0.1 Applications of Theorems 3 and 4

An interesting application of Theorem 3 consists in analysing the properties of time series obtained by summing two autoregressive processes defined on different time grids. The idea is to construct the dynamics of the resulting process scale by scale, by summing the persistent components of the given autoregressive processes. The autocorrelation function (ACF) of the final time series depends on the parameters of the two addends. We show by means of a numerical example that different choices of such parameters have different impacts on the short and long lags of the resulting ACF.

Given a common innovation process $\boldsymbol{\varepsilon} = \{\varepsilon\}_{t\in\mathbb{Z}}$, we let the coefficients $\delta_i^{(j)}$ in Theorem 3 be the Haar ones, i.e.

$$\delta_i^{(j)} = \begin{cases} \frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{0, \ldots, 2^{j-1}-1\}, \\ -\frac{1}{\sqrt{2^j}} & \text{if} \quad i \in \{2^{j-1}, \ldots, 2^j - 1\}. \end{cases}$$

We start from two weakly stationary purely non-deterministic time series, $\mathbf{x} = \{x_t\}_{t\in\mathbb{Z}}$ and $\mathbf{y} = \{y_t\}_{t\in\mathbb{Z}}$, defined by two families of multiscale impulse responses: $\left\{\beta_{x,k}^{(j)}\right\}_{j,k}$ and $\left\{\beta_{y,k}^{(j)}\right\}_{j,k}$ with $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Theorem 4 ensures that the cone C_t^δ coincides with the space $\mathcal{H}_t(\boldsymbol{\varepsilon})$, hence $z_t = x_t + y_t$ belongs to $\mathcal{H}_t(\boldsymbol{\varepsilon})$.

We assume the multiscale impulse response functions of x_t to be

$$\beta_{x,k}^{(j)} = \frac{\rho_x^{k2^j}}{\sqrt{2^j}} \frac{\left(1-\rho_x^{2^{j-1}}\right)^2}{1-\rho_x}, \qquad j \in \mathbb{N}, k \in \mathbb{N}_0,$$

i.e. the extended Wold coefficients of a weakly stationary $AR(1)$ process with parameter ρ_x, with $|\rho_x| < 1$.[1]

We now fix a scale J and we define the multiscale impulse responses of y_t by setting

$$\beta_{y,k}^{(j)} = \begin{cases} 0 & \text{if} \quad j \in \{0, \ldots, J\}, \\ \frac{\rho_y^{k2^{j-J}}}{\sqrt{2^{j-J}}} \frac{\left(1-\rho_y^{2^{j-J-1}}\right)^2}{1-\rho_y} & \text{if} \quad j \geqslant J+1 \end{cases}$$

[1] See Appendix **??** for the computation of the multiscale impulse responses of an $AR(1)$ process.

with $|\rho_y| < 1$. A simple comparison with $\beta^{(j)}_{x,k}$ shows that the coefficients $\beta^{(j)}_{y,k}$ identify an autoregressive process defined on the grid $S^{(J)}_t$. Therefore,

$$y_t = \sum_{k=0}^{+\infty} \rho_y^k \tilde{\varepsilon}^{(J)}_{t-k2^J},$$

where $\tilde{\boldsymbol{\varepsilon}}^{(\boldsymbol{J})} = \left\{ \tilde{\varepsilon}^{(J)}_{t-k2^J} \right\}_{k\in\mathbb{Z}}$ is the unit variance white noise[2]

$$\tilde{\varepsilon}^{(J)}_t = \frac{1}{\sqrt{2^J}} \sum_{i=0}^{2^J-1} \varepsilon_{t-i}.$$

In contrast with $\mathbf{x}$, which is a standard $AR(1)$ process, we call $\mathbf{y}$ an $AR(1)$ process *with horizon* 2^J. From the standpoint of interpretation we can think of $\mathbf{x}$ as a daily process, while $\mathbf{y}$ is a time series acting on longer lags (monthly, yearly...), depending on the choice of J.

