EXAMPLE OF AUTOREGRESSIVE AT SCALE 3

Consider $x_t = \phi x_{t-8} + \varepsilon_t$ with ε unit variance white noise

For convenience, denote $\phi = \rho^8$ so that $\boxed{x_t = \rho^8 x_{t-8} + \varepsilon_t}$

- CLASSICAL WOLD DECOMPOSITION of X

$$(I - \rho^8 L^8) x_t = \varepsilon_t \Rightarrow x_t = (I - \rho^8 L^8)^{-1} \varepsilon_t = \sum_{m=0}^{\infty} \rho^{8m} \varepsilon_{t-8m} = \varepsilon_t + \rho^8 \varepsilon_{t-8} + \rho^{16} \varepsilon_{t-16} + \dots$$

$$\Rightarrow x_t = \sum_{h=0}^{\infty} \alpha_h \varepsilon_{t-h} \quad \text{with} \quad \alpha_h = \begin{cases} \rho^h & \text{if } h = 8m \text{ for some } m \\ 0 & \text{otherwise} \end{cases}$$

- EXTENDED WOLD DECOMPOSITION of X

$\boxed{j=1}$: $\beta_k^{(1)} = \frac{1}{\sqrt{2}} (\alpha_{2k} - \alpha_{2k-1})$ $\quad k \in \mathbb{N}_0$

If $2k = 8m$ for some m, that is $k = 4m$, $\beta_{4m}^{(1)} = \frac{1}{\sqrt{2}} \rho^{8m}$

$$\Rightarrow \beta_k^{(1)} = \begin{cases} \frac{1}{\sqrt{2}} \rho^{2k} & \text{if } k = 4m \text{ for some } m \\ 0 & \text{otherwise} \end{cases} = \left\{ \frac{1}{\sqrt{2}}; 0; 0; 0; \frac{\rho^8}{\sqrt{2}}; 0; 0; 0; \frac{\rho^{16}}{\sqrt{2}}; \dots \right.$$

$k \in \mathbb{N}_0$

$\boxed{j=2}$ $\beta_k^{(2)} = \frac{1}{2} (\alpha_{4k} + \alpha_{4k+1} - \alpha_{4k+2} - \alpha_{4k+3})$

If $4k = 8m$ for some m, that is $k = 2m$, $\beta_{2m}^{(2)} = \frac{1}{2} \rho^{8m}$

$$\Rightarrow \beta_k^{(2)} = \begin{cases} \frac{1}{2} \rho^{4k} & \text{if } k = 2m \text{ for some } m \\ 0 & \text{otherwise} \end{cases} = \left\{ \frac{1}{2}; 0; \frac{\rho^8}{2}; 0; \frac{\rho^{16}}{2}; 0 \dots \right\}$$

$\boxed{j=3}$ $\beta_k^{(3)} = \frac{1}{\sqrt{2^3}} (\alpha_{8k} + \alpha_{8k+1} + \alpha_{8k+2} + \alpha_{8k+3} - \alpha_{8k+4} - \alpha_{8k+5} - \alpha_{8k+6} - \alpha_{8k+7})$

$$= \frac{1}{\sqrt{2^3}} \rho^{8k} \Rightarrow \beta_k^{(3)} = \frac{1}{\sqrt{2^3}} \rho^{8k} \qquad k \in \mathbb{N}_0$$

$\boxed{j \geq 4}$ $\beta_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right)$ $\quad k \in \mathbb{N}_0$

Note that 2^{j-1} is multiple of 8 and that we always find m such that $k2^j = 8m$. Hence $k2^j + 2^{j-1} = 8m + 2^{j-1}$

Also, $\alpha_{k2^j} = \rho^{k2^j} = \rho^{8m}$, $\alpha_{k2^j+2^{j-1}} = \rho^{k2^j+2^{j-1}} = \rho^{8m+2^{j-1}}$

$$\beta_k^{(j)} = \frac{1}{\sqrt{2^j}}\left(\rho^{8m} + \rho^{8m+8} + \dots + \rho^{8m+2^{j-1}-8} - \rho^{8m+2^{j-1}} - \dots - \rho^{8m+2^{j-1}+2^{j-1}-8}\right)$$

$$= \frac{1}{\sqrt{2^j}}\rho^{8m}\left(\sum_{i=0}^{\frac{2^{j-1}}{8}-1}\rho^{8i} - \rho^{2^{j-1}}\sum_{i=0}^{\frac{2^{j-1}}{8}-1}\rho^{8i}\right)$$

