \left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y) dy \right) \right) \delta\alpha(z)U'(z-z) \\
&\quad \cdot \left[\delta \left(\alpha(z)U'(z-z) + \int_z^z U'(z-y)h(y) dy \right) \right] \\
&= U^{-1'}(0)\delta\alpha(z)U''(0) + U^{-1''}(0)\delta\alpha(z)U'(0)\delta\alpha(z)U'(0) \\
&= \delta\alpha(z)U''(0) - U''(0)\delta\alpha(z)\delta\alpha(z) \\
&= U''(0)\delta\alpha(z)(1 - \delta\alpha(z)).
\end{aligned}$$

Proof of Proposition 8

Since the hazard rate of the first action (accept or reject the BP) is observed in the data and satisfies

$$\theta(t_1|p, r, \tau_0) = \lambda(t_1)(1 - F_V(r)), \quad (\text{SA.2})$$

it is clear that $\lambda(t_1)(1 - F_V(r))$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, \bar{T}]$.

We next show that this implies that $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$. By definition

$$\alpha(r, \tau_0, t_1) = \exp(\gamma F_V(r) - \gamma),$$

where

$$\gamma = \int_t^{t+\tau_0} \lambda(s) ds.$$

Therefore

$$\begin{aligned}
\alpha(r, \tau_0, t_1) &= \exp \left(-(1 - F_V(r)) \int_t^{t+\tau_0} \lambda(s) ds \right) \\
&= \exp \left(- \int_t^{t+\tau_0} \lambda(s)(1 - F_V(r)) ds \right).
\end{aligned}$$

Since $\lambda(t_1)(1 - F_V(r))$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$, this implies that $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.

Next we show that $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$. Again, by definition

$$\begin{aligned}
h(y, \tau_0, t_1) &= \exp(\gamma F_V(y) - \gamma) \gamma f_V(y) \\
&= \alpha(r, \tau_0, t_1) \int_t^{t+\tau_0} \lambda(s) f_V(y) ds.
\end{aligned}$$

Since $\lambda(t_1)(1 - F_V(y))$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$, its derivative $-\lambda(t_1)f_V(y)$ is also identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$. This implies that $\int_t^{t+\tau_0} \lambda(s)f_V(y) ds$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$. Therefore, $h(y, \tau_0, t)$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.

Next, we consider identification of $c(p, r, \tau_0, t_1)$. From Section 3.3, we know that

$$\Pr(B = 1|p, r, \tau_0, t_1) = \frac{1 - F_V(c(p, r, \tau_0, t_1))}{1 - F_V(r)},$$

where $\Pr(B = 1|p, r, \tau_0, t_1)$ is observed on the support $r \in [\underline{r}, \bar{r}]$, $p \in [r, \bar{p}]$, and $t_1 \in [0, T)$ (at τ_0). Therefore,

$$\begin{aligned} \Pr(B = 1|p, r, \tau_0, t_1) &= \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\lambda(t_1)(1 - F_V(r))} \\ &= \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\theta(t_1|p, r, \tau_0)}, \end{aligned}$$

and therefore $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$ is identified on the same support. Note that this term is the hazard rate of the BP being accepted.

Since we have already identified $\lambda(t_1)(1 - F_V(r))$ on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T)$, this implies that

$$c(p, r, \tau_0, t_1) = z,$$

where z satisfies

$$\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1))) = \lambda(t_1)(1 - F_V(z)). \quad (\text{SA.3})$$

Intuitively, this says that the cutoff at (p, r, τ_0, t_1) is equal to the hypothetical reserve price that would imply that the hazard rate of the first action is equal $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$.

It remains to be verified that we can identify the z that satisfies (SA.3). Note that the right-hand side (r.h.s.) of (SA.3) is strictly decreasing in z . Since $c(p, r, \tau_0, t_1) \geq \underline{r}$, the left-hand side (l.h.s.) is less than or equal to the r.h.s. at $z = \underline{r}$. Hence, we want to increase z above \underline{r} to satisfy (SA.3). The problem is that we only observe the r.h.s. for $z \in [\underline{r}, \bar{r}]$. However, as long as $c(p, r, \tau_0, t_1) \leq \bar{r}$, we can find a $z \in [\underline{r}, \bar{r}]$ that satisfies (SA.3). This implies that $c(p, r, \tau_0, t_1)$ is identified on the set (p, r, t_1) such that $c(p, r, \tau_0, t_1) \leq \bar{r}$.¹ This immediately implies that the inverse cutoff function $p(c, r, \tau_0, t_1)$ is identified on the set $r \in [\underline{r}, \bar{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \bar{r}]$.

Thus, we have shown that the following statements hold:

- (i) The function $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.
- (ii) The function $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, T - \tau_0)$.