By construction, the multiscale impulse responses of $\mathbf{z} = \{z_t\}_{t\in\mathbb{Z}}$ are the sum of the extended Wold coefficients of $\mathbf{x}$ and $\mathbf{y}$:

$$\beta^{(j)}_{z,k} = \beta^{(j)}_{x,k} + \beta^{(j)}_{y,k}, \qquad j \in \mathbb{N}, k \in \mathbb{N}_0.$$

By Theorem 3, therefore, $\mathbf{z}$ is weakly stationary and purely non-deterministic.[3] In general, the sum of two autoregressive processes with different horizons is not stationary, but the structure of the shocks at different scales required by Theorem 3 ensures that this is the case. Hereafter we plot the ACF of $\mathbf{z}$ for some choices of the parameters ρ_x, ρ_y and of the scale J.

We start by setting $\rho_x = 0.7$, $\rho_y = 0.9$ and $J = 3$. Overall the process $\mathbf{z}$ is more persistent than either $\mathbf{x}$ or $\mathbf{y}$ taken alone and, in Figure 1, we see that its ACF is piecewise approximated by the ACFs of $AR(1)$ processes with different parameters. In particular, the ACF of an $AR(1)$ with parameter 0.9 provides a good approximation for the short lags, but not for the others. However, for the intermediate lags we can employ an $AR(1)$ with parameter

[2] Note that the innovation process $\tilde{\boldsymbol{\varepsilon}}^{(\boldsymbol{J})}$ is different from the detail process $\boldsymbol{\varepsilon}^{(\boldsymbol{J})}$.

[3] All the assumptions of Theorem 3 are satisfied. Indeed, Assumption 1) holds by construction and assumption 3) follows from the use of the Haar coefficients. As for 2), the square-summability of the $\beta^{(j)}_{z,k}$ is ensured by the square-summability of the $\beta^{(j)}_{x,k}$ and the $\beta^{(j)}_{y,k}$, that come from the well-defined Extended Wold Decomposition of the processes $\mathbf{x}$ and $\mathbf{y}$ respectively.

0.97 and for the long lags we can take the parameter 0.98. Moreover, the persistence of $\mathbf{z}$ increases with the scale J, as we discuss in Appendix A.

In addition, alternative choices of ρ_y allow to modify the short lags of the ACF of $\mathbf{x}$, while keeping the long lags unchanged. For instance, this is possible by setting $\rho_y = \rho_x^{2^J}$, as we show in Appendix A. Summing up, the scale-by-scale construction of $\mathbf{z}$ and the choices of the parameters ρ_x, ρ_y and of the scale J allow us to obtain ACFs with predetermined features. Further numerical examples are in Appendix A.