$$= \frac{1}{\sqrt{2^j}}\rho^{8m}\left(1-\rho^{2^{j-1}}\right)\sum_{i=0}^{\frac{2^{j-1}}{8}-1}(\rho^8)^i$$

$$= \frac{1}{\sqrt{2^j}}\rho^{8m}\left(1-\rho^{2^{j-1}}\right)\frac{1-\rho^{2^{j-1}}}{1-\rho^8}$$

$$= \frac{\rho^{k2^j}}{\sqrt{2^j}}\frac{\left(1-\rho^{2^{j-1}}\right)^2}{1-\rho^8} \qquad j \geq 4, \qquad k \in \mathbb{N}_0$$

- COMPARISON WITH EXTENDED WOLD DECOMPOSITION OF ARMA(1,1)

Consider the ARMA(1,1) process $x_t = r x_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$, where ε is the same unit variance white noise as before. The E.W.D provides

$$\hat{\beta}_0^{(j)} = \frac{1}{\sqrt{2^j}}\left\{1 + (r+\theta)\frac{1-2r^{2^{j-1}-1}+r^{2^j-1}}{1-r}\right\} \qquad \forall j \in \mathbb{N}$$

$$\hat{\beta}_k^{(j)} = \frac{1}{\sqrt{2^j}}(r+\theta)\frac{\left(1-r^{2^{j-1}}\right)^2}{1-r}r^{k2^j-1} \qquad k \in \mathbb{N},\ j \in \mathbb{N}$$

$$= \frac{r^{k2^j}}{\sqrt{2^j}}\frac{r+\theta}{r}\frac{\left(1-r^{2^{j-1}}\right)^2}{1-r}$$

We look for r, θ such that $\hat{\beta}_k^{(j)} = \beta_k^{(j)} \quad \forall k \in \mathbb{N},\ j \geq 4$

First of all, $\rho^{k2^j} = r^{k2^j} \Rightarrow \boxed{r = \rho}$

Then,
$$\frac{\rho+\theta}{\rho}\frac{\left(1-\rho^{2^{j-1}}\right)^2}{1-\rho} = \frac{\left(1-\rho^{2^{j-1}}\right)^2}{1-\rho^8}$$

$$\Rightarrow \rho+\theta = \frac{\rho(1-\rho)}{1-\rho^8} \Rightarrow \theta = \frac{\rho(1-\rho)}{1-\rho^8} - \rho = \frac{\cancel{\rho}-\rho^2-\cancel{\rho}+\rho^9}{1-\rho^8}$$

$$\Rightarrow \boxed{\theta = \rho^2\cdot\frac{\rho^7-1}{1-\rho^8}}$$

If $\phi = 0{,}9$,
$r = \rho = 0.9869$
$\theta = -0.8578$

With this choice of parameters,

$$\hat{\beta}_0^{(j)} = \frac{1}{\sqrt{2^j}}\left\{1 + \left(\rho + \rho^2 \cdot \frac{\rho^7 - 1}{1-\rho^8}\right) \frac{1 - 2\rho^{2^{j-1}-1} + \rho^{2^j-1}}{1-\rho}\right\}$$

$$= \frac{1}{\sqrt{2^j}}\left\{1 + \rho\, \frac{1-\rho^8+\rho^8-\rho}{1-\rho^8} \cdot \frac{1 - 2\rho^{2^{j-1}-1} + \rho^{2^j-1}}{1-\rho}\right\}$$

$$= \frac{1}{\sqrt{2^j}}\left\{1 + \frac{\rho - 2\rho^{2^{j-1}} + \rho^{2^j}}{1-\rho^8}\right\} \qquad \forall j \in \mathbb{N}$$

$$\hat{\beta}_k^{(j)} = \frac{\rho^{k2^j}}{\sqrt{2^j}} \frac{(1-\rho^{2^{j-1}})^2}{1-\rho^8} \qquad k \in \mathbb{N},\ j \in \mathbb{N}$$

Although $\hat{\beta}_k^{(j)} = \beta_k^{(j)}$ $\forall k \in \mathbb{N}$, $j \geq 4$, we have