¹This set exists. To show this, consider a situation where $r = \underline{r}$ and $p = \underline{r} + \varepsilon$ for some arbitrarily small ε . For small enough ε , $c(p, r, \tau_0, t_1)$ will be below \bar{r} (since c is continuous and $c(\bar{r}, \bar{r}, \tau_0, t_1) = \bar{r}$). Obviously the size of this set will depend on the range $[\underline{r}, \bar{r}]$.

(iii) The function $p(c, r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \bar{r}]$.

Recall that our integral equation

$$U(c - p(c, r, \tau_0, t_1)) = \delta(\tau_0) \left(\alpha(r, \tau_0, t_1) U(c - r) + \int_r^c U(c - y) h(y, \tau_0, t_1) dy \right) \quad (\text{SA.4})$$

can be reduced to

$$U''(c - r) = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)} U'(c - r), \quad (\text{SA.5})$$

where

$$\Phi(c, r, \tau_0, t_1) = \alpha(r, \tau_0, t_1) \left[\frac{(1 - p_c(c, r, \tau_0, t_1))}{p_r(c, r, \tau_0, t_1)} - 1 \right].$$

Identification of $\alpha(r, \tau_0, t_1)$, $h(y, \tau_0, t_1)$, and $p(c, r, \tau_0, t_1)$ implies that we can identify $\frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)}$ on $r \in [\underline{r}, \bar{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \bar{r}]$. Hence, by arguments similar to Proposition 3, equation (SA.5) identifies $U(\cdot)$ on $[0, \underline{r} - \bar{r}]$. By the same arguments as in Section 3.4, $\delta(\cdot)$ is identified at τ_0 .

Proof of Proposition 9

Assumption 8 further restricts the support of p to $[p_0 - \varepsilon, p_0 + \varepsilon]$. We also assume that p_0 is such that there exists a $r^* \in (\underline{r}, \bar{r})$ and a t_1^* such that $c(p_0, r^*, \tau_0, t_1^*) \in (\underline{r}, \bar{r})$. By the same arguments as in the proof of Proposition 8, we know that the following statements hold:

- (i) The function $\alpha(r, \tau, t_1)$ is identified on $r \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, \bar{T} - \tau_0)$.
- (ii) The function $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \bar{r}]$ and $t_1 \in [0, \bar{T} - \tau_0)$.

By the same arguments as above (and the condition that $c \in (\underline{r}, \bar{r})$), one can see that $c(p, r, \tau_0, t_1)$ will be identified for $p \in (p_0 - \varepsilon, p_0 + \varepsilon)$, $r \in (r - \eta, r + \eta)$, $t_1 = t_1^*$, and $\tau = \tau_0$ for η sufficiently small. Therefore, the inverse cutoff function $p(c, r, \tau_0, t_1)$ will be identified at $t_1 = t_1^*$, and $\tau = \tau_0$ in a ball centered at $(c(p_0, r^*, \tau_0, t_1^*), r^*)$. This implies that $p_r(c, r, \tau_0, t_1)$ and $p_c(c, r, \tau_0, t_1)$ are identified over that same region, as are $\Phi(c, r, \tau_0, t_1)$ and $\Phi_r(c, r, \tau_0, t_1)$. We have

$$\frac{U''(c - r)}{U'(c - r)} = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)}. \quad (\text{SA.6})$$

Hence, the Arrow–Pratt measure of risk aversion $\frac{U''}{U'}$ is identified at the point $c(p_0, r^*, \tau_0, t_1^*) - r^*$. Again, by the same arguments as Section 3.4, $\delta(\cdot)$ is identified at τ_0 .

APPENDIX SB: PROOF THAT $U''' \leq 0$ IS A SUFFICIENT CONDITION FOR ASSUMPTION 3

We have

$$M(v, r, \tau, t) = U^{-1} \left(\delta(\tau) \left(\alpha(r, \tau, t) U(v - r) + \int_r^v U(v - y) h(y, \tau, t) dy \right) \right)$$

so

$$\begin{aligned}
M_v(v, r, \tau, t) &= U^{-1} \left(\delta(\tau) \left(\alpha(r, \tau, t) U(v-r) + \int_r^v U(v-y) h(y, \tau, t) dy \right) \right) \\
&\quad \cdot \delta(\tau) \left(\alpha(r, \tau, t) U'(v-r) + \int_r^v U'(v-y) h(y, \tau, t) dy \right) \\
&= \frac{\delta(\tau) \left(\alpha(r, \tau, t) U'(v-r) + \int_r^v U'(v-y) h(y, \tau, t) dy \right)}{U' \left(U^{-1} \left(\delta(\tau) \left(\alpha(r, \tau, t) U(v-r) + \int_r^v U(v-y) h(y, \tau, t) dy \right) \right) \right)} \\
&< \left(\delta(\tau) \left(\alpha(\cdot) U'(v-r) + \int_r^v U'(v-y) h(\cdot) dy \right) \right. \\
&\quad \left. + \left(1 - \alpha(\cdot) - \int_r^v h(\cdot) dy \right) U'(0) \right) \\
&\quad / \left(U' \left(U^{-1} \left(\delta(\tau) \left(\alpha(\cdot) U(v-r) + \int_r^v U(v-y) h(\cdot) dy \right) \right) \right. \right. \\
&\quad \left. \left. + \left(1 - \alpha(\cdot) - \int_r^v h(\cdot) dy \right) U(0) \right) \right).
\end{aligned}$$