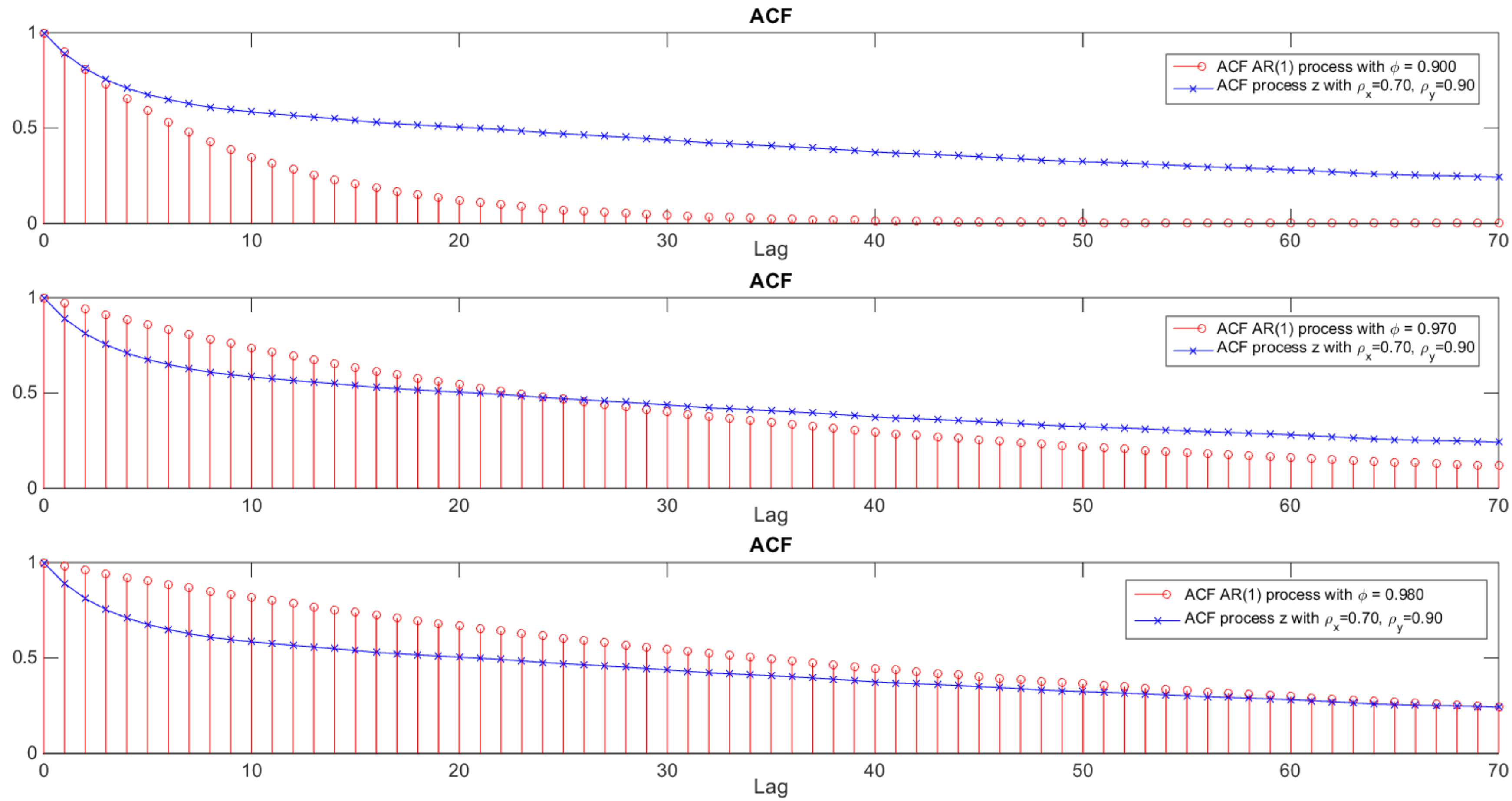

Figure 1: Comparison between the ACF of the process $\mathbf{z}$ with $\rho_x = 0.7$, $\rho_y = 0.9$, $J = 3$ and the ACF of an $AR(1)$ process with parameter $\phi = 0.9, 0.97, 0.98$.

A Further applications of Theorems 3 and 4

We provide some complementary simulations related to the examples discussed in Section 0.1, where $\mathbf{x}$ is a standard $AR(1)$ process and $\mathbf{y}$ is an $AR(1)$ with horizon 2^J.

First, we focus on the choice of the scale J. Indeed, as J increases, the process $\mathbf{y}$ concentrates on more and more persistent time scales. In Figure 2 we plot the ACF of $\mathbf{z}$ in case $J = 3, 4, 5$ by setting the parameters $\rho_x = 0.7$ and $\rho_y = 0.9$. We see that the tail of the ACF increases as J goes up, revealing more and more persistence.

These examples illustrate that the persistence of a standard $AR(1)$ process may increase when we sum an $AR(1)$ with horizon 2^J. However, other choices of ρ_y allow to modify the short lags of the ACF of $\mathbf{x}$, while keeping the long lags unchanged. Indeed, by setting $\rho_y = \rho_x^{2^J}$ we have, for any $j \geqslant J+1$ and $k \in \mathbb{N}_0$,

$$\beta_{y,k}^{(j)} = \frac{\rho_x^{k2^j}}{\sqrt{2^j}} \frac{\left(1 - \rho_x^{2^{j-1}}\right)^2}{1 - \rho_x^{2^J}}.$$

Then, for all $j \geqslant J+1$ and $k \in \mathbb{N}_0$, the multiscale impulse responses of $\mathbf{z}$ are

$$\beta_{z,k}^{(j)} = \frac{\rho_x^{k2^j}}{\sqrt{2^j}} \frac{\left(1 - \rho_x^{2^{j-1}}\right)^2}{1 - \rho_x} \left\{ 1 + \frac{\sqrt{2^J}\,(1 - \rho_x)}{1 - \rho_x^{2^J}} \right\}.$$

As $1 + \frac{\sqrt{2^J}(1-\rho_x)}{1-\rho_x^{2^J}}$ is not dependent on j or k, at scales $j \geqslant J+1$ we see that $\beta_{z,k}^{(j)}$ are proportional to the extended Wold coefficients of an $AR(1)$ with parameter ρ_x. Differently stated, at scales $j \geqslant J+1$, the coefficients $\beta_{z,k}^{(j)}$ coincide with the multiscale impulse responses of an $AR(1)$ with parameter ρ_x, whose error variance is

$$1 + \frac{\sqrt{2^J}\,(1 - \rho_x)}{1 - \rho_x^{2^J}}.$$

As the ACF of an $AR(1)$ does not depend on the variance of the innovations and the scales $j \geqslant J+1$ characterize the long-run behaviour of $\mathbf{z}$, we find that the long lags of the ACF of $\mathbf{z}$ coincide with the ones of the process $\mathbf{x}$. Nevertheless, the short lags of the ACF of $\mathbf{z}$ are modified, as we see in Figure 3.