1) $\cdot\ \hat{\beta}_0^{(j)} \neq \beta_0^{(j)}$ $\quad j \geq 4$

2) $\cdot\ \hat{\beta}_k^{(j)} \neq \beta_k^{(j)}$ $\quad j = 1,2,3,\ k \in \mathbb{N}_0$

1) $$\hat{\beta}_0^{(j)} = \frac{1}{\sqrt{2^j}}\left\{\frac{1-\rho^8+\rho-2\rho^{2^{j-1}}+\rho^{2^j}}{1-\rho^8}\right\} = \frac{1}{\sqrt{2^j}}\left\{\frac{1-2\rho^{2^{j-1}}+\rho^{2^j}}{1-\rho^8} + \rho\,\frac{1-\rho^7}{1-\rho^8}\right\}$$

$$= \frac{(1-\rho^{2^{j-1}})^2}{\sqrt{2^j}\,(1-\rho^8)} + \frac{\rho}{\sqrt{2^j}}\frac{1-\rho^7}{1-\rho^8} = \beta_0^{(j)} + \frac{\rho}{\sqrt{2^j}}\frac{1-\rho^7}{1-\rho^8}$$

$$\Rightarrow \hat{\beta}_0^{(j)} - \beta_0^{(j)} = \frac{\rho}{\sqrt{2^j}}\frac{1-\rho^7}{1-\rho^8} \qquad j \geq 4$$

2) $\boxed{j=1}$ $$\hat{\beta}_0^{(1)} = \frac{1}{\sqrt{2}}\left\{1 + \frac{\rho - 2\rho + \rho^2}{1-\rho^8}\right\} = \frac{1}{\sqrt{2}} + \frac{\rho}{\sqrt{2}} \cdot \frac{\rho-1}{1-\rho^8} = \beta_0^{(1)} + \frac{\rho}{\sqrt{2}}\frac{\rho-1}{1-\rho^8}$$

$$\Rightarrow \hat{\beta}_0^{(1)} - \beta_0^{(1)} = \frac{\rho}{\sqrt{2}}\frac{\rho-1}{1-\rho^8}$$

$$\hat{\beta}_k^{(1)} = \frac{\rho^{2k}}{\sqrt{2}}\frac{(1-\rho)^2}{1-\rho^8} \neq \beta_k^{(1)} \quad \text{since} \quad \beta_k^{(1)} = \begin{cases} \frac{\rho^{2k}}{\sqrt{2}} & \text{if } k = 4m \\ 0 & \text{otherwise} \end{cases}$$

$k \in \mathbb{N}$.

$\boxed{j=2}$ $\hat{\beta}_0^{(2)} = \frac{1}{2}\left\{1 + \frac{\rho - 2\rho^2 + \rho^4}{1-\rho^8}\right\} = \frac{1}{2} + \frac{\rho}{2}\,\frac{1-2\rho+\rho^3}{1-\rho^8} = \beta_0^{(2)} + \frac{\rho}{2}\,\frac{1-2\rho+\rho^3}{1-\rho^8}$

For $k \in \mathbb{N}$, $\hat{\beta}_k^{(2)} = \frac{\rho^{4k}}{2}\,\frac{(1-\rho^2)^2}{1-\rho^8} \neq \beta_k^{(2)} = \begin{cases} \frac{\rho^{4k}}{2} & \text{if } k = 2m \\ 0 & \text{otherwise} \end{cases}$

$\boxed{j=3}$ $\hat{\beta}_0^{(3)} = \frac{1}{\sqrt{2^3}}\left\{1 + \frac{\rho - 2\rho^4 + \rho^8}{1-\rho^8}\right\} = \frac{1}{\sqrt{2^3}} + \frac{\rho}{\sqrt{2^3}}\,\frac{1-2\rho^3+\rho^7}{1-\rho^8} = \beta_0^{(3)} + \frac{\rho}{\sqrt{2^3}}\,\frac{1-2\rho^3+\rho^7}{1-\rho^8}$

$\Rightarrow \hat{\beta}_0^{(3)} - \beta_0^{(3)} = \frac{\rho}{\sqrt{2^3}} \cdot \frac{1-2\rho^3+\rho^7}{1-\rho^8}$

For $k \in \mathbb{N}$, $\hat{\beta}_k^{(3)} = \frac{\rho^{8k}}{\sqrt{2^3}}\,\frac{(1-\rho^4)^2}{1-\rho^8} \neq \beta_k^{(3)} = \frac{\rho^{8k}}{\sqrt{2^3}}$