The strict inequality holds because of our normalizations that $U(0) = 0$ and $U'(0) = 1$, and because $1 - \alpha(r, \tau, t) - \int_r^v h(y, \tau, t) dy > 0$ for any finite v .

Therefore, we have

$$M_v(v, r, \tau, t) < \frac{\delta(\tau) EU'(x)}{U'(U^{-1}(\delta(\tau) EU(x)))},$$

where the random variable x has a mixed discrete-continuous distribution, taking the value 0 with probability $1 - \alpha(r, \tau, t) - \int_r^v h(y, \tau, t) dy$ and the value $v - r$ with probability $\alpha(r, \tau, t)$, and having density $h(y, \tau, t)$ over the interval $(0, v - r)$. Because $U'' \leq 0$ and $\delta(\tau) < 1$, Jensen's inequality implies that $\delta(\tau) EU(x) < U(Ex)$. Therefore,

$$\begin{aligned}
M_v(v, r, \tau, t) &< \frac{\delta(\tau) EU'(x)}{U'(U^{-1}(U(Ex)))} \\
&= \frac{\delta(\tau) EU'(x)}{U'(Ex)}.
\end{aligned}$$

Since $U''' \leq 0$, Jensen's inequality implies $EU'(x) \leq U'(Ex)$. Hence,

$$M_v(v, r, \tau, t) < \delta(\tau) < 1.$$

APPENDIX SC: IDENTIFICATION OF UTILITY FUNCTIONS FROM CERTAINTY EQUIVALENTS

Suppose that U is a utility function defined on $\mathcal{X} \subset \mathbb{R}$ and that \mathcal{F} is a collection of distributions with supports contained in \mathcal{X} . This generates a certainty equivalent functional

(also called a quasilinear mean)

$$m(F) = U^{-1}\left(\int U(x) dF(x)\right), \quad F \in \mathcal{F}.$$

Now suppose that we are given a collection of lotteries \mathcal{F} and a quasilinear mean functional m . If \mathcal{F} is sufficiently rich, it is plausible that the utility function U is uniquely determined (up to affine transformations) by m . We show that this is true even for a well chosen one-dimensional family of lotteries.

Our example is adapted from the proof of Theorem 83 in [Hardy, Littlewood, and Polya \(1952\)](#). Let $\mathcal{X} = [a, b]$ and consider the collection of lotteries $\mathcal{F} = \{F_t(x), t \in [0, 1]\}$, where the F_t are mixtures of point masses at the endpoints a and b :

$$F_t(x) = (1 - t)\delta_a(x) + t\delta_b(x).$$

Note that

$$m(F_0) = m(\delta_a) = a,$$

$$m(F_1) = m(\delta_b) = b,$$

and since m is continuous and strictly increasing, $m(F_t)$ takes every value in $[a, b]$.

Suppose that there is another function V satisfying

$$m(F) = V^{-1}\left(\int V(x) dF(x)\right), \quad F \in \mathcal{F}.$$

Let

$$\tilde{x}(t) = m(F_t) = U^{-1}[(1 - t)U(a) + tU(b)] = V^{-1}[(1 - t)V(a) + tV(b)].$$

We have

$$U(\tilde{x}(t)) = (1 - t)U(a) + tU(b),$$

and we can solve for t and $(1 - t)$:

$$t = \frac{U(\tilde{x}(t)) - U(a)}{U(b) - U(a)},$$

$$(1 - t) = \frac{U(b) - U(\tilde{x}(t))}{U(b) - U(a)}.$$

Now

$$\begin{aligned} V(\tilde{x}(t)) &= (1 - t)V(a) + tV(b) \\ &= \frac{U(b) - U(\tilde{x}(t))}{U(b) - U(a)} \cdot V(a) + \frac{U(\tilde{x}(t)) - U(a)}{U(b) - U(a)} \cdot V(b). \end{aligned}$$

This is a linear (in fact, affine) function of $U(\tilde{x}(t))$, so we can write

$$V(\tilde{x}(t)) = \alpha + \beta U(\tilde{x}(t)),$$

where α and β do not depend on t and $\beta > 0$. Since this holds for all $t \in [0, 1]$, we have

$$V(x) = \alpha + \beta U(x) \quad \forall x \in [a, b].$$

Thus V must be an affine transformation of U .

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