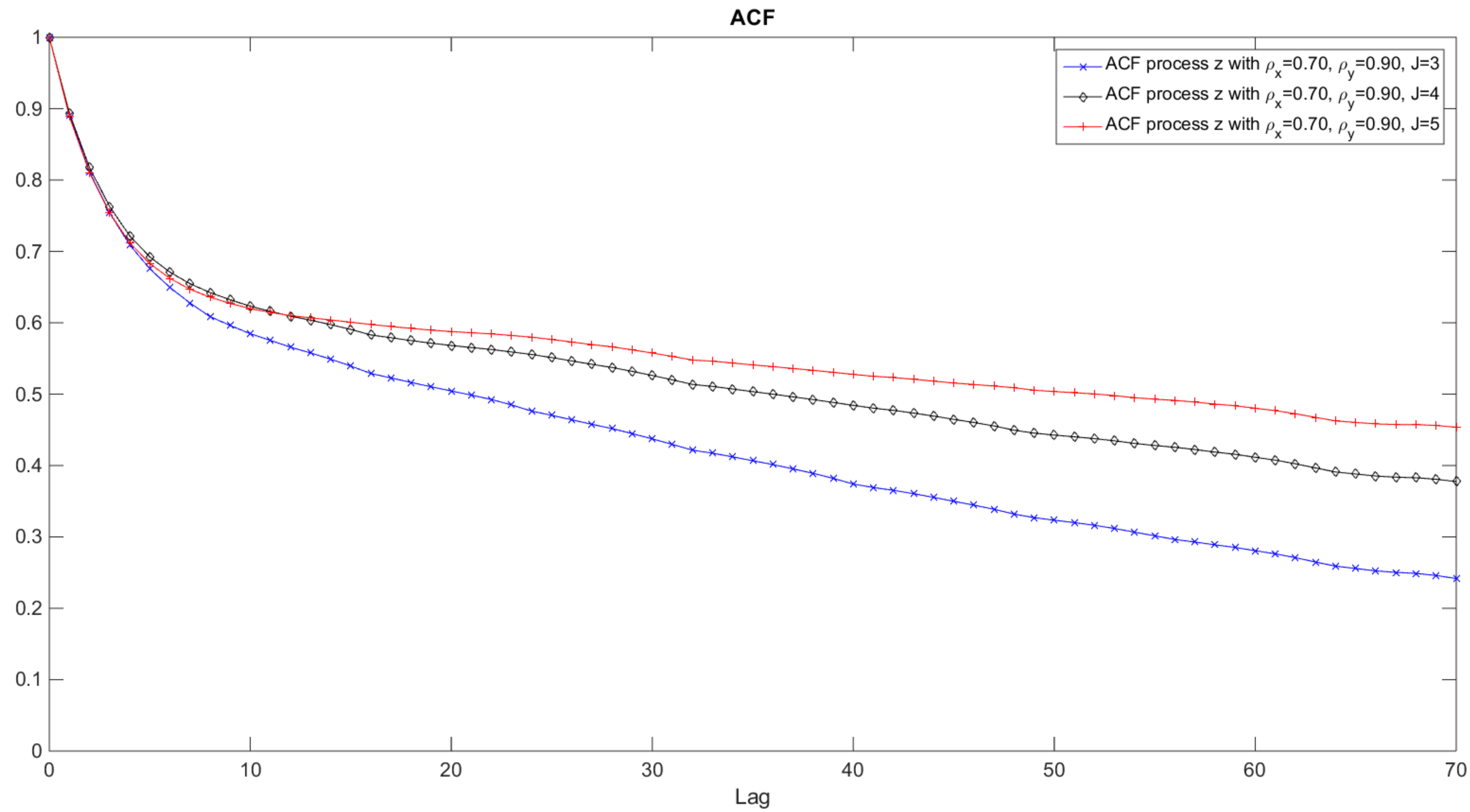

Figure 2: ACF of the process $\mathbf{z}$ with $\rho_x = 0.7$, $\rho_y = 0.9$ and $J = 3, 4, 5$.

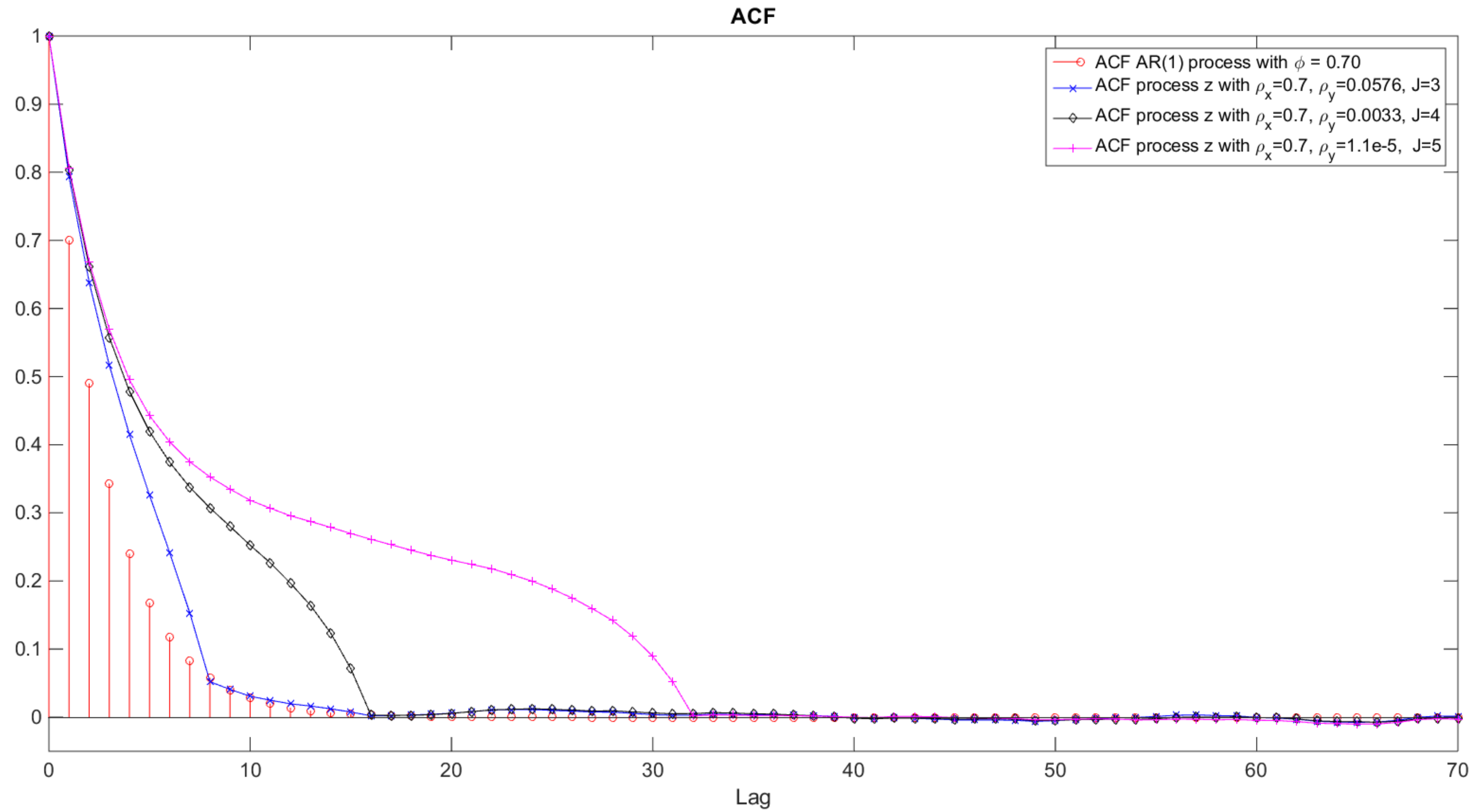

Figure 3: ACF of the process $\mathbf{z}$ with $\rho_x = 0.7$, $\rho_y = \rho_x^{2^J}$ and $J = 3, 4, 5$.